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LARGE DEVIATIONS FOR THE SQUARED RADIAL
ORNSTEIN–UHLENBECK PROCESS¹⁾

Рассматривается обобщенный радиальный процесс Орнштейна–Уленбека (процесс Кокса–Ингерсолла–Росса), и для оценок максимального правдоподобия его пространственного коэффициента и коэффициента сноса устанавливаются принципы больших уклонений. Мы сосредоточиваем внимание на наиболее доступной для изучения ситуации, когда пространственный параметр строго больше двух, а параметр сноса отрицателен. В отличие от предыдущих работ по этой теме, принципы больших уклонений устанавливаются в случае, когда оба параметра оцениваются одновременно.

Ключевые слова и фразы: принцип больших уклонений, функция уклонений, оценка максимального правдоподобия, принцип сжатия, теорема Гертнера–Эллиса, процесс Кокса–Ингерсолла–Росса.

1. Introduction. The generalized squared radial Ornstein–Uhlenbeck process, also known as the Cox–Ingersoll–Ross process, is the strong solution of the stochastic differential equation

$$dX_t = (a + bX_t) dt + 2\sqrt{X_t} dB_t \quad (1.1)$$

with the initial state $X_0 = x \geq 0$, the dimensional parameter $a > 0$, the drift coefficient $b \in \mathbf{R}$ and (B_t) a standard Brownian motion. The behavior of the process was widely investigated and depends on the values of both coefficients a and b . We restrict ourself to the most tractable situation, where $a > 2$ and $b < 0$. In this case, the process is ergodic and never reaches zero.

We estimate parameters a and b at the same time using a trajectory of the process over the time interval $[0, T]$. The maximum likelihood estimators (MLE) of a and b are given by

$$\begin{aligned} \hat{a}_T &= \frac{\int_0^T X_t dt \int_0^T \frac{1}{X_t} dX_t - T(X_T - x)}{\int_0^T X_t dt \int_0^T \frac{1}{X_t} dt - T^2}, \\ \hat{b}_T &= \frac{(X_T - x) \int_0^T \frac{1}{X_t} dt - T \int_0^T \frac{1}{X_t} dX_t}{\int_0^T X_t dt \int_0^T \frac{1}{X_t} dt - T^2}. \end{aligned} \quad (1.2)$$

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Overbeck [14] showed that $\hat{a}_T$ and $\hat{b}_T$ converge almost surely to a and b . In addition, he proved that

$$\sqrt{T} \begin{pmatrix} \hat{a}_T - a \\ \hat{b}_T - b \end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}(0, 4C^{-1}),$$

where

$$C = \begin{pmatrix} -b/(a-2) & 1 \\ 1 & -a/b \end{pmatrix}.$$

Moderate deviation results for $\hat{a}_T$ and $\hat{b}_T$ are achieved in [8]. In addition, Zani [15] established large deviation principles (LDP) for the MLE of a assuming b known and, conversely, for the MLE of b assuming a known. Our goal is to extend her results to the case, where both parameters are estimated simultaneously. Our method is also different and we explain how we have simplified her approach at the beginning of subsection 2.2 and section 4, using a new strategy introduced by Bercu and Richou in [3] for the study of the Ornstein–Uhlenbeck process with shift.

The paper is organized as follows. Section 2 is devoted to an LDP for the couple $(\hat{a}_T, \hat{b}_T)$, which is obtained via LDPs for two other couples of estimators constructed on the MLE. Before we prove those results, which is respectively the aim of sections 5 and 6, we investigate in section 3 LDPs for some useful functionals of the process and compute in section 4 the normalized cumulant generating function of a given quadruplet, which is a keystone for every LDP we establish in this paper. Technical proofs are postponed to the appendix A to E (sections 7–11).

2. Main results. We start by rewriting the estimators $\hat{a}_T$ and $\hat{b}_T$ in such a way that they are much easier to handle. We need the assumption that the starting point x is positive to apply the well-known Itô formula to $\ln X_T$. We obtain that

$$\int_0^T \frac{1}{X_t} dX_t = \ln X_T - \ln x + 2 \int_0^T \frac{1}{X_t} dt \quad (2.1)$$

which leads to

$$\hat{a}_T = \frac{S_T(2\Sigma_T + L_T) - (X_T - x)/T}{V_T} \quad \text{and} \quad \hat{b}_T = \frac{((X_T - x)/T - 2)\Sigma_T - L_T}{V_T},$$

where $V_T = S_T \Sigma_T - 1$ with

$$S_T = \frac{1}{T} \int_0^T X_t dt \quad \text{and} \quad \Sigma_T = \frac{1}{T} \int_0^T \frac{1}{X_t} dt,$$

and

$$L_T = \frac{\ln X_T - \ln x}{T}.$$

For the remaining part of the paper, we remove the parts involving the starting point x . It does not change the large deviation results because both couple of estimators, with and without x and $\ln x$, are exponentially equivalent so that they share the same LDP. Thus, we now suppose that the estimators are respectively given by

$$\hat{a}_T = \frac{S_T(2\Sigma_T + L_T) - X_T/T}{V_T} \quad \text{and} \quad \hat{b}_T = \frac{(X_T/T - 2)\Sigma_T - L_T}{V_T} \quad (2.2)$$

where $L_T = \ln X_T/T$. As the rate function of the LDP for the MLE will turn out to be not directly computable, we first consider two couples of simplified estimators constructed from $(\hat{a}_T, \hat{b}_T)$ using the fact that L_T and X_T/T both tend to zero almost surely for T going to infinity. For those estimators LDPs are more straightforward and will finally be involved in the computation of the rate function of the MLE $(\hat{a}_T, \hat{b}_T)$. All the LDPs established in this paper are satisfied with speed T .

2.1. Simplified estimators. A first strategy to propose simplified estimators of a and b is to remove the logarithmic term L_T in the expression of $\hat{a}_T$ and $\hat{b}_T$ given by (2.2). This way, we obtain a new couple $(\tilde{a}_T, \tilde{b}_T)$ defined by

$$\tilde{a}_T = \frac{2S_T\Sigma_T - X_T/T}{V_T} \quad \text{and} \quad \tilde{b}_T = \frac{(X_T/T - 2)\Sigma_T}{V_T}. \quad (2.3)$$

It is clear that $\tilde{a}_T$ and $\tilde{b}_T$ converge almost surely to a and b . Moreover, we also have the same central limit theorem (CLT)

$$\sqrt{T} \begin{pmatrix} \tilde{a}_T - a \\ \tilde{b}_T - b \end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}(0, 4C^{-1}).$$

The proof of this result can be found in appendix A. We also state an LDP for the couple $(\tilde{a}_T, \tilde{b}_T)$ assuming both parameters a and b unknown.

Theorem 2.1. *The couple $(\tilde{a}_T, \tilde{b}_T)$ satisfies an LDP with good rate function*

$$J_{a,b}(\alpha, \beta) = \begin{cases} \frac{(a-2)^2\beta}{8(2-\alpha)} \left(1 + \frac{(2-\alpha)b}{\beta(a-2)}\right)^2 + 2\beta - b & \text{if } \alpha > 2, \ b/3 \leq \beta < 0 \text{ or if } \alpha < 2, \ \beta > 0, \\ \frac{(a-2)^2\beta}{8(2-\alpha)} \left(1 + \frac{(2-\alpha)b}{\beta(a-2)}\right)^2 - \frac{\beta}{4} \left(1 - \frac{b}{\beta}\right)^2 & \text{if } \alpha > 2, \ \beta \leq b/3, \\ -b & \text{if } (\alpha, \beta) = (2, 0), \\ +\infty & \text{otherwise.} \end{cases}$$

P r o o f. The proofs of this theorem and the two following corollaries are postponed to section 5.

We give the shape of this rate function in Figure 1 in the particular case $(a, b) = (4, -1)$ and over $[3, 5] \times [-4, -0.5]$. One can notice that the rate function reaches zero at point $(4, -1)$.

Fig. 1. Rate function for the couple of simplified estimators $(\tilde{a}_T, \tilde{b}_T)$.

LDPs for each estimator $\tilde{a}_T$ and $\tilde{b}_T$ immediately follow from the contraction principle (see [5, Theorem 4.2.1 and the subsequent remarks]) which is recalled here for completeness.

Lemma 2.1 (contraction principle). *Let $(Z_T)_T$ be a sequence of random variables in $\mathbf{R}^d$ satisfying an LDP with good rate function I , and let $g: \mathbf{R}^d \rightarrow \mathbf{R}^n$ be a continuous function over $\mathcal{D}_I = \{x \in \mathbf{R}^d \mid I(x) < +\infty\}$. The sequence $(g(Z_T))_T$ satisfies an LDP with good rate function J defined for all $y \in \mathbf{R}^n$ by*

$$J(y) = \inf_{\{x \in \mathcal{D}_I \mid g(x)=y\}} I(x).$$

Corollary 2.1. *The sequence $(\tilde{a}_T)$ satisfies an LDP with good rate function*

$$J_a(\alpha) = \begin{cases} \frac{b}{4} \left(a - 6 - \sqrt{(a-2)^2 + 16(2-\alpha)} \right) & \text{if } \alpha \leq \ell_a, \\ \frac{b}{4} \left(a - \sqrt{\alpha \left(\frac{(a-2)^2}{\alpha-2} + 2 \right)} \right) & \text{if } \alpha \geq \ell_a, \end{cases}$$

where $\ell_a = 10/9 + (1/9)\sqrt{64 + 9(a-2)^2}$.

Corollary 2.2. *The sequence $(\tilde{b}_T)$ satisfies an LDP with good rate function*

$$J_b(\beta) = \begin{cases} -\frac{\beta}{4} \left(1 - \frac{b}{\beta}\right)^2 & \text{if } \beta \leq \frac{b}{3}, \\ 2\beta - b & \text{if } \beta \geq \frac{b}{3}. \end{cases}$$

R e m a r k 2.1. This rate function is the same as the one obtained by Zani [15] for the MLE of b assuming a known.

Figure 3 displays the rate functions J_a and J_b in the particular case, where $(a, b) = (4, -1)$.

A second strategy to propose simplified estimators of a and b is to remove the term X_T/T in the expression of $\hat{a}_T$ and $\hat{b}_T$ given by (2.2). Then, we obtain a new couple $(\check{a}_T, \check{b}_T)$ defined by

$$\check{a}_T = \frac{S_T(2\Sigma_T + L_T)}{V_T} \quad \text{and} \quad \check{b}_T = \frac{-2\Sigma_T - L_T}{V_T}. \quad (2.4)$$

As previously, $\check{a}_T$ and $\check{b}_T$ converge almost surely to a and b , and

$$\sqrt{T} \begin{pmatrix} \check{a}_T - a \\ \check{b}_T - b \end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}(0, 4C^{-1}).$$

The proof of this result is given in appendix A (section 7). Again, we establish an LDP for the couple $(\check{a}_T, \check{b}_T)$ and deduce as corollaries LDPs for both estimators assuming a and b unknown.

Theorem 2.2. *The couple $(\check{a}_T, \check{b}_T)$ satisfies an LDP with good rate function*

$$K_{a,b}(\alpha, \beta) = \begin{cases} \frac{a}{4}(b - \beta) - \frac{\alpha}{8\beta}(b^2 - \beta^2) - \frac{\beta}{\alpha}(\sqrt{2} + \sqrt{C_\alpha})^2 & \text{if } \beta < 0, 0 < \alpha \leq \alpha_a \text{ or if } \beta > 0, \alpha < 0, \\ \frac{a}{4}(b - \beta) - \frac{\alpha}{8\beta}(b^2 - \beta^2) - \frac{\beta(a - \alpha)^2}{8(\alpha - 2)} & \text{if } \beta < 0, \alpha \geq \alpha_a, \\ -\frac{b}{4}(4 - a + \sqrt{a^2 + 16}) & \text{if } (\alpha, \beta) = (0, 0), \\ +\infty & \text{otherwise,} \end{cases}$$

where $C_\alpha = (a - \alpha)^2/8 + 2 - \alpha$ and $\alpha_a = -2(a/2 - 2 - \sqrt{a^2 - 2a + 4})/3$.

One can observe that the rate functions $J_{a,b}$ and $K_{a,b}$ are equal over some domain of $\mathbf{R}^2$. It is possible to see it on Figure 2 which is quite similar to the previous one and displays the rate function $K_{a,b}$ in the particular case where $(a, b) = (4, -1)$.

Fig. 2. Rate function for the couple of simplified estimators $(\check{a}_T, \check{b}_T)$.

Corollary 2.3. *The sequence $(\check{a}_T)$ satisfies an LDP with good rate function*

$$K_a(\alpha) = \begin{cases} -\frac{b}{4}(4 - a + \sqrt{a^2 + 16}) & \text{if } \alpha = 0, \\ K_{a,b}(\alpha, \beta_b) & \text{if } \alpha < \alpha_a, \alpha \neq 0, \\ \frac{b}{4}\left(a - \sqrt{\alpha\left(\frac{(a-2)^2}{\alpha-2} + 2\right)}\right) & \text{if } \alpha \geq \alpha_a, \end{cases}$$

where $\beta_b = b\alpha(16\sqrt{2C_\alpha} + a^2 - 8\alpha + 32)^{-1/2}$ and α_a, C_α are defined in Theorem 2.2.

Corollary 2.4. *The sequence $(\check{b}_T)$ satisfies an LDP with good rate function*

$$K_b(\beta) = \inf\{K_{a,b}(\alpha, \beta) \mid \alpha \in \mathbf{R}\}.$$

In particular,

$$K_b(0) = K_{a,b}(0, 0) = -\left(\frac{b}{4}\right)(4 - a + \sqrt{a^2 + 16}).$$

Figure 3 displays the rate functions K_a and K_b in the particular case where $(a, b) = (4, -1)$.

2.2. Large deviation results for the MLE. The following theorem gives a large deviation principle for the MLE $(\hat{a}_T, \hat{b}_T)$ of the couple (a, b) . In contrast with the previous literature, we consider both parameters unknown and estimate them simultaneously. We also simplified the approach of the previous literature as our proofs only rely on the Gärtner–Ellis theorem and do not need, for example, accurate time-depending changes of probability.

Theorem 2.3. *The couple $(\hat{a}_T, \hat{b}_T)$ satisfies an LDP with good rate function $I_{a,b}$ given over $\mathbf{R}^2$ by*

$$I_{a,b}(\alpha, \beta) = \min\{J_{a,b}(\alpha, \beta); K_{a,b}(\alpha, \beta)\}.$$

Section 6 is devoted to the proof of this result.

Making use of the contraction principle (see Lemma 2.1), once again, we obtain straightforwardly the two following corollaries.

Corollary 2.5. *The sequence $(\hat{a}_T)$ satisfies an LDP with good rate function*

$$I_a(\alpha) = \min\{J_a(\alpha); K_a(\alpha)\}.$$

Corollary 2.6. *The sequence $(\hat{b}_T)$ satisfies an LDP with good rate function*

$$I_b(\beta) = \min\{J_b(\beta); K_b(\beta)\}.$$

Figure 3 displays the rate functions I_a and I_b in the particular case where $(a, b) = (4, -1)$.

Fig. 3. Rate functions for dimensional and drift parameters a and b in the case $(a, b) = (4, -1)$; J_a, J_b ($\cdots$), K_a, K_b ($-\cdot-\cdot-$), I_a, I_b ($-$).

