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Следовательно, $v(\theta) \geq (q - \theta)(1 - \theta)/(1 + q)$. Очевидно также, что $v(\theta) \leq q - \theta$. Таким образом, получено двустороннее неравенство для $v(\theta)$: $(q - \theta)(1 - \theta)/(1 + q) \leq v(\theta) \leq q - \theta$. Решая дифференциальное неравенство $(q - \theta)(1 - \theta)/(1 + q) \leq \dot{\theta} \leq q - \theta$ при начальном условии $\theta|_{t=0} = 0$, получим

$$\frac{q(g(t, q) - 1)}{g(t, q) - q} \leq \theta(t) \leq q(1 - e^{-t}), \quad (3)$$

где функция $g(t, q)$ определена в формулировке теоремы и $\theta(t) = f^t(0)$. Из неравенства (3) и определения функции $F_r(t)$ следует утверждение теоремы.

З а м е ч а н и е. Правая часть неравенства теоремы 2 стремится к правой части неравенства теоремы 1 при $q \rightarrow 1$.

В заключение автор выражает глубокую признательность В.В. Горайнову за постановку задачи и обсуждение результатов.

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RARELY OBSERVED SAMPLE MAXIMA¹⁾

Для выборочных максимумов, наблюдаемых в почти геометрические моменты времени, определяются предельные распределения и их области притяжения.

Ключевые слова и фразы: выборочный максимум, область притяжения, геометрические моменты времени.

1. Introduction. Suppose $X_1, X_2, \dots$ are independent and identically distributed random variables with distribution function F . Suppose we observe $M_{k_n} := \max\{X_1, X_2, \dots, X_{k_n}\}$ for $n = 1, 2, \dots$, where $\{k_n\}$ is a nondecreasing integer sequence satisfying

$$\lim_{n \rightarrow +\infty} \frac{k_{n+1}}{k_n} = e^d \quad (1)$$

with $d > 0$.

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Assume there exist sequences of norming constants $\{a_n\}$ positive and $\{b_n\}$ real such that

$$\lim_{n \rightarrow +\infty} F^{k_n}(a_n x + b_n) = G(x), \quad (2)$$

where G is a nondegenerate distribution function and the convergence is for all but countably many x .

We want to characterize the class of distribution functions G occurring in (2) and for each of them we want necessary and sufficient conditions on F such that (2) holds. A different and somewhat more involved solution to this problem can be found in [1].

Section 2 contains the results. The proofs and some examples can be found in Section 3.

An approach similar to the one offered here can also be used to provide an alternative for Kruglov's [2] results for a sequence of partial sums rather than sample maxima.

2. Results. Before stating the results we formulate the problem somewhat differently in order to make their formulation more transparent.

Clearly (2) is equivalent to

$$\lim_{n \rightarrow +\infty} (u(a_n x + b_n) - \ln k_n) = \Phi(x), \quad (3)$$

with $u(x) := -\ln(-\ln F(x))$ and $\Phi(x) := -\ln(-\ln G(x))$, or equivalently,

$$\lim_{n \rightarrow +\infty} \frac{v(x + \ln k_n) - b_n}{a_n} = \Psi(x), \quad (4)$$

with $v(x) := (-\ln(-\ln F))^{\leftarrow}(x)$ and $\Phi(x) := (-\ln(-\ln G))^{\leftarrow}(x)$, where the arrow means generalized inverse function and the convergence is for all but countably many x . For a nondecreasing right-continuous function f , $f^{\leftarrow}(y) = \inf\{x: f(x) \geq y\}$.