R e m a r k 2.2. Both couples of simplified estimators perform better than the MLE in terms of large deviations, as their rate functions are always greater.

3. Some results about the process. The aim of this section is to establish LDPs with speed T for S_T , Σ_T , and V_T , which will be involved in the proof of the main theorem.

Lemma 3.1. *The couple (S_T, Σ_T) satisfies an LDP with good rate function*

$$I(x, y) = \begin{cases} \frac{y}{2(xy-1)} + \frac{b^2}{8}x + \frac{(a-2)^2}{8}y + \frac{ab}{4} & \text{if } x > 0, y > 0, \text{ and } xy - 1 > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

For the proof see appendix B (section 8).

The following result can be proved either directly with the same method or using the previous lemma together with the contraction principle recalled in Lemma 2.1.

Theorem 3.1. *The sequence (S_T) satisfies an LDP with good rate function*

$$I(x) = \begin{cases} \frac{(a+bx)^2}{8x} & \text{if } x > 0, \\ +\infty & \text{if } x \leq 0. \end{cases}$$

In addition, the sequence (Σ_T) satisfies an LDP with good rate function

$$J(x) = \begin{cases} \frac{((a-2)x+b)^2}{8x} & \text{if } x > 0, \\ +\infty & \text{if } x \leq 0. \end{cases}$$

It is now easy to establish an LDP for V_T . We recall that $V_T = h(S_T, \Sigma_T)$, where h is the function defined on $\mathbf{R}^2$ by $h(x, y) = xy - 1$.

Theorem 3.2. *The sequence (V_T) satisfies an LDP with good rate function*

$$K(x) = \begin{cases} -\frac{b}{4} \sqrt{(x+1) \left((a-2)^2 + \frac{4}{x} \right)} + \frac{ab}{4} & \text{if } x > 0, \\ +\infty & \text{if } x \leq 0. \end{cases}$$

P r o o f. It follows immediately from Lemma 3.1 together with the contraction principle (see Lemma 2.1). It only remains to explicitly evaluate the rate function K given, for all real z , by

$$K(z) = \inf_{\{(x,y) \mid z=xy-1\}} I(x, y),$$

where I is defined in Lemma 3.1.

4. Cumulant generating function for the quadruplet. To establish LDPs for the estimators $(\hat{a}_T, \hat{b}_T)$, we need to compute the normalized cumulant generating function of the quadruplet $(X_T/T, S_T, \Sigma_T, L_T)$. However, this does not lead to a steep function (see [5] for the definition), which is a necessary condition to apply the Gärtner–Ellis theorem. In contrast with the previous literature, we will not search another method to obtain large deviation results. Following the strategy of [3], the idea to overcome this difficulty is to consider instead the quadruplet $\mathcal{Q}_T = (\sqrt{X_T/T}, S_T, \Sigma_T, \mathcal{L}_T)$, where

$$\mathcal{L}_T = -\sqrt{\frac{-\ln X_T}{T}} \mathbf{1}_{X_T < 1} + \frac{\ln X_T}{T} \mathbf{1}_{X_T \geq 1}. \quad (4.1)$$

Proposition 4.1. Let $\Lambda_T(\lambda, \mu, \nu, \gamma)$ be the normalized cumulant generating function of the quadruplet $\mathcal{Q}_T$ given over $\mathbf{R}^4$ by

$$\Lambda_T(\lambda, \mu, \nu, \gamma) = \frac{1}{T} \ln \mathbf{E} \left[\exp \left(\lambda \sqrt{T} \sqrt{X_T} + \gamma T \mathcal{L}_T + \mu \int_0^T X_t dt + \nu \int_0^T \frac{1}{X_t} dt \right) \right].$$

Denote by Λ its pointwise limit as T tends to $+\infty$. For all $\lambda, \gamma \in \mathbf{R}$, $\mu < b^2/8$, and $\nu < (a-2)^2/8$,

$$\Lambda(\lambda, \mu, \nu, \gamma) = \begin{cases} -\frac{d}{2}(1+f) - \frac{ab}{4} + \frac{\lambda^2}{d-b} & \text{if } \lambda > 0 \text{ and } \gamma \geq 0 \text{ or if } \gamma < 0, \lambda > 0, \\ & \text{and } \frac{\gamma^2}{\lambda^2} < \frac{2f+a+2}{d-b}, \\ -\frac{d}{2}(1+f) - \frac{ab}{4} + \frac{\gamma^2}{2f+a+2} & \text{if } \lambda \leq 0 \text{ and } \gamma < 0 \text{ or if } \gamma < 0, \lambda > 0, \\ & \text{and } \frac{\gamma^2}{\lambda^2} \geq \frac{2f+a+2}{d-b}, \\ -\frac{d}{2}(1+f) - \frac{ab}{4} & \text{if } \lambda \leq 0 \text{ and } \gamma \geq 0, \end{cases}$$

where $d = \sqrt{b^2 - 8\mu}$ and $f = (\sqrt{(a-2)^2 - 8\nu})/2$.

Lemma 4.1. The function Λ is steep.

P r o o f. The function Λ is differentiable over its domain $\mathcal{D}_\Lambda = \mathbf{R} \times]-\infty, b^2/8[\times]-\infty, (a-2)^2/8[\times \mathbf{R}$ and its gradient is given by

$$\nabla \Lambda = \left(\frac{2\lambda}{d-b} \mathbf{1}_{\Delta_1}, \frac{2(1+f)}{d} + \frac{4\lambda^2}{d(d-b)^2} \mathbf{1}_{\Delta_1}, \frac{d}{2f} + \frac{2\gamma^2}{f(2f+a+2)^2} \mathbf{1}_{\Delta_2}, \frac{2\gamma}{2f+a+2} \mathbf{1}_{\Delta_2} \right)^T, \quad (4.2)$$

where

$$\begin{aligned} \Delta_1 &= \left\{ (\lambda, \mu, \nu, \gamma) \in \mathcal{D}_\Lambda \mid \lambda > 0 \text{ and } \gamma \geq 0 \text{ or } \gamma < 0, \lambda > 0 \right. \\ &\quad \left. \text{and } \frac{\gamma^2}{\lambda^2} < \frac{2f+a+2}{d-b} \right\}, \\ \Delta_2 &= \left\{ (\lambda, \mu, \nu, \gamma) \in \mathcal{D}_\Lambda \mid \lambda \leq 0 \text{ and } \gamma < 0 \text{ or } \gamma < 0, \lambda > 0 \right. \\ &\quad \left. \text{and } \frac{\gamma^2}{\lambda^2} \geq \frac{2f+a+2}{d-b} \right\}. \end{aligned}$$

We easily obtain that the norm of (4.2) goes to infinity for any sequence in the interior of $\mathcal{D}_\Lambda$ converging to a boundary point. Lemma 4.1 is proved.

P r o o f o f P r o p o s i t i o n 4.1. We want to find the limit of $\Lambda_T(\lambda, \mu, \nu, \gamma)$ as $T \rightarrow +\infty$. It follows from Girsanov's theorem (see [10]) that for all $\lambda, \gamma \in \mathbf{R}$, $\mu < b^2/8$ and $\nu < (a-2)^2/8$,

$$\Lambda_T(\lambda, \mu, \nu, \gamma) = \frac{c-a}{4T} \ln x + \frac{d-b}{4T} x + \frac{dc-ab}{4} + \frac{1}{T} \ln \mathbb{E}_{c,d} \left[X_T^{(a-c)/4} \exp \left(\frac{b-d}{4} X_T + \lambda \sqrt{T} \sqrt{X_T} + \gamma T \mathcal{L}_T \right) \right], \quad (4.3)$$

where we changed parameters a and b to new parameters c and d , respectively, given by $c = 2 + \sqrt{(a-2)^2 - 8\nu}$ and $d = \sqrt{b^2 - 8\mu}$. Besides, we know that X_T follows a noncentral chi-squared distribution and its transition density $p_{c,d}$ is given for any $T \in \mathbf{R}_*^+$, $x > 0$, and $y \in \mathbf{R}_*^+$ by

$$p_{c,d}(T, x, y) = \frac{d}{4 \sinh(dT/2)} \left(\frac{y}{x} \right)^{f/2} I_f \left(\frac{d\sqrt{xy}}{2 \sinh(dT/2)} \right) \times \exp \left(-\frac{d}{4} \left(cT + (x+y) \coth \left(\frac{dT}{2} \right) + x - y \right) \right)$$

with $f = (c-2)/2$ and I_f being the modified Bessel function of the first kind. Thus, equation (4.3) becomes

$$\Lambda_T(\lambda, \mu, \nu, \gamma) = \frac{c-a}{4T} \ln x + \frac{d-b}{4T} x + \frac{dc-ab}{4} + \frac{1}{T} \ln \int_0^\infty y^{(a-c)/4} e^{((b-d)/4)y + \lambda \sqrt{T}y + \gamma T l(T,y)} p_{c,d}(T, x, y) dy, \quad (4.4)$$

where $l(T, y) = -\sqrt{-T^{-1} \log y} \mathbf{1}_{y < 1} + T^{-1} \log y \mathbf{1}_{y \geq 1}$. We take out of the integral all the terms that do not depend on y . This leads to

$$\Lambda_T(\lambda, \mu, \nu, \gamma) = \frac{1}{T} \left(\ln \mathcal{J}_T + \ln \frac{d\sqrt{x}}{4 \sinh(dT/2)} - \frac{1}{4} \left(abT + dx \coth \frac{dT}{2} + a \ln x + bx \right) \right),$$

where

$$\mathcal{J}_T = \int_0^\infty \exp \left(\lambda \sqrt{T}y + \gamma T l(T, y) - \frac{y}{4} \left(d \coth \frac{dT}{2} - b \right) \right) y^{(a-2)/4} \times I_f \left(\frac{d\sqrt{xy}}{2 \sinh(dT/2)} \right) dy. \quad (4.5)$$

However, as soon as T tends to infinity, $\coth(dT/2)$ goes to 1, which implies that

$$\lim_{T \rightarrow +\infty} -\frac{1}{4T} \left(abT + dx \coth \frac{dT}{2} + a \ln x + bx \right) = -\frac{ab}{4}. \quad (4.6)$$

On the other hand,

$$\frac{1}{T} \ln \sinh \frac{dT}{2} = \frac{1}{T} \frac{dT}{2} + \frac{1}{T} \ln \frac{1 - e^{-dT}}{2},$$

which clearly leads to

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \ln \frac{d\sqrt{x}}{4 \sinh(dT/2)} = -\frac{d}{2}. \quad (4.7)$$

We have to establish the asymptotic behavior of $(1/T) \ln \mathcal{J}_T$. We split $\mathcal{J}_T$ into two terms: $\mathcal{J}_T = H_T + K_T$, where

$$H_T = \int_0^1 e^{\lambda \sqrt{Ty} - \gamma \sqrt{-T \ln y} - \alpha_T y} y^{(a-2)/4} I_f(\beta_T \sqrt{y}) dy, \quad (4.8)$$

$$K_T = \int_1^\infty e^{\lambda \sqrt{Ty} - \alpha_T y} y^{\gamma + (a-2)/4} I_f(\beta_T \sqrt{y}) dy \quad (4.9)$$

with

$$\alpha_T = \frac{d \coth(dT/2) - b}{4} \quad \text{and} \quad \beta_T = \frac{d\sqrt{x}}{2 \sinh(dT/2)}. \quad (4.10)$$

We need the four following lemmas, whose proofs are postponed to appendix C (section 9).

Lemma 4.2. *For all $\gamma < 0$ and $\lambda \in \mathbf{R}$, one can find the following bounds for H_T as T goes to infinity:*

$$H_T \leq \frac{2^{2-f}}{\Gamma(f+1)} \sqrt{\pi} g^{-3/2} |\gamma| \sqrt{T} e^{|\lambda| + \alpha_T/T + \beta_T/\sqrt{T}} \beta_T^f \exp\left(\frac{\gamma^2 T}{4g}\right),$$

$$H_T \geq \frac{2^{-1-f}}{\Gamma(f+1)} \sqrt{\pi} g^{-3/2} |\gamma| \sqrt{T} e^{-|\lambda| - \alpha_T/T} \beta_T^f \left(\frac{\gamma^2 T}{4g}\right),$$

where $g = (2f + a + 2)/4$.

Lemma 4.3. *For all $\gamma \geq 0$ and $\lambda \in \mathbf{R}$, bounds for H_T are given by*

$$H_T \leq \frac{(\beta_T)^f}{\Gamma(f+1)2^f} \exp\left(|\lambda| \sqrt{T} + \beta_T\right),$$

$$H_T \geq \frac{(\beta_T)^f \varepsilon_T}{2^f \Gamma(1+f)} \exp\left(-\gamma \sqrt{T} \sqrt{\ln T} - g \ln T\right),$$

where

$$\varepsilon_T = e^{-\alpha_T} \left(\varepsilon + \frac{e^{-|\lambda|}}{g} \left(1 - \frac{\gamma \sqrt{T}}{2g \sqrt{\ln T} + \gamma \sqrt{T}} \right) \right),$$

$$\varepsilon = \exp\left(\lambda \mathbf{1}_{\lambda \geq 0} + \lambda \sqrt{T} \mathbf{1}_{\lambda < 0}\right).$$

Lemma 4.4. For all $\lambda \leq 0$ and $\gamma \in \mathbf{R}$, T tending to infinity, $K_T = O((\beta_T)^f)$. Moreover if $\gamma \geq 0$, then we have the following lower bound:

$$K_T \geq \frac{2^{1-f}(\beta_T)^f}{\Gamma(1+f)} \frac{e^{-\alpha_T}}{2\alpha_T} \frac{1}{1 - \lambda\sqrt{T}/(2\alpha_T)} \exp(\lambda\sqrt{T}).$$

Lemma 4.5. For all $\lambda > 0$, $\gamma \in \mathbf{R}$, K_T is bounded as follows for T going to infinity:

$$K_T \leq \frac{2^{2-f}\sqrt{2\pi}(\beta_T)^f}{\Gamma(1+f)} (\beta_T + \lambda\sqrt{T})^{2\gamma+2g-1} \exp\left(\frac{(\lambda\sqrt{T} + \beta_T)^2}{4\alpha_T}\right),$$

$$K_T \geq 2^{1-f} \sqrt{\frac{\pi}{d-b}} \frac{(\beta_T)^f}{\Gamma(1+f)} m_{\gamma,\lambda,T} \exp\left(\frac{\lambda^2 T}{4\alpha_T}\right),$$

where $m_{\gamma,\lambda,T} = \min\{(\lambda\sqrt{T}/\alpha_T)^{2(\gamma+g-1)}; \lambda\sqrt{T}/\alpha_T\}$.

It is clear with those lemmas that the asymptotic behavior of $\mathcal{J}_T$ depends on the sign of λ and γ .

Case 1: $\lambda > 0$ and $\gamma \geq 0$. We directly deduce from Lemmas 4.5 and 4.3 that, for T large enough,

$$H_T + K_T \leq \frac{2^{2-f}\sqrt{2\pi}(\beta_T)^f}{\Gamma(1+f)} (\beta_T + \lambda\sqrt{T})^{2\gamma+2g-1} \times \left(\exp\left(\frac{(\lambda\sqrt{T} + \beta_T)^2}{4\alpha_T}\right) + \exp(|\lambda|\sqrt{T} + \beta_T) \right)$$

and thus

$$\begin{aligned} \overline{\lim}_{T \rightarrow \infty} \frac{1}{T} \ln \mathcal{J}_T &\leq f \lim_{T \rightarrow \infty} \frac{1}{T} \ln \beta_T + \lim_{T \rightarrow \infty} \frac{1}{T} \frac{(\lambda\sqrt{T} + \beta_T)^2}{4\alpha_T} \\ &= -f \frac{d}{2} + \frac{\lambda^2}{d-b}. \end{aligned} \quad (4.11)$$

We show alike by using the lower bounds of Lemmas 4.5 and 4.3 that

$$\underline{\lim}_{T \rightarrow \infty} \frac{1}{T} \ln \mathcal{J}_T \geq -f \frac{d}{2} + \frac{\lambda^2}{d-b}, \quad (4.12)$$

and we finally obtain

$$\lim_{T \rightarrow \infty} \frac{1}{T} \ln \mathcal{J}_T = -f \frac{d}{2} + \frac{\lambda^2}{d-b}. \quad (4.13)$$

Case 2: $\lambda \leq 0$ and $\gamma < 0$. With Lemma 4.4, we know that $K_T = O((\beta_T)^f)$. Thus, $H_T + K_T = H_T + O((\beta_T)^f)$. Lemma 4.2 gives bounds for H_T , which lead to

$$\begin{aligned} \overline{\lim}_{T \rightarrow \infty} \frac{1}{T} \ln \mathcal{J}_T &\leq \lim_{T \rightarrow \infty} \frac{1}{T} \ln \left((\beta_T)^f h_{T,\lambda,\gamma} \exp\left(\frac{\gamma^2 T}{4g}\right) \right) \\ &\quad + \lim_{T \rightarrow \infty} \frac{1}{T} \ln \left(1 + Ch_{T,\lambda,\gamma}^{-1} e^{-\gamma^2 T/(4g)} \right), \end{aligned}$$

where C is some positive constant and

$$h_{T,\lambda,\gamma} = \frac{2^{2-f}}{\Gamma(f+1)} \sqrt{\pi} g^{-3/2} |\gamma| \sqrt{T} e^{|\lambda|+\alpha_T/T+\beta_T/\sqrt{T}}.$$

Using the fact that $h_{T,\lambda,\gamma}^{-1} e^{-\gamma^2 T/(4g)}$ tends to zero as T goes to infinity, we obtain

$$\overline{\lim}_{T \rightarrow \infty} \frac{1}{T} \ln \mathcal{J}_T \leq -f \frac{d}{2} + \frac{\gamma^2}{4g}. \quad (4.14)$$

We obtain the same lower bound by using the lower bound in Lemma 4.2.

Case 3: $\lambda \leq 0$ and $\gamma \geq 0$. Lemma 4.3 gives $H_T = O((\beta_T)^f \exp(|\lambda|\sqrt{T}))$ and by Lemma 4.4, we know that $K_T = O((\beta_T)^f)$. Consequently,

$$\overline{\lim}_{T \rightarrow \infty} \frac{1}{T} \ln \mathcal{J}_T \leq f \lim_{T \rightarrow \infty} \frac{1}{T} \ln \beta_T = -f \frac{d}{2}. \quad (4.15)$$

And the lower bounds given in Lemmas 4.3 and 4.4 lead to

$$\underline{\lim}_{T \rightarrow \infty} \frac{1}{T} \ln \mathcal{J}_T \geq f \lim_{T \rightarrow \infty} \frac{1}{T} \ln \beta_T = -f \frac{d}{2}. \quad (4.16)$$

Case 4: $\lambda > 0$ and $\gamma < 0$. Using Lemmas 4.5 and 4.2, we show that

$$\begin{aligned} H_T + K_T &\leq C(\beta_T)^f \left(\bar{h}_{T,\lambda} \exp\left(\frac{\gamma^2 T}{4g}\right) \right. \\ &\quad \left. + \left(\beta_T + \lambda\sqrt{T}\right)^{2\gamma+2g-1} \exp\left(\frac{(\lambda\sqrt{T} + \beta_T)^2}{4\alpha_T}\right) \right), \end{aligned}$$

where $\bar{h}_{T,\lambda} = \sqrt{T} e^{|\lambda|+\alpha_T/T+\beta_T/\sqrt{T}}$ and C is some positive constant. Thus

$$\begin{aligned} \overline{\lim}_{T \rightarrow \infty} \frac{1}{T} \ln \mathcal{J}_T &\leq \lim_{T \rightarrow \infty} \frac{1}{T} \ln \left((\beta_T)^f \exp\left(T \times \max\left(\frac{\gamma^2}{4g}; \frac{\lambda^2}{d-b}\right)\right) \right) \\ &= -f \frac{d}{2} + \max\left\{\frac{\gamma^2}{4g}; \frac{\lambda^2}{d-b}\right\}. \end{aligned}$$

We also show that

$$H_T + K_T \geq C(\beta_T)^f \left(\underline{h}_{T,\lambda} \exp\left(\frac{\gamma^2 T}{4g}\right) + m_{\gamma,\lambda,T} \exp\left(\frac{\lambda^2 T}{4\alpha_T}\right) \right),$$

where C is still some positive constant and $\underline{h}_{T,\lambda} = \sqrt{T} e^{-|\lambda|-\alpha_T/T}$. It leads to

$$\underline{\lim}_{T \rightarrow \infty} \frac{1}{T} \ln \mathcal{J}_T \geq -f \frac{d}{2} + \max\left\{\frac{\gamma^2}{4g}; \frac{\lambda^2}{d-b}\right\}. \quad (4.17)$$

Proposition 4.1 is proved.

5. Proofs of the LDPs for the couples of simplified estimators.

5.1. Proof of Theorem 2.1. We will now establish an LDP for the first couple of simplified estimators. We notice that $(\tilde{a}_T, \tilde{b}_T) = f(\sqrt{X_T/T}, S_T, \Sigma_T)$, where f is the function defined on $\{(x, y, z) \in \mathbf{R}^3 \mid yz \neq 1\}$ by

$$f(x, y, z) = \left(\frac{2zy - x^2}{yz - 1}, \frac{(x^2 - 2)z}{yz - 1} \right).$$

Thus, we first compute an LDP for the triplet $(\sqrt{X_T/T}, S_T, \Sigma_T)$ and then apply the contraction principle to the obtained rate function.

Lemma 5.1. *The sequence $\{(\sqrt{X_T/T}, S_T, \Sigma_T)\}$ satisfies an LDP with good rate function*

$$I(x, y, z) = \begin{cases} \frac{ab}{4} + \frac{b^2}{8}y + \frac{(a-2)^2}{8}z - \frac{b}{4}x^2 + \frac{(x^2+2)^2z}{8(yz-1)} \\ \quad \text{if } x \geq 0, y, z, yz-1 > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

The proof is postponed to appendix D (section 10).

Since f is continuous over $\mathcal{D}_I = \{(x, y, z) \in \mathbf{R}^3 \mid I(x, y, z) < +\infty\}$, we deduce from Lemma 5.1 together with the contraction principle that $(\tilde{a}_T, \tilde{b}_T)$ satisfies an LDP with good rate function $J_{a,b}$ given by

$$J_{a,b}(\alpha, \beta) = \inf_{\mathcal{D}_I} \{I(x, y, z) \mid f(x, y, z) = (\alpha, \beta)\} \quad (5.1)$$

which reduces to

$$J_{a,b}(\alpha, \beta) = \inf_{\mathcal{D}_I} \left\{ \frac{ab}{4} + \frac{b^2}{8}y + \frac{(a-2)^2}{8}z - \frac{b}{4}x^2 + \frac{(x^2+2)^2z}{8(yz-1)} \mid f(x, y, z) = (\alpha, \beta) \right\},$$

where the infimum over the empty set is equal to the infinity. One easily see that $J_{a,b}(\alpha, \beta) = +\infty$ as soon as $\alpha = 2$ and $\beta \neq 0$ or $\beta = 0$ and $\alpha \neq 2$, since we take the infimum over the empty set. For the remaining particular case $(\alpha, \beta) = (2, 0)$, as $z > 0$ on $\mathcal{D}_I$, the only way of satisfying $f(x, y, z) = (2, 0)$ is to take $x^2 = 2$. Therefore,

$$J_{a,b}(2, 0) = \inf_{y>0, z>0, yz-1>0} \left\{ \frac{ab}{4} + \frac{b^2}{8}y + \frac{(a-2)^2}{8}z - \frac{b}{2} + \frac{2z}{yz-1} \right\} = -b. \quad (5.2)$$

Otherwise, the condition $f(x, y, z) = (\alpha, \beta)$ implies that

$$z = \frac{\beta}{2-\alpha}, \quad yz-1 = \frac{\beta y + \alpha - 2}{2-\alpha}, \quad x^2 = \beta y + \alpha.$$

Thus, if $\beta/(2 - \alpha) < 0$, then $J_{a,b}(\alpha, \beta) = +\infty$, otherwise

$$J_{a,b}(\alpha, \beta) = \inf_{\mathcal{D}_y} \left\{ \frac{ab}{4} + \frac{(a-2)^2}{8} \frac{\beta}{2-\alpha} - \frac{b}{4} \alpha + (b^2 - 2b\beta) \frac{y}{8} + \frac{\beta}{8} \frac{(\beta y + \alpha + 2)^2}{\beta y + \alpha - 2} \right\},$$

where $\mathcal{D}_y = \{y > (2 - \alpha)/\beta \mid \beta y + \alpha \geq 0\}$. Depending on the values of β and α , the infimum will be reached either at the critical point $y = (2 - \alpha)/\beta - 4/(b - \beta)$ or at the boundary $y = -\alpha/\beta$ of the domain $\mathcal{D}_y$. For detailed computations, we refer to the arXiv version of this paper (arXiv:1407.4949).

5.2. Proofs of Corollaries 2.1 and 2.2. Using the contraction principle again, we deduce from Theorem 2.1 LDPs for both estimators. We begin with $\tilde{b}_T$ because the calculations are really straightforward.

Proof of Corollary 2.2. From the contraction principle, we know that for any $\beta \in \mathbf{R}$,

$$J_b(\beta) = \inf_{\alpha \in \mathbf{R}} I_{a,b}(\alpha, \beta).$$

We have directly that $J_b(0) = J_{a,b}(2, 0) = -b$ and that, for $\beta \neq 0$,

$$J_b(\beta) = J_{a,b}\left(2 + \beta \frac{a-2}{b}, \beta\right).$$

This leads to the result noticing that it is continuous at point zero. Corollary 2.2 is proved.

Proof of Corollary 2.1. With the contraction principle again, we have, for all $\alpha \in \mathbf{R}$,

$$J_a(\alpha) = \inf_{\beta \in \mathbf{R}} J_{a,b}(\alpha, \beta).$$

We easily show that $J_a(2) = J_{a,b}(2, 0) = -b$. For any $\alpha < 2$, we easily obtain that

$$J_a(\alpha) = \inf_{\beta > 0} \left\{ \frac{(a-2)^2 \beta}{8(2-\alpha)} \left(1 + \frac{(2-\alpha)b}{\beta(a-2)}\right)^2 + 2\beta - b \right\} = J_{a,b}(\alpha, \beta_0),$$

where β_0 is the critical point given by

$$\beta_0 = -(2 - \alpha) b ((a - 2)^2 + 16(2 - \alpha))^{-1/2}.$$

The case $\alpha > 2$ is somewhat longer to handle. Indeed, $J_{a,b}(\alpha, \beta) = \min\{I_1; I_2\}$, where

$$I_1 = \inf_{\beta \leq b/3} \left\{ \frac{(a-2)^2 \beta}{8(2-\alpha)} \left(1 + \frac{(2-\alpha)b}{\beta(a-2)}\right)^2 - \frac{\beta}{4} \left(1 - \frac{b}{\beta}\right)^2 \right\}, \quad (5.3)$$

$$I_2 = \inf_{b/3 \leq \beta < 0} \left\{ \frac{(a-2)^2 \beta}{8(2-\alpha)} \left(1 + \frac{(2-\alpha)b}{\beta(a-2)}\right)^2 + 2\beta - b \right\}. \quad (5.4)$$

For I_2 we have already computed the critical point β_0 , while investigating the case $a < 2$. We similarly obtain the critical point β_1 for I_2 given by

$$\beta_1 = b \sqrt{\alpha(\alpha - 2) ((a - 2)^2 + 2(\alpha - 2))^{-1/2}}.$$

After carefully investigating the conditions on α and β under which the critical points are inside the respective domains, we obtain that

$$J_{a,b}(\alpha, \beta) = \begin{cases} J_{a,b}(\alpha, \beta_0) & \text{if } 2 < \alpha \leq \ell_a, \\ J_{a,b}(\alpha, \beta_1) & \text{if } \alpha > \ell_a, \end{cases} \quad (5.5)$$

where $\ell_a := 10/9 + (1/9)\sqrt{64 + 9(a - 2)^2}$. This leads to the conclusion, noticing that it is continuous at the point $\alpha = 2$. Corollary 2.1 is proved.

5.3. Proof of Theorem 2.2. We consider the second couple of simplified estimators defined by

$$\check{a}_T = \frac{S_T(2\Sigma_T + L_T)}{V_T} \quad \text{and} \quad \check{b}_T = \frac{-2\Sigma_T - L_T}{V_T}.$$

We notice that $(\check{a}_T, \check{b}_T) = h(S_T, \Sigma_T, \mathcal{L}_T)$, where h is the function defined on $\{(y, z, t) \in \mathbf{R}^3 \mid yz - 1 \neq 0\}$ by

$$h(y, z, t) = \left(\frac{y(2z - t^2 \mathbf{1}_{t \leq 0} + t \mathbf{1}_{t > 0})}{yz - 1}, \frac{t^2 \mathbf{1}_{t \leq 0} - t \mathbf{1}_{t > 0} - 2z}{yz - 1} \right). \quad (5.6)$$

Once again, we start by computing an LDP for the triplet $(S_T, \Sigma_T, \mathcal{L}_T)$ and then we deduce an LDP for the couple of estimators applying the contraction principle to the obtained rate function.

Lemma 5.2. *The sequence $\{(S_T, \Sigma_T, \mathcal{L}_T)\}$ satisfies an LDP with good rate function*

$$\tilde{I}(y, z, t) = \begin{cases} \frac{ab}{4} + \frac{b^2}{8}y + \frac{(a-2)^2}{8}z + \frac{a}{4}t^2 + \frac{4z(yt^2 + 1) + t^4y}{8(yz - 1)} & \text{if } t \leq 0, y > 0, z > 0, \text{ and } yz - 1 > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

The proof is postponed to Appendix E (section 11).