Let x_0 be a continuity point of Ψ . We have

$$\lim_{n \rightarrow +\infty} \frac{v(x + \ln k_n^*) - b_n^*}{a_n^*} = \Psi^*(x),$$

with

$$\Psi^*(x) = \frac{\Psi(x + x_0) - \Psi(x_0)}{\Psi(d + x_0) - \Psi(x_0)},$$

$$k_n^* := [k_n e^{x_0}], \quad a_n^* := a_n (\Psi(d + x_0) - \Psi(x_0)), \quad b_n^* := b_n + a_n \Psi(x_0).$$

Note that Ψ^* is continuous at $x = 0$, $\Psi^*(0) = 0$ and $\Psi^*(d) = 1$. From now on we shall suppose that (4) holds with Ψ continuous at $x = 0$, $\Psi(0) = 0$ and $\Psi(d) = 1$. That is, we suppose without loss of generality

$$\begin{cases} G(0) = e^{-1}, \\ G(1) = \exp(-e^{-d}), \\ G(x) \text{ is continuous at } x = 0. \end{cases} \quad (5)$$

In what follows we consider $s_n := 1 + a + a^2 + \dots + a^{n-1}$, $n \geq 1$, and $s_0 = 0$. Hence $s_n = n$ if $a = 1$ and $s_n = (a^n - 1)/(a - 1)$ if $a \neq 1$.

Theorem 1. 1. Any limit distribution function G in (2) satisfying (5) has the following representation: for some $a > 0$ and all $0 \leq x \leq 1$, $n = 0, \pm 1, \pm 2, \dots$

$$-\ln(-\ln G(a^n x + s_n)) = nd + y(x), \quad (6)$$

and the function $y: [0, 1] \rightarrow [0, d]$ is nondecreasing, right-continuous and satisfies $y(x) > 0$ for $x > 0$.

2. Any distribution function with the stated representation can occur as a limit distribution in (2).

Remark 1. Relation (6) has two aspects:

(i) the graph of the function $-\ln(-\ln G)$ goes through the points (s_n, nd) for $n = 0, \pm 1, \pm 2, \dots$;

(ii) on the interval between any of the points $\{s_n\}$ the shape of the function $-\ln(-\ln G)$ is the same, modulo a scale constant.

A somewhat simpler relation holds for the inverse function of $-\ln(-\ln G)$.

Proposition 1. Any limit distribution function G in (2) satisfying (5) has the following representation: for some $a > 0$ and all $0 \leq x \leq d$, $n = 0, \pm 1, \pm 2, \dots$

$$\left(-\ln(-\ln G)\right)^{\leftarrow}(nd+x) = s_n + a^n w(x), \quad (7)$$

and the function $w: [0, d] \rightarrow [0, 1]$ is nondecreasing, left-continuous and continuous at $x = 0$.

Next we discuss the domains of attraction.

Theorem 2. Let G be a limit law in (2) characterized by a, d and y via Theorem 1. In order that (2) holds for some distribution function F and sequence $\{k_n\}$ satisfying (1) it is necessary and sufficient that there exists a sequence $\{b_n\}$ (reflecting the discrete behaviour, cf. Remark 1(i)) with

$$\lim_{n \rightarrow +\infty} \frac{b_{n+1} - b_n}{b_n - b_{n-1}} = a \quad (8)$$

such that

$$\lim_{n \rightarrow +\infty} u(b_{n+1}) - u(b_n) = d \quad (9)$$

and moreover (reflecting the repetitiveness of the function between the points $\{s_n\}$, cf. Remark 1(ii))

$$\lim_{n \rightarrow +\infty} u(b_n + x(b_{n+1} - b_n)) - u(b_n) = y(x) \quad (10)$$

for all continuity point x of y in $[0, 1]$. Here a and the function y are as in Theorem 1. Then (2) holds with the b_n given, $a_n := b_{n+1} - b_n$ and $k_n := [e^{u(b_n)}]$.

Remark 2. For a fixed sequence $\{k_n\}$ satisfying (1) the domain of attraction is characterized by: for $b_n := v(\ln k_n)$ we should have (8), (9) and (10).

Again the conditions for v are somewhat more natural:

Proposition 2. In order to have (2) for a sequence $\{k_n\}$ satisfying (1) and G satisfying (7) it is necessary and sufficient that

$$\lim_{n \rightarrow +\infty} \frac{v(\ln k_{n+1}) - v(\ln k_n)}{v(\ln k_n) - v(\ln k_{n-1})} = a \quad (11)$$

and

$$\lim_{n \rightarrow +\infty} \frac{v(x + \ln k_n) - v(\ln k_n)}{v(\ln k_{n+1}) - v(\ln k_n)} = w(x), \quad (12)$$

for continuity points x of w in $[0, d]$. Here a and the function w are as in Proposition 1.