Since the function h given by (5.6) is continuous over the domain, where the rate function $\tilde{I}$ of Lemma 5.2 is finite, the contraction principle applied to $\tilde{I}$ shows that $(\check{a}_T, \check{b}_T)$ satisfies an LDP with good rate function $K_{a,b}$ given by

$$K_{a,b}(\alpha, \beta) = \inf_{\mathcal{D}_{\alpha,\beta}} \left\{ \frac{ab}{4} + \frac{b^2}{8}y + \frac{(a-2)^2}{8}z + \frac{a}{4}t^2 + \frac{4z(yt^2 + 1) + t^4y}{8(yz - 1)} \right\},$$

where

$$\mathcal{D}_{\alpha,\beta} = \{(y, z, t) \in \mathbf{R}^3 \mid y > 0, z > 0, yz - 1 > 0, t \leq 0, h(y, z, t) = (\alpha, \beta)\}$$

and the infimum over the empty set is equal to infinity. The condition $h(y, z, t) = (\alpha, \beta)$ implies that

$$\beta y = -\alpha \quad \text{and} \quad t^2 = (2 - \alpha)z - \beta.$$

It gives us some additional conditions on the parameters. First of all, we notice that for α or β equal to zero, $\mathcal{D}_{\alpha,\beta}$ is not empty if and only if the other one is also zero. Thus, $K_{a,b}(\alpha, \beta) = +\infty$ over $\{0\} \times \mathbf{R}_*^+$ and $\mathbf{R}_*^+ \times \{0\}$. If $\alpha = \beta = 0$, then necessarily $t^2 = 2z$ and we easily obtain the critical points leading to

$$K_{a,b}(0, 0) = -\frac{b}{4} \left(4 - a + \sqrt{a^2 + 16} \right).$$

Moreover, for $\beta \neq 0$, as $y = -\alpha/\beta$, $\mathcal{D}_{\alpha,\beta}$ is empty as soon as α and β have the same sign. So $K_{a,b}(\alpha, \beta) = +\infty$ over $\mathbf{R}_*^+ \times \mathbf{R}_*^+$ and $\mathbf{R}_*^- \times \mathbf{R}_*^-$. Besides, both expressions will give us boundaries for z depending on the sign of α , β , and $2 - \alpha$, because t^2 must be positive and z must be greater than $1/y$. Assuming that all conditions are fulfilled, we derive from Lemma 5.2 that

$$\tilde{I}\left(-\frac{\alpha}{\beta}, z, \sqrt{(2 - \alpha)z - \beta}\right) = A_{\alpha,\beta} + C_\alpha z - \frac{2\beta z}{\alpha z + \beta},$$

where C_α and $A_{\alpha,\beta}$ do not depend on z , and are defined by

$$C_\alpha = \frac{1}{8} (a - \alpha)^2 + 2 - \alpha \quad \text{and} \quad A_{\alpha,\beta} = -\frac{\alpha}{\beta} \frac{b^2}{8} + \frac{ab}{4} - \frac{a\beta}{4} + \frac{\alpha\beta}{8}. \quad (5.7)$$

Thus,

$$K_{a,b}(\alpha, \beta) = \inf_{\mathcal{D}_z} \left\{ A_{\alpha,\beta} + C_\alpha z - \frac{2\beta z}{\alpha z + \beta} \right\}, \quad (5.8)$$

where $\mathcal{D}_z = \{z > 0 \mid z > -\beta/\alpha \text{ and } (2 - \alpha)z - \beta \geq 0\}$. Depending on the values of α and β , the infimum will be reached either at the critical point $z_0 = -(\beta/\alpha)(1 + \sqrt{2/C_\alpha})$ or at the boundary $z = \beta/(2 - \alpha)$ of the domain $\mathcal{D}_z$. Theorem 2.2 is proved.

5.4. Proofs of Corollaries 2.3 and 2.4.

Proof of Corollary 2.4. The result is a direct application of the contraction principle to the rate function $K_{a,b}$ of Theorem 2.2. We did not obtain an explicit expression of the infimum.

Proof of Corollary 2.3. With the contraction principle again, we know that $(\check{a}_T)$ satisfies an LDP with good rate function

$$K_a(\alpha) = \inf_{\beta \in \mathbf{R}} K_{a,b}(\alpha, \beta).$$

First of all, we easily show that $K_a(0) = K_{a,b}(0, 0)$. Then, for any $\alpha < \alpha_a$, simple critical points computations show that

$$K_a(\alpha) = K_{a,b}(\alpha, \beta_b), \quad (5.9)$$

where $\beta_b = b\alpha(16\sqrt{2}\sqrt{C_\alpha} + a^2 - 8\alpha + 32)^{-1/2}$. Finally, in the case $\alpha \geq \alpha_a$,

$$K_a(\alpha) = \inf_{\beta < 0} \left\{ \frac{(a-2)^2\beta}{8(2-\alpha)} \left(1 + \frac{(2-\alpha)b}{\beta(a-2)} \right)^2 - \frac{\beta}{4} \left(1 - \frac{b}{\beta} \right)^2 \right\} = J_a(\alpha)$$

and we conclude using the proof of Corollary 2.1. Corollary 2.3 is proved.

6. Proof of Theorem 2.3. We now come back to the MLE $(\hat{a}_T, \hat{b}_T)$. As we did for the couples of simplified estimators, we first establish an LDP for the quadruplet $\mathcal{Q}_T$ and we deduce an LDP for the MLE via the contraction principle.

6.1. Existence of an LDP.

Lemma 6.1. *The quadruplet $\mathcal{Q}_T = (\sqrt{X_T/T}, S_T, \Sigma_T, \mathcal{L}_T)$ satisfies an LDP with good rate function Λ^* given by*

$$\Lambda^*(x, y, z, t) = +\infty \quad \text{for } x < 0, t > 0, y \leq 0, z \leq 0 \text{ or } yz - 1 \leq 0, \quad (6.1)$$

and otherwise

$$\Lambda^*(x, y, z, t) = \sup_{\mathcal{D}_{a,f}} h(d, f), \quad (6.2)$$

where $\mathcal{D}_{a,f} = \{d > 0, f > 0\}$ and, with $\varphi(f) = 2f + a + 2$,

$$\begin{aligned} h(d, f) = & \frac{1}{4} \left(t\sqrt{\varphi(f)} - x\sqrt{d-b} \right)^2 + y \frac{b^2 - d^2}{8} + \frac{(a-2)^2 - 4f^2}{8} z \\ & + \frac{d}{2} (1+f) + \frac{ab}{4}. \end{aligned}$$

P r o o f. Using the Gärtner–Ellis theorem, we have to compute the Fenchel–Legendre transform Λ^* of the cumulant generating function Λ defined in Proposition 4.1:

$$\Lambda^*(x, y, z, t) = \sup_{\mathcal{D}} \{x\lambda + y\mu + z\nu + t\gamma - \Lambda(x, y, z, t)\}, \quad (6.3)$$

where $\mathcal{D} = \{\lambda \in \mathbf{R}, \gamma \in \mathbf{R}, \mu < b^2/8, \nu < (a-2)^2/8\}$. We show with the same arguments as for the other LDP proofs that

$$\Lambda^*(x, y, z, t) = +\infty \quad \text{for } x < 0, t > 0, y \leq 0, z \leq 0 \text{ or } yz - 1 \leq 0. \quad (6.4)$$

Besides, for $x \geq 0$, the part involving λ in the function we want to optimize is always negative for $\lambda \leq 0$ and sometimes positive for $\lambda > 0$. Thus the supremum is necessarily reached for some $\lambda > 0$. With the same argument

for $t \leq 0$, we show that we only have to consider $\gamma < 0$. Replacing μ and ν by their expressions in d and f , the domain $\mathcal{D}$ over which we optimize reduces to $\mathcal{D} = \{\lambda > 0, \gamma < 0, d > 0, f > 0\}$. Replacing Λ by its value leads to

$$\Lambda^*(x, y, z, t) = \max\{S_1; S_2\},$$

where

$$\begin{aligned} S_1 &= \sup_{\mathcal{D} \cap \{\gamma^2/\lambda^2 \leq \varphi(f)/(d-b)\}} \left\{ x\lambda + y \frac{b^2 - d^2}{8} + z \frac{(a-2)^2 - 4f^2}{8} + t\gamma \right. \\ &\quad \left. + \frac{d}{2}(1+f) + \frac{ab}{4} - \frac{\lambda^2}{d-b} \right\}, \\ S_2 &= \sup_{\mathcal{D} \cap \{\gamma^2/\lambda^2 \geq \varphi(f)/(d-b)\}} \left\{ x\lambda + y \frac{b^2 - d^2}{8} + z \frac{(a-2)^2 - 4f^2}{8} + t\gamma \right. \\ &\quad \left. + \frac{d}{2}(1+f) + \frac{ab}{4} - \frac{\gamma^2}{\varphi(f)} \right\}. \end{aligned}$$

We first consider S_1 . The domain over which we take the supremum is given by

$$\mathcal{D} \cap \left\{ \frac{\gamma^2}{\lambda^2} \leq \frac{\varphi(f)}{d-b} \right\} = \left\{ \lambda < 0, d > 0, f > 0, 0 > \gamma \geq -\sqrt{\frac{\varphi(f)}{d-b}} \lambda \right\}.$$

Over this domain, since $t \leq 0$, we have $0 \leq t\gamma \leq -t\sqrt{\varphi(f)/(d-b)}\lambda$, so that the supremum of $t\gamma$ is equal to $-t\lambda\sqrt{\varphi(f)/(d-b)}$. Thus, if we set $\mathcal{D}_1 = \{\lambda < 0, d > 0, f > 0\}$, then

$$\begin{aligned} S_1 &= \sup_{\mathcal{D}_1} \left\{ x\lambda + y \frac{b^2 - d^2}{8} + z \frac{(a-2)^2 - 4f^2}{8} \right. \\ &\quad \left. - t\lambda \sqrt{\frac{\varphi(f)}{d-b}} + \frac{d}{2}(1+f) + \frac{ab}{4} - \frac{\lambda^2}{d-b} \right\}. \end{aligned}$$

The supremum over λ is easy to compute. Indeed, the function is concave on λ and the critical point is given by

$$\lambda = \frac{d-b}{2} \left(x - t \sqrt{\frac{\varphi(f)}{d-b}} \right).$$

Finally, with $\mathcal{D}_{d,f} = \{d > 0, f > 0\}$, we obtain

$$\begin{aligned} S_1 &= \sup_{\mathcal{D}_{d,f}} \left\{ \frac{1}{4} \left(t\sqrt{\varphi(f)} - x\sqrt{d-b} \right)^2 + y \frac{b^2 - d^2}{8} \right. \\ &\quad \left. + \frac{(a-2)^2 - 4f^2}{8} z + \frac{d}{2}(1+f) + \frac{ab}{4} \right\}. \end{aligned}$$

We do the same thing with S_2 , computing first the supremum over λ and then over γ . We obtain $S_1 = S_2$, so that

$$\Lambda^*(x, y, z, t) = S_1 = \sup_{\mathcal{D}_{d,f}} h(d, f), \quad (6.5)$$

where

$$h(d, f) = \frac{1}{4} \left(t\sqrt{\varphi(f)} - x\sqrt{d-b} \right)^2 + y \frac{b^2 - d^2}{8} + \frac{(a-2)^2 - 4f^2}{8} z + \frac{d}{2} (1+f) + \frac{ab}{4}.$$

Remark 6.1. The supremum in (6.2) is not explicitly computable but, since the function h is concave, it is reached for some (d^*, f^*) and this gives the rate function of the LDP satisfied by the quadruplet $\mathcal{Q}_T$.

Lemma 6.2. *The couple $(\hat{a}_T, \hat{b}_T)$ satisfies an LDP with good rate function $I_{a,b}$ given over $\mathbf{R}^2$ by*

$$I_{a,b}(\alpha, \beta) = \begin{cases} K_{a,b}(0, 0) & \text{if } (\alpha, \beta) = (0, 0), \\ J_{a,b}(2, 0) & \text{if } (\alpha, \beta) = (2, 0), \\ \inf_{\mathcal{D}_{x,t}} \sup_{\mathcal{D}_{d,f}} H(x, t, d, f) & \text{if } (\alpha, \beta) \in \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3, \\ +\infty & \text{otherwise,} \end{cases}$$

where $\mathcal{D}_1 = \mathbf{R}^- \times \mathbf{R}_*^+$, $\mathcal{D}_2 =]0, 2[\times \mathbf{R}$, $\mathcal{D}_3 = [2, +\infty[\times \mathbf{R}_*^-$, $\mathcal{D}_{d,f} = \{d > 0, f > 0\}$,

$$\mathcal{D}_{x,t} = \left\{ x \geq 0, t \leq 0 \mid \frac{x^2 - \alpha}{\beta} \frac{t^2 + \beta}{2 - \alpha} > 1 \right\},$$

and

$$H(x, t, d, f) = \frac{1}{4} \left(t\sqrt{2f + a + 2} - x\sqrt{d-b} \right)^2 + \frac{b^2 - d^2}{8} \frac{x^2 - \alpha}{\beta} + \frac{(a-2)^2 - 4f^2}{8} \frac{t^2 + \beta}{2 - \alpha} + \frac{d}{2} (1+f) + \frac{ab}{4}.$$

Proof. $(\hat{a}_T, \hat{b}_T) = g(\mathcal{Q}_T)$, where g is the function defined on $\{(x, y, z, t) \in \mathbf{R}^4 \mid yz - 1 \neq 0\}$ by

$$g(x, y, z, t) = \left(\frac{y(2z - t^2 \mathbf{1}_{t \leq 0} + t \mathbf{1}_{t > 0}) - x^2}{yz - 1}, \frac{t^2 \mathbf{1}_{t \leq 0} - t \mathbf{1}_{t > 0} + (x^2 - 2)z}{yz - 1} \right).$$

Since g is continuous over the domain $\mathcal{D}_{\Lambda^*}$, where the rate function Λ^* of Lemma 6.1 is finite, the contraction principle applies and give us that the couple $(\hat{a}_T, \hat{b}_T)$ satisfies an LDP with good rate function $I_{a,b}$ given by

$$I_{a,b}(\alpha, \beta) = \inf_{\mathcal{D}_{\Lambda^*}} \{ \Lambda^*(x, y, z, t) \mid g(x, y, z, t) = (\alpha, \beta) \}. \quad (6.6)$$

The condition $g(x, y, z, t) = (\alpha, \beta)$ gives us a link between x and y and one between t and z :

$$\beta y = x^2 - \alpha \quad \text{and} \quad (2 - \alpha)z = t^2 + \beta. \quad (6.7)$$

We first notice that if $\alpha \leq 0$ and $\beta < 0$, then y is negative and for all x, z, t , we have $\Lambda^*(x, y, z, t) = +\infty$, thus, $I_{a,b}(\alpha, \beta) = +\infty$. Similarly, if $\beta \geq 0$ and $\alpha > 2$, then z is negative and $\Lambda^*(x, y, z, t) = +\infty$ for all x, y, t , thus, $I_{a,b}(\alpha, \beta) = +\infty$. If $\beta = 0$ and $\alpha < 0$, the first condition in (6.7) leads to x^2 negative and if $\alpha = 2$ and $\beta > 0$, then the second condition gives t^2 negative. So, in both cases, we get $I_{a,b}(\alpha, \beta) = +\infty$. We now focus on the values of α and β for which $I_{a,b}$ is not clearly infinite. We first consider the two remaining limit cases: $(0, 0)$ and $(2, 0)$. If $\alpha = \beta = 0$, then the first condition of (6.7) gives $x^2 = \alpha = 0$ so that

$$I_{a,b}(0, 0) = K_{a,b}(0, 0). \quad (6.8)$$

Similarly, if $\alpha = 2$ and $\beta = 0$, then the second condition implies that $t^2 = (2 - \alpha)z = 0$ and consequently

$$I_{a,b}(2, 0) = J_{a,b}(2, 0). \quad (6.9)$$

For all remaining values of (α, β) , we define the function

$$\begin{aligned} H(x, t, d, f) = & \frac{1}{4} \left(t\sqrt{2f+a+2} - x\sqrt{d-b} \right)^2 + \frac{b^2 - d^2}{8} \frac{x^2 - \alpha}{\beta} \\ & + \frac{(a-2)^2 - 4f^2}{8} \frac{t^2 + \beta}{2 - \alpha} + \frac{d}{2}(1+f) + \frac{ab}{4} \end{aligned} \quad (6.10)$$

and obtain the announced result:

$$I_{a,b}(\alpha, \beta) = \inf_{\mathcal{D}_{x,t}} \sup_{\mathcal{D}_{d,f}} H(x, t, d, f), \quad (6.11)$$

where

$$\mathcal{D}_{x,t} = \left\{ x \geq 0, t \leq 0 \mid \frac{x^2 - \alpha}{\beta} \frac{t^2 + \beta}{2 - \alpha} > 1 \right\} \quad \text{and} \quad \mathcal{D}_{d,f} = \{d > 0, f > 0\}.$$

We were not able to compute $I_{a,b}$ explicitly at this stage. It is the aim of the next subsection. Lemma 6.2 is proved.

6.2. Evaluating the rate function $I_{a,b}$. Our goal is to show that

$$I_{a,b}(\alpha, \beta) = \min\{J_{a,b}(\alpha, \beta); K_{a,b}(\alpha, \beta)\}, \quad (6.12)$$

where $J_{a,b}$ and $K_{a,b}$ are the rate functions for the two couples of simplified estimators (see Theorems 2.1 and 2.2) and $I_{a,b}$ is given by Lemma 6.2. We notice that

$$K_{a,b}(\alpha, \beta) = \inf_{t \leq 0} \sup_{\mathcal{D}_{d,f}} H(0, t, d, f) \quad \text{and} \quad J_{a,b}(\alpha, \beta) = \inf_{x \geq 0} \sup_{\mathcal{D}_{d,f}} H(x, 0, d, f).$$

Thus it easily follows that

$$I_{a,b}(\alpha, \beta) \leq \min\{J_{a,b}(\alpha, \beta); K_{a,b}(\alpha, \beta)\}. \quad (6.13)$$

So, we just have to show the inequality in the other side.

We denote $\hat{\theta}_T = (\hat{a}_T, \hat{b}_T)$ and $\bar{\theta}_T = (\tilde{a}_T, \tilde{b}_T) \mathbf{1}_{X_T \geq 1} + (\check{a}_T, \check{b}_T) \mathbf{1}_{X_T < 1}$.

Lemma 6.3. *The estimators $\bar{\theta}_T$ and $\hat{\theta}_T$ are exponentially equivalent, which means that for all $\varepsilon > 0$,*

$$\limsup_{T \rightarrow +\infty} \frac{1}{T} \ln \mathbf{P} \left(\|\hat{\theta}_T - \bar{\theta}_T\| > \varepsilon \right) = -\infty.$$

In particular, since the sequence $(\hat{\theta}_T)$ satisfies an LDP with good rate function $I_{a,b}$, the same LDP holds true for $(\bar{\theta}_T)$.

P r o o f. From the definition of each estimator, we get that

$$\begin{aligned} \hat{a}_T - (\tilde{a}_T \mathbf{1}_{X_T \geq 1} + \check{a}_T \mathbf{1}_{X_T < 1}) &= \frac{S_T L_T \mathbf{1}_{X_T \geq 1} - (X_T/T) \mathbf{1}_{X_T < 1}}{V_T}, \\ \hat{b}_T - (\tilde{b}_T \mathbf{1}_{X_T \geq 1} + \check{b}_T \mathbf{1}_{X_T < 1}) &= \frac{(X_T/T) \Sigma_T \mathbf{1}_{X_T < 1} - L_T \mathbf{1}_{X_T \geq 1}}{V_T}. \end{aligned}$$

Thus, for all $\varepsilon > 0$,

$$\mathbf{P} \left(\|\hat{\theta}_T - \bar{\theta}_T\| > \varepsilon \right) \leq P_T^\varepsilon + Q_T^\varepsilon + p_T^\varepsilon + q_T^\varepsilon,$$

where

$$\begin{aligned} P_T^\varepsilon &= \mathbf{P} \left(\left| \frac{S_T L_T \mathbf{1}_{X_T \geq 1}}{V_T} \right| \geq \frac{\varepsilon}{2\sqrt{2}} \right), \\ Q_T^\varepsilon &= \mathbf{P} \left(\left| \frac{(X_T/T) \mathbf{1}_{X_T < 1}}{V_T} \right| \geq \frac{\varepsilon}{2\sqrt{2}} \right), \\ p_T^\varepsilon &= \mathbf{P} \left(\left| \frac{(X_T/T) \Sigma_T \mathbf{1}_{X_T < 1}}{V_T} \right| \geq \frac{\varepsilon}{2\sqrt{2}} \right), \\ q_T^\varepsilon &= \mathbf{P} \left(\left| \frac{L_T \mathbf{1}_{X_T \geq 1}}{V_T} \right| \geq \frac{\varepsilon}{2\sqrt{2}} \right). \end{aligned}$$

For all $\eta > 0$, we have the following upper bounds:

$$\begin{aligned} P_T^\varepsilon &\leq \mathbf{P} \left(|S_T| \geq \frac{\varepsilon}{2\eta\sqrt{2}} \right) + \mathbf{P} \left(\frac{|L_T \mathbf{1}_{X_T \geq 1}|}{V_T} \geq \eta \right) \\ &\leq \mathbf{P} \left(S_T \geq \frac{\varepsilon}{2\eta\sqrt{2}} \right) + \mathbf{P}(L_T \mathbf{1}_{X_T \geq 1} \geq \eta^2) + \mathbf{P}(V_T \leq \eta), \end{aligned} \quad (6.14)$$

$$p_T^\varepsilon \leq \mathbf{P} \left(\Sigma_T \geq \frac{\varepsilon}{2\eta\sqrt{2}} \right) + \mathbf{P} \left(\frac{X_T}{T} \mathbf{1}_{X_T < 1} \geq \eta^2 \right) + \mathbf{P}(V_T \leq \eta), \quad (6.15)$$

$$q_T^\varepsilon \leq \mathbf{P} \left(L_T \mathbf{1}_{X_T \geq 1} \geq \frac{\varepsilon\eta}{2\sqrt{2}} \right) + \mathbf{P}(V_T \leq \eta), \quad (6.16)$$

$$Q_T^\varepsilon \leq \mathbf{P} \left(\frac{X_T}{T} \mathbf{1}_{X_T < 1} \geq \frac{\varepsilon\eta}{2\sqrt{2}} \right) + \mathbf{P}(V_T \leq \eta). \quad (6.17)$$

First of all, using Theorem 3.1, we show that for all $c > -a/b$,

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \ln \mathbf{P}(S_T \geq c) = -I(c) \quad (6.18)$$

and for all $c > -b/(a-2)$,

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \ln \mathbf{P}(\Sigma_T \geq c) = -J(c), \quad (6.19)$$

where I and J are given in Theorem 3.1. Likewise, we deduce from Theorem 3.2 that for any $c > 0$ small enough

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \ln \mathbf{P}(V_T \leq c) = -K(c). \quad (6.20)$$

We now consider the parts involving L_T . For all $c > 0$ and $\lambda > 0$,

$$\mathbf{P}(L_T \mathbf{1}_{X_T \geq 1} \geq c) = \mathbf{P}(\ln X_T \mathbf{1}_{X_T \geq 1} \geq cT) \leq \mathbf{E} [e^{\lambda \ln X_T}] e^{-\lambda cT}.$$

Hence $(1/T) \ln \mathbf{P}(L_T \mathbf{1}_{X_T \geq 1} \geq c) \leq -\lambda c + (1/T) \ln \mathbf{E} X_T^\lambda$. Asymptotic properties of the moments of the process X_T as T tends to infinity can be found in [2, Proposition 3], and give that the second term tends to zero for T going to infinity. Thus, for any $\lambda > 0$ and $c > 0$, we have the following upper bound:

$$\limsup_{T \rightarrow +\infty} \frac{1}{T} \ln \mathbf{P}(L_T \mathbf{1}_{X_T \geq 1} \geq c) \leq -\lambda c.$$

Consequently, letting λ go to infinity, we obtain that for all $c > 0$,

$$\limsup_{T \rightarrow +\infty} \frac{1}{T} \ln \mathbf{P}(L_T \mathbf{1}_{X_T \geq 1} \geq c) = -\infty. \quad (6.21)$$

Finally, we consider the terms involving X_T/T . For all $c > 0$ and $\lambda > 0$,

$$\mathbf{P}\left(\frac{X_T}{T} \mathbf{1}_{X_T < 1} \geq c\right) \leq \mathbf{E} \left[e^{\lambda X_T \mathbf{1}_{X_T < 1}} \right] e^{-\lambda cT} \leq e^{\lambda - \lambda cT}.$$

Hence $(1/T) \ln \mathbf{P}((X_T/T) \mathbf{1}_{X_T < 1} \geq c) \leq -\lambda c + \lambda/T$. Thus, for any $\lambda > 0$ and $c > 0$, we have the following upper bound:

$$\limsup_{T \rightarrow +\infty} \frac{1}{T} \ln \mathbf{P}\left(\frac{X_T}{T} \mathbf{1}_{X_T < 1} \geq c\right) \leq -\lambda c.$$

Letting λ go to infinity, we obtain that for all $c > 0$,

$$\limsup_{T \rightarrow +\infty} \frac{1}{T} \ln \mathbf{P}\left(\frac{X_T}{T} \mathbf{1}_{X_T < 1} \geq c\right) = -\infty. \quad (6.22)$$

Consequently, combining the limits (6.18)–(6.22), we are able to compute the asymptotic behavior of the bounds (6.14)–(6.17) and we show that for all $\varepsilon > 0$ and all $\eta > 0$ small enough,

$$\limsup_{T \rightarrow +\infty} \frac{1}{T} \ln \mathbf{P} \left(\|\hat{\theta}_T - \bar{\theta}_T\| > \varepsilon \right) \leq -M_{\varepsilon, \eta},$$

where

$$M_{\varepsilon, \eta} = \min \left\{ I \left(\frac{\varepsilon}{2\eta\sqrt{2}} \right), J \left(\frac{\varepsilon}{2\eta\sqrt{2}} \right); K(\eta) \right\}.$$

Each term in this minimum tends to infinity as η goes to zero, so that $M_{\varepsilon, \eta}$ itself tends to infinity. This gives the announced result.

P r o o f o f T h e o r e m 2.3. We have already shown in Lemma 6.2 that $I_{a,b}(2, 0) = J_{a,b}(2, 0)$ and $I_{a,b}(0, 0) = K_{a,b}(0, 0)$ and that, except at this two points, $I_{a,b}$ is infinite over $\mathbf{R}^- \times \mathbf{R}^-$ and over $[2, +\infty[\times \mathbf{R}^+$. We also know by (6.13) that

$$I_{a,b}(\alpha, \beta) \leq \min\{J_{a,b}(\alpha, \beta); K_{a,b}(\alpha, \beta)\},$$

so we still need to establish the other inequality over the remaining domain. In the sequel, we show that, for all compact subsets $C \subset \mathbf{R}^2$,

$$\limsup_{T \rightarrow +\infty} \frac{1}{T} \ln \mathbf{P}(\bar{\theta}_T \in C) \leq - \inf_{(\alpha, \beta) \in C} \min\{J_{a,b}(\alpha, \beta); K_{a,b}(\alpha, \beta)\}. \quad (6.23)$$

It is sufficient to consider compact subsets of $\mathbf{R}^2$ instead of closed ones, since we already know that the sequence $\bar{\theta}_T$ satisfies an LDP with good rate function $I_{a,b}$ and $\mathbf{R}^2$ is locally compact so that the family $(\mathbf{P}(\bar{\theta}_T \in \cdot))_T$ is exponentially tight (see [5, Lemma 1.2.18 and Exercise 1.2.19]). This will prove the announced result as, by Lemma 6.3, the sequences $(\bar{\theta}_T)_T$ and $(\hat{\theta}_T)_T$ share the same LDP.

First of all, we notice that $\bar{\theta}_T = g(\mathcal{Q}_T)$, where g is the function defined over $\{(x, y, z, t) \in \mathbf{R}^4 \mid yz - 1 \neq 0\}$ by

$$g(x, y, z, t) = \left(\frac{y(2z - t^2 \mathbf{1}_{t < 0}) - x^2 \mathbf{1}_{t \geq 0}}{yz - 1}, \frac{t^2 \mathbf{1}_{t < 0} + (x^2 \mathbf{1}_{t \geq 0} - 2)z}{yz - 1} \right), \quad (6.24)$$

and the quadruplet $\mathcal{Q}_T$ satisfies an LDP with good rate function Λ^* given by Lemma 6.1. Since the function g given by (6.24) is not continuous, we cannot apply directly the contraction principle. However,

$$\limsup_{T \rightarrow +\infty} \frac{1}{T} \ln \mathbf{P}(\bar{\theta}_T \in C) \leq - \inf_{g^{-1}(C)} \Lambda^*. \quad (6.25)$$

We need to describe the subset $\overline{g^{-1}(C)}$. A quadruplet (x, y, z, t) of $\mathbf{R}^4$ belongs to $\overline{g^{-1}(C)}$ if and only if there exist a sequence $(x_n, y_n, z_n, t_n)_n$ and a sequence $(\alpha_n, \beta_n)_n \in C$ such that, as n tends to infinity,

$$(x_n, y_n, z_n, t_n) \rightarrow (x, y, z, t) \quad (6.26)$$

and for all n

$$g(x_n, y_n, z_n, t_n) = (\alpha_n, \beta_n). \quad (6.27)$$

Since C is a compact subset, up to a subsequence, there exists $(\alpha, \beta) \in C$ such that (α_n, β_n) converges to (α, β) as n goes to infinity. Moreover, (6.27) is equivalent to the following conditions for all n :

$$\beta_n y_n = x_n^2 \mathbf{1}_{t_n \geq 0} - \alpha_n \quad \text{and} \quad (2 - \alpha_n) z_n = t_n^2 \mathbf{1}_{t_n < 0} + \beta_n.$$

Up to a subsequence again, both indicator functions converge toward 1 or 0. Thus, letting n go to infinity, we obtain conditions on (x, y, z, t) which lead to

$$\overline{g^{-1}(C)} = \bigcup_{(\alpha, \beta) \in C} \mathcal{D}_{\alpha, \beta}^+ \cup \mathcal{D}_{\alpha, \beta}^-,$$

where

$$\begin{aligned} \mathcal{D}_{\alpha, \beta}^+ &= \{(x, y, z, t) \in \mathbf{R}^3 \times \mathbf{R}^+ \mid \beta y = x^2 - \alpha, \quad (2 - \alpha)z = \beta\}, \\ \mathcal{D}_{\alpha, \beta}^- &= \{(x, y, z, t) \in \mathbf{R}^3 \times \mathbf{R}^- \mid \beta y = -\alpha, \quad (2 - \alpha)z = t^2 + \beta\}. \end{aligned}$$