Remark 3. If we do not require $w(0) = 0$ and $w(d) = 1$ the limit in (12) should be $(w(x) - w(0))/(w(d) - w(0))$.

3. Proofs and examples. It is easily seen the statements of Theorems 1 and 2 and the corresponding propositions are equivalent. So it is sufficient to prove the statements of the propositions.

Proof of Proposition 1. Relations (1) and (2) imply

$$\lim_{n \rightarrow +\infty} F^{k_{n+1}}(a_n x + b_n) = G^{e^d}(x). \quad (13)$$

Khintchine's Convergence of Types Theorem then tells us that there exist a positive and b real such that

$$G^{e^d}(x) = G\left(\frac{x-b}{a}\right) \quad (14)$$

for all x , and

$$\lim_{n \rightarrow +\infty} \frac{a_{n+1}}{a_n} = a, \quad \lim_{n \rightarrow +\infty} \frac{b_{n+1} - b_n}{a_n} = b. \quad (15)$$

Equivalently we have

$$\Psi(x+d) = a\Psi(x) + b \quad (16)$$

for all x . Hence

$$\Psi(x+d) - \Psi(d) = a(\Psi(x) - \Psi(0)). \quad (17)$$

By writing $\Psi(nd+x) - \Psi(nd) = \Psi(nd+x) - \Psi(d) - (\Psi(nd) - \Psi(d))$ and applying (17) repeatedly, we find that for $n = \pm 1, \pm 2, \dots$ and all x

$$\Psi(nd+x) - \Psi(nd) = a^n(\Psi(x) - \Psi(0)). \quad (18)$$

By applying (17) with $x = d$ we get $\Psi(2d) = \Psi(0) + s_2(\Psi(d) - \Psi(0))$ and by iteration

$$\Psi(nd) = \Psi(0) + s_n(\Psi(d) - \Psi(0)). \quad (19)$$

Combination of (18) and (19) gives the statement of the proposition, since Ψ satisfies $\Psi(0) = 0$ and $\Psi(d) = 1$.

Proof of Proposition 2. The weak convergence of the sequence of monotone functions to the left of (13) is equivalent to the weak convergence of the sequence of inverse functions. Hence (13) is equivalent to

$$\lim_{n \rightarrow +\infty} \frac{v(x + \ln k_n) - b_n}{a_n} = \Psi(x), \quad (20)$$

for continuity points of Ψ . Since Ψ is continuous at $x = 0$ and at $x = d$ while $\Psi(0) = 0$ and $\Psi(d) = 1$, relation (20) holds with $b_n := v(\ln k_n)$ and $a_n := v(\ln k_{n+1}) - v(\ln k_n)$. Relation (15) implies (11). The rest is obvious.

Example 1. Suppose $a = 1$ and $d > 0$. Consider $w(x) = 0$ for $0 \leq x \leq d/2$ and $w(x) = 1$ for $d/2 < x \leq d$. The geometric distribution belongs to the domain of attraction of G .

Example 2. Consider $w(x) = x/d$ with $d > 0$. In this case G is a continuous distribution function.

Example 3. Consider $w(x) = (e^{\gamma x} - 1)/(e^{\gamma d} - 1)$ for real γ , $\gamma \neq 0$ and $w(x) = x$ for $\gamma = 0$. Then the representation holds with $a = e^{\gamma d}$. The result is an extreme value distribution, the only one through the points $((a^n - 1)/(a - 1), nd)$.

Example 4. R. von Mises [3] gave the following example of a distribution function that is not in the domain of attraction of any extreme-value distribution: $F(x) = 1 - \exp(-x - \frac{1}{2} \sin x)$. However, relation (2) holds with, for instance, $k_n = [e^{2n\pi}]$, $a_n = 2\pi$ and $b_n = 2\pi n$. In the representation of the limit distribution function G we have $a = 1$, $d = 2\pi$ and $y(x) = 2\pi x + \frac{1}{2} \sin(2\pi x)$.

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