Thus, (6.25) becomes

$$\limsup_{T \rightarrow +\infty} \frac{1}{T} \ln \mathbf{P}(\bar{\theta}_T \in C) \leq - \inf_{(\alpha, \beta) \in C} \min \left\{ \inf_{\mathcal{D}_{\alpha, \beta}^+} \Lambda^*; \inf_{\mathcal{D}_{\alpha, \beta}^-} \Lambda^* \right\}. \quad (6.28)$$

For all $\alpha \neq 2$ and $\beta \neq 0$, we easily rewrite

$$\begin{aligned} \mathcal{D}_{\alpha, \beta}^+ &= \left\{ \left(x, \frac{x^2 - \alpha}{\beta}, \frac{\beta}{2 - \alpha}, t \right), x \in \mathbf{R}, t \geq 0 \right\}, \\ \mathcal{D}_{\alpha, \beta}^- &= \left\{ \left(x, -\frac{\alpha}{\beta}, \frac{t^2 + \beta}{2 - \alpha}, t \right), x \in \mathbf{R}, t \leq 0 \right\}. \end{aligned}$$

Since the rate function Λ^* given by Lemma 6.1 is infinite for $t > 0$, the infimum over $\mathcal{D}_{\alpha, \beta}^+$ reduces to the infimum over $\{(x, (x^2 - \alpha)/\beta, \beta/(2 - \alpha), 0), x \in \mathbf{R}\}$ which is equal to $J_{a, b}(\alpha, \beta)$. Over $\mathcal{D}_{\alpha, \beta}^-$, we know by Lemma 6.1 that

$$\Lambda^* \left(x, -\frac{\alpha}{\beta}, \frac{t^2 + \beta}{2 - \alpha}, t \right) = \sup_{\mathcal{D}_{d, f}} \left\{ \frac{1}{4} \left(x\sqrt{d - b} - t\sqrt{\varphi(f)} \right)^2 + \psi_{d, f}(t) \right\},$$

where $\psi_{d, f}(t)$ does not depend on x . Thus,

$$\Lambda^* \left(x, -\frac{\alpha}{\beta}, \frac{t^2 + \beta}{2 - \alpha}, t \right) \geq \Lambda^* \left(0, -\frac{\alpha}{\beta}, \frac{t^2 + \beta}{2 - \alpha}, t \right)$$

and the infimum over $\mathcal{D}_{\alpha, \beta}^-$ is greater than the infimum over $\{(0, -\alpha/\beta, (t^2 + \beta)/(2 - \alpha), t), t \leq 0\}$, which is equal to $K_{a, b}(\alpha, \beta)$.

We now need to investigate the cases $\alpha = 2$ and $\beta = 0$ before concluding. For $\beta = 0$ and $\alpha \notin]0, 2[$, we already know the value of $I_{a,b}$. If $\alpha \in]0, 2[$, then

$$\mathcal{D}_{\alpha,0}^+ = \{(x, y, z, t) \in \mathbf{R}^3 \times \mathbf{R}^+ | x^2 = \alpha, z = 0\} \quad \text{and} \quad \mathcal{D}_{\alpha,0}^- = \emptyset.$$

And with the same argument as before we obtain that for all $\alpha \in]0, 2[$,

$$I_{a,b}(\alpha, 0) = J_{a,b}(\alpha, 0) = \min\{J_{a,b}(\alpha, 0); K_{a,b}(\alpha, 0)\},$$

since $K_{a,b}(\alpha, 0)$ is equal to infinity. Now, for $\alpha = 2$, we have already computed $I_{a,b}$ for all $\beta \geq 0$. For $\beta < 0$,

$$\mathcal{D}_{2,\beta}^+ = \emptyset \quad \text{and} \quad \mathcal{D}_{2,\beta}^- = \{(x, y, z, t) \in \mathbf{R}^3 \times \mathbf{R}^- | \beta y = -2, t^2 = -\beta\}.$$

We obtain that

$$\mathcal{D}_{2,\beta}^- = \left\{ \left(x, -\frac{2}{\beta}, z, t \right), x \in \mathbf{R}, z \in \mathbf{R}, t \leq 0 \right\}$$

and with the same argument as before, the infimum over this subset is greater than the infimum over $\{(0, -2/\beta, z, t), z \in \mathbf{R}, t \leq 0\}$, which is equal to $K_{a,b}(2, \beta)$. Thus, as $J_{a,b}(2, \beta)$ is equal to infinity, we can conclude that

$$I_{a,b}(2, \beta) = \min\{J_{a,b}(2, \beta); K_{a,b}(2, \beta)\}.$$

Theorem 2.3 is proved.

7. Appendix A: Proofs of the CLT for the two couples of simplified estimators. The key to obtain those results is the Slutsky lemma. Indeed, we have

$$\sqrt{T} \begin{pmatrix} \tilde{a}_T - a \\ \tilde{b}_T - b \end{pmatrix} = \sqrt{T} \begin{pmatrix} \hat{a}_T - a \\ \hat{b}_T - b \end{pmatrix} + \sqrt{T} \begin{pmatrix} -S_T L_T / V_T \\ L_T / V_T \end{pmatrix}, \quad (7.1)$$

where

$$\sqrt{T} \begin{pmatrix} \hat{a}_T - a \\ \hat{b}_T - b \end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}(0, 4C^{-1}), \quad C = \begin{pmatrix} -b/(a-2) & 1 \\ 1 & -a/b \end{pmatrix},$$

and we show that the right-hand side of (7.1) converges to zero in probability. Namely, it is well known (see, for instance, [14, Lemma 3]) that S_T converges almost surely to $-a/b$ and V_T to $2/(a-2)$. And for all $\varepsilon > 0$, we have

$$\begin{aligned} \mathbf{P} \left(\left| \frac{\ln X_T}{\sqrt{T}} \right| \geq \varepsilon \right) &= \mathbf{P} \left(|\ln X_T| \geq \sqrt{T} \varepsilon \right) \\ &\leq \mathbf{P} \left(\ln X_T \geq \sqrt{T} \varepsilon \right) + \mathbf{P} \left(-\ln X_T \geq \sqrt{T} \varepsilon \right) \\ &\leq \mathbf{E} (X_T) e^{-\sqrt{T} \varepsilon} + \mathbf{E} (X_T^{-1}) e^{-\sqrt{T} \varepsilon} \xrightarrow{T \rightarrow +\infty} 0 \end{aligned}$$

because $\mathbf{E} X_T$ converges to $\mathbf{E}(X_\infty) = -a/b$ (see [14, Lemma 3]), and, since the parameter a is supposed greater than 2, we obtain from [2, Proposition 3] that

$$\mathbf{E} X_T^{-1} \rightarrow -\frac{b}{2} \frac{\Gamma(a/2 - 1)}{\Gamma(a/2)} = \mathbf{E} X_\infty^{-1}.$$

This gives the announced convergence in probability to zero. Thus, with the Slutsky lemma, the simplified estimators $(\tilde{a}_T, \tilde{b}_T)$ satisfy the same asymptotic normality result as the MLE.

Similarly, for the second couple of simplified estimators, we have

$$\sqrt{T} \begin{pmatrix} \check{a}_T - a \\ \check{b}_T - b \end{pmatrix} = \sqrt{T} \begin{pmatrix} \hat{a}_T - a \\ \hat{b}_T - b \end{pmatrix} + \sqrt{T} \begin{pmatrix} X_T/(TV_T) \\ -X_T \Sigma_T/(TV_T) \end{pmatrix}. \quad (7.2)$$

Since Σ_T converges almost surely to $-b/(a-2)$, we only have to show that $X_T/\sqrt{T}$ converges to zero in probability. For all $\varepsilon > 0$,

$$\mathbf{P}\left(\left|\frac{X_T}{\sqrt{T}}\right| \geq \varepsilon\right) = \mathbf{P}\left(X_T \geq \sqrt{T}\varepsilon\right) \leq \frac{\mathbf{E} X_T}{\sqrt{T}\varepsilon} \xrightarrow{T \rightarrow +\infty} 0$$

with the same argument as before. Thus $(\check{a}_T, \check{b}_T)$ also satisfies the same asymptotic normality result.

8. Appendix B: Proof of Lemma 3.1. We apply the Gärtner–Ellis theorem (see [5]). It is easy to deduce from Proposition 4.1 that the pointwise limit $\tilde{\Lambda}$ of the normalized cumulant generating function $\tilde{\Lambda}_T$ of the couple (S_T, Σ_T) is given by

$$\tilde{\Lambda}(\mu, \nu) = -\frac{d}{2}(1+f) - \frac{ab}{4}, \quad (8.1)$$

where $d = \sqrt{b^2 - 8\mu}$ and $f = \sqrt{(a/2 - 1)^2 - 2\nu}$. We easily get from Lemma 4.1 that the function $\tilde{\Lambda}$ is steep. To obtain the rate function I , we just have to compute the Fenchel–Legendre transform of $\tilde{\Lambda}$:

$$I(x, y) = \sup_{\mu < b^2/8, \nu < (a-2)^2/8} \{x\mu + y\nu - \tilde{\Lambda}(\mu, \nu)\}. \quad (8.2)$$

First we note that

$$\text{for } x \leq 0 \text{ or } y \leq 0, \quad I(x, y) = +\infty, \quad (8.3)$$

because $\tilde{\Lambda}(\mu, \nu)$ tends to $-\infty$ as ν or μ tends to $-\infty$. Only the case $x > 0$ and $y > 0$ remains to be studied.

We look for the critical points. If $xy - 1 \neq 0$, we obtain

$$\begin{pmatrix} d_0 \\ f_0 \end{pmatrix} = \frac{1}{xy - 1} \begin{pmatrix} 2y \\ 1 \end{pmatrix}. \quad (8.4)$$

Since d_0 and f_0 must both be positive, the solution is in the domain if and only if $xy - 1 > 0$. It is easy to check that this critical point corresponds to a maximum of $\tilde{\Lambda}$. Using the equalities $\mu = (b^2 - d^2)/8$, $\nu = ((a/2 - 1)^2 - f^2)/2$ and replacing them into (8.2), we get

$$I(x, y) = \frac{y}{2(xy - 1)} + \frac{b^2}{8}x + \frac{(a - 2)^2}{8}y + \frac{ab}{4}. \quad (8.5)$$

To conclude, we need to examine the case $x > 0$, $y > 0$, $xy - 1 < 0$. We already know that for ν or μ tending to $-\infty$, $-\tilde{\Lambda}(\mu, \nu)$ tends to $+\infty$. However, since x and y are nonnegative, we cannot conclude directly. But it is possible to find a direction in which $\tilde{\Lambda}$ dominates the expression. Note that, for $-\nu$ and $-\mu$ large enough, $x\mu + y\nu - \tilde{\Lambda}(\mu, \nu) \sim x\mu + y\nu + 2\sqrt{\mu\nu}$. Let $k > 0$ and $\nu = k\mu$. We just have to find a $k > 0$ that satisfies $k^2y^2 + 2k(xy - 2) + x^2 < 0$. The above polynomial has two positive roots. Any k between these two roots will satisfy the condition. For example, $k = (2 - xy)/y^2$ fits. We have found directions for which $x\mu + y\nu - \tilde{\Lambda}(\mu, \nu)$ tends to $+\infty$, so that the supremum itself is equal to $+\infty$. And,

$$\text{for } x > 0, y > 0 \text{ such that } xy - 1 < 0, \quad I(x, y) = +\infty. \quad (8.6)$$

Combining (8.3), (8.5), and (8.6), we obtain the announced result.

9. Appendix C: Proofs of Lemmas 4.2–4.5. The four following proofs rely on lower and upper bounds for the modified Bessel function of the first kind, given by formula (6.25) of [13]. More precisely, for all $z > 0$ and $\nu > -1/2$, we have

$$1 < \left(\frac{2}{z}\right)^\nu \Gamma(\nu + 1) I_\nu(z) < e^z. \quad (9.1)$$

9.1. Proof of Lemma 4.2. It is easy to deduce the following upper and lower bounds from (9.1):

$$\frac{(\beta_T \sqrt{y})^f}{2^f \Gamma(f + 1)} \leq I_f(\beta_T \sqrt{y}) \leq \frac{(\beta_T \sqrt{y})^f}{2^f \Gamma(f + 1)} e^{\beta_T \sqrt{y}}. \quad (9.2)$$

Substituting it into the expression of H_T given by (4.8) leads to

$$\frac{2^f \Gamma(1 + f)}{\beta_T^f} H_T \geq \int_0^1 \exp\left(\lambda \sqrt{Ty} - \gamma \sqrt{-T \ln y} - \alpha_T y\right) \times y^{(2f+a-2)/4} dy, \quad (9.3)$$

$$\frac{2^f \Gamma(1 + f)}{\beta_T^f} H_T \leq \int_0^1 \exp\left(\beta_T \sqrt{y} + \lambda \sqrt{Ty} - \gamma \sqrt{-T \ln y} - \alpha_T y\right) \times y^{(2f+a-2)/4} dy. \quad (9.4)$$

We consider separately the integrals over $[0, 1/T]$ and over $[1/T, 1]$. On the one hand,

$$\begin{aligned}
 & \int_{1/T}^1 \exp \left(\beta_T \sqrt{y} + \lambda \sqrt{T y} - \gamma \sqrt{-T \ln y} - \alpha_T y \right) y^{(2f+a-2)/4} dy \\
 & \leq \int_{1/T}^1 \exp \left(\beta_T \sqrt{y} + \lambda \sqrt{T y} - \gamma \sqrt{-T \ln y} \right) dy \\
 & \leq 1 \times \sup_{[1/T, 1]} \exp \left(\beta_T \sqrt{y} + \lambda \sqrt{T y} - \gamma \sqrt{-T \ln y} \right) \\
 & \leq \exp \left(\lambda \sqrt{T} + \beta_T - \gamma \sqrt{-T \ln \frac{1}{T}} \right) \\
 & = O \left(\exp \left((\lambda - \gamma) \sqrt{T} \sqrt{\ln T} \right) \right).
 \end{aligned}$$

On the other hand, with the same argument, we show that

$$\begin{aligned}
 & \int_{1/T}^1 \exp \left(\lambda \sqrt{T y} - \gamma \sqrt{-T \ln y} - \alpha_T y \right) y^{(2f+a-2)/4} dy \\
 & = O \left(\exp \left((\lambda - \gamma) \sqrt{T} \sqrt{\ln T} \right) \right).
 \end{aligned}$$

Over $[0, 1/T]$, the functions $e^{\lambda \sqrt{T y}}$, $e^{\beta_T \sqrt{y}}$ and $e^{-\alpha_T y}$ are bounded. So, since α_T and β_T are always positive, we have the following bounds:

$$\begin{aligned}
 & \int_0^{1/T} \exp \left(\lambda \sqrt{T y} - \gamma \sqrt{-T \ln y} - \alpha_T y \right) y^{(2f+a-2)/4} dy \\
 & \geq \exp \left(-\frac{\alpha_T}{T} - |\lambda| \right) \int_0^{1/T} \exp \left(-\gamma \sqrt{-T \ln y} \right) y^{(2f+a-2)/4} dy, \\
 & \int_0^{1/T} \exp \left(\beta_T \sqrt{y} + \lambda \sqrt{T y} - \gamma \sqrt{-T \ln y} - \alpha_T y \right) y^{(2f+a-2)/4} dy \\
 & \leq \exp \left(|\lambda| + \frac{\alpha_T}{T} + \frac{\beta_T}{\sqrt{T}} \right) \int_0^{1/T} \exp \left(-\gamma \sqrt{-T \ln y} \right) y^{(2f+a-2)/4} dy.
 \end{aligned}$$

Using the change of variable given by $z = \sqrt{-\ln y} + \gamma \sqrt{T}/(2g)$, where $g = (2f + a + 2)/4$, we obtain

$$\begin{aligned}
 & \int_0^{1/T} \exp \left(-\gamma \sqrt{-T \ln y} \right) y^{g-1} dy \\
 & = 2e^{\gamma^2 T/(4g)} \int_{\sqrt{\ln T} + \gamma \sqrt{T}/(2g)}^{+\infty} e^{-g z^2} \left(z - \frac{\gamma \sqrt{T}}{2g} \right) dz \\
 & = 2e^{\gamma^2 T/(4g)} \left(\int_{\sqrt{\ln T} + \gamma \sqrt{T}/(2g)}^{+\infty} e^{-g z^2} z dz - \frac{\gamma \sqrt{T}}{2g} \int_{\sqrt{\ln T} + \gamma \sqrt{T}/(2g)}^{+\infty} e^{-g z^2} dz \right).
 \end{aligned}$$

By dominated convergence as $\gamma < 0$, the first integral tends to zero when T goes to infinity and the second one tends to the positive constant $\sqrt{\pi/g}$.

This leads to the following bounds, for T large enough:

$$\begin{aligned} \int_0^{1/T} \exp\left(-\gamma\sqrt{-T\ln y}\right) y^{g-1} dy &\leq 2\sqrt{\frac{\pi}{g}} \frac{|\gamma|\sqrt{T}}{g} \exp\left(\frac{\gamma^2 T}{4g}\right), \\ \int_0^{1/T} \exp\left(-\gamma\sqrt{-T\ln y}\right) y^{g-1} dy &\geq \frac{1}{2}\sqrt{\frac{\pi}{g}} \frac{|\gamma|\sqrt{T}}{g} \exp\left(\frac{\gamma^2 T}{4g}\right). \end{aligned}$$

Combined with the result over $[1/T, 1]$, it gives the announced result.

9.2. Proof of Lemma 4.3. The upper bound easily follows from (9.2). Actually, for all $\gamma > 0$ and $\lambda \in \mathbf{R}$, we obtain

$$\begin{aligned} H_T &\leq \frac{\beta_T^f}{\Gamma(f+1) \cdot 2^f} \int_0^1 \exp\left((\lambda\sqrt{T} + \beta_T)\sqrt{y} - \gamma\sqrt{-T\ln y}\right) y^{(2f+a-2)/4} dy \\ &\leq \frac{\beta_T^f}{\Gamma(f+1) \cdot 2^f} \exp\left(|\lambda|\sqrt{T} + \beta_T\right). \end{aligned} \quad (9.5)$$

Besides, using the lower bound of (9.2), we clearly have, for all $\gamma > 0$ and $\lambda \in \mathbf{R}$,

$$H_T \geq \frac{\beta_T^f e^{-\alpha_T}}{\Gamma(f+1) \cdot 2^f} \int_0^1 \exp\left(\lambda\sqrt{T}\sqrt{y} - \gamma\sqrt{-T\ln y}\right) y^{(2f+a-2)/4} dy. \quad (9.6)$$

To obtain the announced lower bound, we need to consider separately the integral over $[0, 1/T]$ and over $[1/T, 1]$. On the one hand, the integral over $[1/T, 1]$ is easy to handle:

$$\begin{aligned} \int_{1/T}^1 \exp\left(\lambda\sqrt{T}y - \gamma\sqrt{-T\ln y}\right) y^{(2f+a-2)/4} dy \\ \geq \varepsilon \exp\left(-\gamma\sqrt{T}\sqrt{\ln T}\right) T^{-(2f+a-2)/4}, \end{aligned} \quad (9.7)$$

where $\varepsilon = \exp(\lambda \mathbf{1}_{\lambda \geq 0} + \lambda\sqrt{T} \mathbf{1}_{\lambda < 0})$. On the other hand,

$$\begin{aligned} \int_0^{1/T} \exp\left(\lambda\sqrt{T}\sqrt{y} - \gamma\sqrt{-T\ln y}\right) y^{(2f+a-2)/4} dy \\ \geq e^{-|\lambda|} \int_0^{1/T} \exp\left(-\gamma\sqrt{-T\ln y}\right) y^{(2f+a-2)/4} dy. \end{aligned}$$

Using the of change variable given by $z = \sqrt{-\ln y} + \gamma\sqrt{T}/(2g)$, where we recall that $g = (2f + a + 2)/4$, we obtain

$$\begin{aligned} \int_0^{1/T} \exp\left(-\gamma\sqrt{-T\ln y}\right) y^{(2f+a-2)/4} dy \\ = 2e^{\gamma^2 T/(4g)} \int_{\sqrt{\ln T} + \gamma\sqrt{T}/(2g)}^{+\infty} e^{-gz^2} \left(z - \frac{\gamma\sqrt{T}}{2g}\right) dz. \end{aligned}$$

First, we clearly have

$$\int_{\sqrt{\ln T} + \gamma\sqrt{T}/(2g)}^{+\infty} e^{-gz^2} z \, dz = \frac{1}{2g} \exp \left(-g \left(\sqrt{\ln T} + \frac{\gamma\sqrt{T}}{2g} \right)^2 \right).$$

Besides, for the second part of the integral, we have

$$\begin{aligned} \int_{\sqrt{\ln T} + \gamma\sqrt{T}/(2g)}^{+\infty} e^{-gz^2} \, dz &= \int_{\sqrt{\ln T} + \gamma\sqrt{T}/(2g)}^{+\infty} e^{-gz^2} z \frac{1}{z} \, dz \\ &\leq \frac{1}{\sqrt{\ln T} + \gamma\sqrt{T}/(2g)} \int_{\sqrt{\ln T} + \gamma\sqrt{T}/(2g)}^{+\infty} e^{-gz^2} z \, dz \\ &= \frac{1}{2g\sqrt{\ln T} + \gamma\sqrt{T}} \exp \left(-g \left(\sqrt{\ln T} + \frac{\gamma\sqrt{T}}{2g} \right)^2 \right). \end{aligned}$$

Thus, for any positive γ , we have the following lower bound:

$$\begin{aligned} \int_0^{1/T} \exp \left(-\gamma\sqrt{-T \ln y} \right) y^{(2f+a-2)/4} \, dy \\ \geq \frac{1}{g} \left(1 - \frac{\gamma\sqrt{T}}{2g\sqrt{\ln T} + \gamma\sqrt{T}} \right) \exp \left(-g \ln T - \gamma\sqrt{T}\sqrt{\ln T} \right). \end{aligned}$$

Combined with (9.7), this leads to

$$\begin{aligned} H_T &\geq \frac{(\beta_T)^f e^{-\alpha_T}}{2^f \Gamma(1+f)} \left(\varepsilon + \frac{e^{-|\lambda|}}{g} \left(1 - \frac{\gamma\sqrt{T}}{2g\sqrt{\ln T} + \gamma\sqrt{T}} \right) \right) \\ &\quad \times \exp \left(-\gamma\sqrt{T}\sqrt{\ln T} - g \ln T \right). \end{aligned}$$

9.3. Proof of Lemma 4.4. Using the inequality (9.2) for the modified Bessel function I_f in (4.9), we obtain

$$K_T \leq \frac{(\beta_T)^f}{2^f \Gamma(1+f)} \int_1^{+\infty} \exp \left(\beta_T \sqrt{y} + \lambda \sqrt{T y} - \alpha_T y \right) y^{\gamma+(2f+a-2)/4} \, dy, \quad (9.8)$$

$$K_T \geq \frac{(\beta_T)^f}{2^f \Gamma(1+f)} \int_1^{+\infty} \exp \left(\lambda \sqrt{T y} - \alpha_T y \right) y^{\gamma+(2f+a-2)/4} \, dy. \quad (9.9)$$

To go further, we need to consider the sign of the exponent $\gamma+(2f+a-2)/4$.

Case 1: $\gamma + (2f+a-2)/4 \leq 0$. Using the fact that $y^{\gamma+(2f+a-2)/4} \leq 1$ and with the change of variable $u = \sqrt{y} - (\lambda\sqrt{T} + \beta_T)/(2\alpha_T)$, we obtain the following asymptotic behavior for the upper bound of K_T :

$$\begin{aligned} K_T &\leq \frac{(\beta_T)^f}{2^f \Gamma(1+f)} \int_1^{+\infty} \exp \left((\beta_T + \lambda\sqrt{T})\sqrt{y} - \alpha_T y \right) \, dy \\ &= \frac{2(\beta_T)^f}{2^f \Gamma(1+f)} \exp \left(\frac{(\lambda\sqrt{T} + \beta_T)^2}{4\alpha_T} \right) \end{aligned}$$

$$\begin{aligned}
& \times \int_{1-(\lambda\sqrt{T}+\beta_T)/(2\alpha_T)}^{+\infty} e^{-\alpha_T u^2} \left(u + \frac{\lambda\sqrt{T} + \beta_T}{2\alpha_T} \right) du \\
& \leq \frac{2(\beta_T)^f}{2^f \Gamma(1+f)} \exp \left(\frac{(\lambda\sqrt{T} + \beta_T)^2}{4\alpha_T} \right) \left(A_1 + \frac{\beta_T}{2\alpha_T} A_2 \right),
\end{aligned}$$

where A_1 and A_2 are given by

$$\begin{aligned}
A_1 &= \int_{1-(\lambda\sqrt{T}+\beta_T)/(2\alpha_T)}^{+\infty} e^{-\alpha_T u^2} u \, du \\
&= \frac{1}{2\alpha_T} \exp \left(-\alpha_T \left(1 - \frac{\lambda\sqrt{T} + \beta_T}{2\alpha_T} \right)^2 \right), \tag{9.10}
\end{aligned}$$

$$\begin{aligned}
A_2 &= \int_{1-(\lambda\sqrt{T}+\beta_T)/(2\alpha_T)}^{+\infty} e^{-\alpha_T u^2} \, du \\
&= \int_{1-(\lambda\sqrt{T}+\beta_T)/(2\alpha_T)}^{+\infty} e^{-\alpha_T u^2} u \frac{1}{u} \, du \\
&\leq \frac{1}{1 - (\lambda\sqrt{T} + \beta_T)/(2\alpha_T)} A_1. \tag{9.11}
\end{aligned}$$

Thus,

$$\begin{aligned}
K_T &\leq \frac{2(\beta_T)^f}{2^f \Gamma(1+f)} \exp \left(\frac{(\lambda\sqrt{T} + \beta_T)^2}{4\alpha_T} \right) \frac{1}{2\alpha_T} \\
&\quad \times \exp \left(-\alpha_T \left(1 - \frac{\lambda\sqrt{T} + \beta_T}{2\alpha_T} \right)^2 \right) \left(1 + \frac{\beta_T/(2\alpha_T)}{1 - (\lambda\sqrt{T} + \beta_T)/(2\alpha_T)} \right) \\
&= \frac{(\beta_T)^f}{2^f \Gamma(1+f) \alpha_T} \left(1 + \frac{\beta_T/(2\alpha_T)}{1 - (\lambda\sqrt{T} + \beta_T)/(2\alpha_T)} \right) \exp \left(\lambda\sqrt{T} + \beta_T - \alpha_T \right). \tag{9.12}
\end{aligned}$$

Case 2: $\gamma + (2f + a - 2)/4 > 0$. With formula 3.462(1) in [9], we get

$$\begin{aligned}
K_T &\leq \frac{(\beta_T)^f}{2^f \Gamma(1+f)} \left(2(2\alpha_T)^{-(\gamma+g)} \Gamma(2\gamma + 2g) \exp \left(\frac{(\lambda\sqrt{T} + \beta_T)^2}{8\alpha_T} \right) \right. \\
&\quad \times D_{-2(\gamma+g)} \left(\frac{-\beta_T - \lambda\sqrt{T}}{\sqrt{2\alpha_T}} \right) \\
&\quad \left. - \int_0^1 \exp \left(\beta_T \sqrt{y} + \lambda\sqrt{T} y - \alpha_T y \right) y^{\gamma+g-1} dy \right) \\
&\leq \frac{2(\beta_T)^f (2\alpha_T)^{-(\gamma+g)} \Gamma(2\gamma + 2g)}{2^f \Gamma(1+f)} \exp \left(\frac{(\lambda\sqrt{T} + \beta_T)^2}{8\alpha_T} \right) \\
&\quad \times D_{-2(\gamma+g)} \left(\frac{-\beta_T - \lambda\sqrt{T}}{\sqrt{2\alpha_T}} \right),
\end{aligned}$$

where $D_{-2(\gamma+g)}$ is the parabolic cylinder function defined by formula 9.250 in [9]. But, if $\lambda < 0$, then for T large enough, $-\beta_T - \lambda\sqrt{T} > 0$ so formula 9.246(1) of [9] gives

$$D_{-2(\gamma+g)}\left(\frac{-\beta_T - \lambda\sqrt{T}}{\sqrt{2\alpha_T}}\right) \sim \left(\frac{-\beta_T - \lambda\sqrt{T}}{\sqrt{2\alpha_T}}\right)^{-2(\gamma+g)} \exp\left(-\frac{(\lambda\sqrt{T} + \beta_T)^2}{8\alpha_T}\right).$$

Thus $K_T = O(\beta_T)^f$ as T goes to infinity. If $\lambda = 0$, we use formula 9.246(2) in [9] to obtain

$$D_{-2(\gamma+g)}\left(\frac{-\beta_T}{\sqrt{2\alpha_T}}\right) \sim \frac{\sqrt{2\pi}}{\Gamma(2\gamma + 2g)} \left(\frac{\beta_T}{\sqrt{2\alpha_T}}\right)^{2\gamma+2g-1} \exp\left(\frac{(\beta_T)^2}{8\alpha_T}\right).$$

This leads to the same conclusion: as T goes to infinity,

$$K_T = O((\beta_T)^f). \quad (9.13)$$

Otherwise, for the proof of Lemma 4.1, we also need a lower bound for K_T when $\gamma \geq 0$. We note that over $[1, +\infty[$, we have $y^{\gamma+g-1} \geq 1$, so that

$$K_T \geq \frac{2^{-f}(\beta_T)^f}{\Gamma(1+f)} \int_1^{+\infty} e^{\lambda\sqrt{T}\sqrt{y}-\alpha_T y} dy. \quad (9.14)$$

With the change of variable given by $u = \sqrt{y} - \lambda\sqrt{T}/(2\alpha_T)$, it becomes

$$\begin{aligned} K_T &\geq \frac{2^{1-f}(\beta_T)^f}{\Gamma(1+f)} \exp\left(\frac{\lambda^2 T}{4\alpha_T}\right) \int_{1-\lambda\sqrt{T}/(2\alpha_T)}^{+\infty} e^{-\alpha_T u^2} \left(u + \frac{\lambda\sqrt{T}}{2\alpha_T}\right) du \\ &\geq \frac{2^{1-f}(\beta_T)^f}{\Gamma(1+f)} \exp\left(\frac{\lambda^2 T}{4\alpha_T}\right) \left(\int_{1-\lambda\sqrt{T}/(2\alpha_T)}^{+\infty} e^{-\alpha_T u^2} u du \right. \\ &\quad \left. + \frac{\lambda\sqrt{T}}{2\alpha_T} \int_{1-\lambda\sqrt{T}/(2\alpha_T)}^{+\infty} e^{-\alpha_T u^2} du \right). \end{aligned} \quad (9.15)$$

However, the first integral is easily computable:

$$\int_{1-\lambda\sqrt{T}/(2\alpha_T)}^{+\infty} e^{-\alpha_T u^2} u du = \frac{1}{2\alpha_T} \exp\left(-\alpha_T \left(1 - \frac{\lambda\sqrt{T}}{2\alpha_T}\right)^2\right) \quad (9.16)$$

and for the second one, we clearly have the following upper bound:

$$\begin{aligned} \int_{1-\lambda\sqrt{T}/(2\alpha_T)}^{+\infty} e^{-\alpha_T u^2} du &= \int_{1-\lambda\sqrt{T}/(2\alpha_T)}^{+\infty} e^{-\alpha_T u^2} u \frac{1}{u} du \\ &\leq \frac{1}{1 - \lambda\sqrt{T}/(2\alpha_T)} \int_{1-\lambda\sqrt{T}/(2\alpha_T)}^{+\infty} e^{-\alpha_T u^2} u du \\ &\leq \frac{1}{1 - \lambda\sqrt{T}/(2\alpha_T)} \frac{1}{2\alpha_T} \exp\left(-\alpha_T \left(1 - \frac{\lambda\sqrt{T}}{2\alpha_T}\right)^2\right). \end{aligned} \quad (9.17)$$

Using the fact that $\lambda < 0$ and combining (9.15), (9.16), and (9.17), we show the announced result:

$$K_T \geq \frac{2^{1-f}(\beta_T)^f}{\Gamma(1+f)} \frac{e^{-\alpha_T}}{2\alpha_T} \left(1 + \frac{\lambda\sqrt{T}/(2\alpha_T)}{1 - \lambda\sqrt{T}/(2\alpha_T)} \right) \exp(\lambda\sqrt{T}). \quad (9.18)$$

All those results are still true for $\lambda = 0$.

9.4. Proof of Lemma 4.5. As previously, we use (9.2) to find lower and upper bounds for K_T as T goes to infinity, for all $\lambda > 0$. We consider two cases depending on the sign of the exponent $\gamma + g - 1$.

Case 1: $\gamma + g - 1 \leq 0$. As in the proof of Lemma 4.4, we have the following upper bound:

$$K_T \leq \frac{2(\beta_T)^f}{2^f\Gamma(1+f)} \exp\left(\frac{(\lambda\sqrt{T} + \beta_T^2)}{4\alpha_T}\right) \left(A_1 + \frac{\beta_T}{2\alpha_T}A_2\right), \quad (9.19)$$

where $A_1 = \int_{1-(\lambda\sqrt{T}+\beta_T)/(2\alpha_T)}^{+\infty} e^{-\alpha_T u^2} u du$ tends to zero for all $\lambda > 0$ by dominated convergence, and $A_2 = \int_{1-(\lambda\sqrt{T}+\beta_T)/(2\alpha_T)}^{+\infty} e^{-\alpha_T u^2} du$ tends to the positive constant $2\sqrt{\pi/d-b}$. Thus, for T large enough,

$$K_T \leq 2^{1-f} \sqrt{\frac{\pi}{d-b}} \frac{(\beta_T)^{1+f}}{\alpha_T \Gamma(1+f)} \exp\left(\frac{(\lambda\sqrt{T} + \beta_T)^2}{4\alpha_T}\right). \quad (9.20)$$

For the lower bound, with the change of variable given by $u = \sqrt{y} - \lambda\sqrt{T}/(2\alpha_T)$, we obtain

$$\begin{aligned} K_T &\geq \frac{2^{-f}(\beta_T)^f}{\Gamma(1+f)} \int_1^{+\infty} e^{\lambda\sqrt{T}\sqrt{y}-\alpha_T y} y^{\gamma+g-1} dy \\ &= \frac{2^{1-f}(\beta_T)^f}{\Gamma(1+f)} e^{\lambda^2 T/(4\alpha_T)} \\ &\quad \times \int_{1-\lambda\sqrt{T}/(2\alpha_T)}^{+\infty} e^{-\alpha_T u^2} \left(u + \frac{\lambda\sqrt{T}}{2\alpha_T}\right)^{2(\gamma+g-1)} \left(u + \frac{\lambda\sqrt{T}}{2\alpha_T}\right) du, \end{aligned}$$

where the integral on the right-hand side is not less than

$$\begin{aligned} &\int_{1-\lambda\sqrt{T}/(2\alpha_T)}^{+\infty} e^{-\alpha_T u^2} \left(2 \max\left\{u, \frac{\lambda\sqrt{T}}{2\alpha_T}\right\}\right)^{2(\gamma+g-1)} du \\ &\geq \left(\frac{2\lambda\sqrt{T}}{2\alpha_T}\right)^{2(\gamma+g-1)} \int_{1-\lambda\sqrt{T}/(2\alpha_T)}^{\lambda\sqrt{T}/(2\alpha_T)} e^{-\alpha_T u^2} du \\ &\quad + \int_{\lambda\sqrt{T}/(2\alpha_T)}^{+\infty} e^{-\alpha_T u^2} (2u)^{2(\gamma+g-1)} du \\ &\geq \left(\frac{\lambda\sqrt{T}}{\alpha_T}\right)^{2(\gamma+g-1)} \int_{1-\lambda\sqrt{T}/(2\alpha_T)}^{\lambda\sqrt{T}/(2\alpha_T)} e^{-\alpha_T u^2} du. \end{aligned}$$

By dominated convergence, the last integral converges as T tends to infinity to the positive constant $2\sqrt{\pi/(d-b)}$. For T large enough, this leads to

$$K_T \geq 2^{1-f} \sqrt{\frac{\pi}{d-b}} \frac{(\beta_T)^f}{\Gamma(1+f)} \left(\frac{\lambda\sqrt{T}}{\alpha_T} \right)^{2(\gamma+g-1)} e^{\lambda^2 T/(4\alpha_T)}. \quad (9.21)$$

Case 2: $\gamma + g - 1 > 0$. Over $[1, +\infty[$, we notice that $y^{\gamma+g-1} \geq 1$. Thus

$$K_T \geq \frac{2^{-f}(\beta_T)^f}{\Gamma(1+f)} \int_1^{+\infty} e^{\lambda\sqrt{T}\sqrt{y}-\alpha_T y} dy. \quad (9.22)$$

With the change of variable given by $u = \sqrt{y} - \lambda\sqrt{T}/(2\alpha_T)$, it becomes

$$\begin{aligned} K_T &\geq \frac{2^{1-f}(\beta_T)^f}{\Gamma(1+f)} e^{\lambda^2 T/(4\alpha_T)} \int_{1-\lambda\sqrt{T}/(2\alpha_T)}^{+\infty} e^{-\alpha_T u^2} \left(u + \frac{\lambda\sqrt{T}}{2\alpha_T} \right) du \\ &\geq \frac{2^{1-f}(\beta_T)^f}{\Gamma(1+f)} e^{\lambda^2 T/(4\alpha_T)} \left(\int_{1-\lambda\sqrt{T}/(2\alpha_T)}^{+\infty} e^{-\alpha_T u^2} u du \right. \\ &\quad \left. + \frac{\lambda\sqrt{T}}{2\alpha_T} \int_{1-\lambda\sqrt{T}/(2\alpha_T)}^{+\infty} e^{-\alpha_T u^2} du \right). \end{aligned}$$

By dominated convergence, the first integral in the last expression tends to zero as T goes to infinity and the second integral converges to the positive constant $2\sqrt{\pi/(d-b)}$. It gives, for T large enough,

$$K_T \geq \sqrt{\frac{\pi}{d-b}} \frac{2^{1-f}(\beta_T)^f}{\Gamma(1+f)} \frac{\lambda\sqrt{T}}{2\alpha_T} e^{\lambda^2 T/(4\alpha_T)}. \quad (9.23)$$

Moreover, we establish the upper bound with [9, formula 3.462(1)] as in the previous proof:

$$\begin{aligned} K_T &\leq \frac{(\beta_T)^f}{2^f \Gamma(1+f)} \left(2(2\alpha_T)^{-(\gamma+g)} \Gamma(2\gamma+2g) \exp\left(\frac{(\lambda\sqrt{T} + \beta_T)^2}{8\alpha_T}\right) \right. \\ &\quad \times D_{-2(\gamma+g)}\left(\frac{-\beta_T - \lambda\sqrt{T}}{\sqrt{2\alpha_T}}\right) \\ &\quad \left. - \int_0^1 \exp\left(\beta_T \sqrt{y} + \lambda\sqrt{T}y - \alpha_T y\right) y^{\gamma+g-1} dy \right) \\ &\leq 2^{1-f}(\beta_T)^f (2\alpha_T)^{-(\gamma+g)} \frac{\Gamma(2\gamma+2g)}{\Gamma(1+f)} \exp\left(\frac{(\lambda\sqrt{T} + \beta_T)^2}{8\alpha_T}\right) \\ &\quad \times D_{-2(\gamma+g)}\left(\frac{-\beta_T - \lambda\sqrt{T}}{\sqrt{2\alpha_T}}\right). \end{aligned}$$

But $-\beta_T - \lambda\sqrt{T} < 0$ so, by [9, formula 9.246(2)]

$$\begin{aligned} D_{-2(\gamma+g)}\left(\frac{-\beta_T - \lambda\sqrt{T}}{\sqrt{2\alpha_T}}\right) &\sim \frac{\sqrt{2\pi}}{\Gamma(2\gamma+2g)} \left(\frac{\beta_T + \lambda\sqrt{T}}{\sqrt{2\alpha_T}} \right)^{2\gamma+2g-1} \\ &\quad \times \exp\left(\frac{(\lambda\sqrt{T} + \beta_T)^2}{8\alpha_T}\right). \end{aligned}$$

This gives, for T large enough,

$$K_T \leq \frac{2^{2-f} \sqrt{2\pi} (\beta_T)^f}{\Gamma(1+f)} \left(\beta_T + \lambda \sqrt{T} \right)^{2\gamma+2g-1} \exp \left(\frac{(\lambda \sqrt{T} + \beta_T)^2}{4\alpha_T} \right). \quad (9.24)$$

This easily leads to the announced result.

10. Appendix D: proof of Lemma 5.1. We call L the pointwise limit of the cumulant generating function of the triplet $(X_T/T, S_T, \Sigma_T)$. We notice that $L(\lambda, \mu, \nu) = \Lambda(\lambda, \mu, \nu, 0)$, where Λ is given by Proposition 4.1. We easily deduce from Lemma 4.1 that the function L is steep. We apply the Gärtner–Ellis theorem: the rate function is given by the Fenchel–Legendre transform of L on its effective domain, namely,

$$I(x, y, z) = \sup_{\lambda \in \mathbf{R}, \mu < b^2/8, \nu < (a-2)^2/8} \{x\lambda + y\mu + z\nu - L(\lambda, \mu, \nu)\}. \quad (10.1)$$

For y and z we recognize exactly the same term as in Appendix B. So with the same argument, we show that

$$\text{for } y \leq 0, z \leq 0 \text{ or } yz - 1 \leq 0, \quad I(x, y, z) = +\infty. \quad (10.2)$$

Moreover, if $x < 0$, then for λ tending to $-\infty$, we have $\lambda x \rightarrow +\infty$ and $I(x, y, z) = +\infty$ since Λ does not depend on λ for $\lambda < 0$. Moreover, for $x > 0$, the term on λ is always negative for λ negative and sometimes positive if λ is positive. So the supremum is necessarily reached for $\lambda > 0$. We finally have to calculate

$$I(x, y, z) = \sup_{\lambda > 0, \mu < b^2/8, \nu < (a-2)^2/8} \{x\lambda + y\mu + z\nu - \Lambda(\lambda, \mu, \nu)\} \quad (10.3)$$

with $x \geq 0, y > 0, z > 0$, and $yz - 1 > 0$. We do exactly as in Appendix A, we are looking for critical points and the calculations are very similar. We find

$$\begin{aligned} \lambda_0 &= \frac{x}{2} \left(\frac{z(x^2 + 2)}{yz - 1} - b \right), & \mu_0 &= \frac{1}{8} \left(b^2 - \frac{z^2(x^2 + 2)^2}{(yz - 1)^2} \right), \\ \nu_0 &= \frac{1}{8} \left((a - 2)^2 - \frac{(x^2 + 2)^2}{(yz - 1)^2} \right), \end{aligned}$$

which lead easily to I .

11. Appendix E: Proof of Lemma 5.2. The pointwise limit of the cumulant generating function of the considered triplet is easily given by $\Lambda(0, \mu, \nu, \gamma)$, where Λ is defined in Lemma 4.1

$$\Lambda(0, \mu, \nu, \gamma) = \begin{cases} -\frac{d}{2}(1+f) - \frac{ab}{4} + \frac{\gamma^2}{2f+a+2} & \text{if } \gamma < 0 \\ -\frac{d}{2}(1+f) - \frac{ab}{4} & \text{else.} \end{cases} \quad (11.1)$$

We recall that $d = \sqrt{b^2 - 8\mu}$ and $f = (1/2)\sqrt{(a-2)^2 - 8\nu}$. Using the Gärtner–Ellis theorem, we deduce that

$$\tilde{I}(y, z, t) = \sup_{\gamma \in \mathbf{R}, \mu < b^2/8, \nu < (a-2)^2/8} \{y\mu + z\nu + t\gamma - \Lambda(0, \mu, \nu, \gamma)\}. \quad (11.2)$$

With the same argument as for the other couple of simplified estimators, we show that $\tilde{I}(y, z, t) = +\infty$ for $y < 0$, $z < 0$ or $yz - 1 < 0$. We also notice that for $t > 0$ the expression inside the supremum tends to infinity as γ goes to infinity. So $\tilde{I}(y, z, t) = +\infty$ for $t > 0$. Besides, for $t \leq 0$, the part involving γ is always negative for $\gamma \geq 0$ and sometimes positive for $\gamma < 0$. It implies that the supremum is necessarily reached for $\gamma < 0$. Replacing μ and ν by their expression on d and f , we obtain, for $y > 0$, $z > 0$, $yz - 1 > 0$, and $t \leq 0$,

$$\begin{aligned} \tilde{I}(y, z, t) = \sup_{\gamma < 0, d > 0, f > 0} \left\{ y \frac{b^2 - d^2}{8} + z \frac{(a-2)^2 - 4f^2}{8} + t\gamma + \frac{ab}{4} \right. \\ \left. + \frac{d}{2}(1+f) - \frac{\gamma^2}{2f+a+2} \right\}. \end{aligned}$$

We investigate critical points. We obtain

$$f_0 = \frac{yt^2 + 2}{2(yz - 1)}, \quad d_0 = \frac{t^2 + 2z}{yz - 1}, \quad \gamma_0 = \frac{yt(t^2 + 2z)}{2(yz - 1)} + \frac{at}{2}. \quad (11.3)$$

Substituting it into the expression of $\tilde{I}$, we easily get the announced result.

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