Math-Net.Ru

Общероссийский математический портал

P. F. Koehl, Pham Huyên, N. Touzi, On super-replication in discrete time under transaction costs, *Теория вероятн. и ее примен.*, 2000, том 45, выпуск 4, 783–788

DOI: <https://doi.org/10.4213/tvp511>

Использование Общероссийского математического портала Math-Net.Ru подразумевает, что вы прочитали и согласны с пользовательским соглашением
<http://www.mathnet.ru/rus/agreement>

Параметры загрузки:

IP: 18.234.156.22

26 апреля 2023 г., 12:48:54

© 2000 г.

KOEHL P.F.*, PHAM H.***, TOUZI N.***

ON SUPER-REPLICATION IN DISCRETE TIME
UNDER TRANSACTION COSTS¹⁾

В общей модели с дискретным временем и пропорциональными операционными издержками выводится двойственное представление для цены репликации, т.е. минимального начального капитала, необходимого для хеджирования функции выплаты в рамках самофинансируемой стратегии. В близкой постановке такое представление известно из работ [3], [7], [1].

Ключевые слова и фразы: операционные издержки, цена репликации, двойственное представление.

1. Introduction and main result. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, T a finite horizon and $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ a family of increasing sub σ -algebras. $\mathcal{F}_t$ represents the information available at time t . We assume as usual that $\mathcal{F}_0$ is trivial and $\mathcal{F}_T = \mathcal{F}$. We fix throughout $p \in [1, \infty)$ and we denote as usual by L^p the set of all random variables X with $\mathbb{E}[|X|^p] < \infty$ and by $\|\cdot\|_p$ the associated norm. We denote by q the conjugate of p (i.e., $1/p + 1/q = 1$); as usual, $q = \infty$ whenever $p = 1$.

The financial market consists in one riskless asset whose price is assumed to be equal to one (without loss of generality) and a risky asset with positive $\mathbb{F}$ -adapted price process $S = \{S_t, t = 0, \dots, T\}$ with $S_t \in L^p$ for all $t = 0, \dots, T$.

The risky asset is subject to proportional transaction costs. Let us introduce the sublinear function defined on $\mathbf{R}$ by:

$$\tau(z) = \begin{cases} (1 + \lambda)z & \text{if } z > 0, \\ (1 - \mu)z & \text{if } z \leq 0, \end{cases}$$

where $(\lambda, \mu) \in (0, \infty) \times (0, 1)$ are, respectively, the transaction cost rates for purchasing and selling. This means that the algebraic cost (respectively, gain) in units of nonrisky asset (cash) induced by a position of z units of risky asset at time t is $\tau(z)S_t$ (respectively, $-\tau(-z)S_t$).

R e m a r k. A trivial but useful property of the function τ is:

$$-\tau(-z) \leq rz \leq \tau(z) \text{ for any } (r, z) \in [1 - \mu, 1 + \lambda] \times \mathbf{R}.$$

A *trading strategy* is a pair $(\eta, \theta) = \{(\eta_t, \theta_t), t = 0, \dots, T-1\}$ of $\mathbb{F}$ -adapted processes. Here, η_t and θ_t are interpreted respectively, as the number of shares of nonrisky and risky asset held at time t .

A trading strategy (η, θ) is said to be *self-financing* if:

$$\eta_t - \eta_{t-1} + \tau(\theta_t - \theta_{t-1})S_t \leq 0, \quad t = 1, \dots, T-1, \quad P\text{-a.s.} \quad (1)$$

which means that, at every trading date, the cost induced by rebalancing the strategy is nonpositive; in other words, sales must finance purchases. We denote by $\mathcal{A}$ the set of all self-financing trading strategies.

* CDC, DABF, Paris; France; e-mail: koehl@laplace.ensae.fr

** (Corresponding author) Laboratoire de Probabilités et Modèles Aléatoires, UFR Mathématiques, Case 7012, Université Paris VII, 2 Place Jussieu, 75251 Paris, Cedex 05, France; e-mail: pham@math.univ-mlv.fr

*** CERMSEM, Université Paris I, 106-112, Blvd. Hôpital, 75013 Paris, France; e-mail: touzi@ceremade.dauphine.fr

¹⁾ This note is based on a previous preprint by the same authors entitled «Option pricing under transaction costs: a martingale approach», CREST Working Paper 9647.

A *contingent claim* is a pair $C = (C^0, C^1)$ of $\mathcal{F}_T$ -measurable random variables, with $(C^0, C^1 S_T) \in L^p \times L^p$, representing a time T portfolio consisting of C^0 units of the nonrisky asset S^0 and C^1 units of the risky one S . We denote by $\mathcal{L}_S^p$ the set of all contingent claims. This set is naturally endowed with the norm $\|C\|_{\mathcal{L}_S^p} = \|C^0\|_p + \|C^1 S_T\|_p$.

We say that a trading strategy (η, θ) dominates some contingent claim C if:

$$\eta_{T-1} - C^0 \geq \tau(C^1 - \theta_{T-1}) S_T \quad P\text{-a.s.}$$

In this note, we are interested in a dual representation of the super-replication cost of a given contingent claim $C \in \mathcal{L}_S^p$, given an initial holding in the risky asset θ_{-1} , defined by:

$$\hat{\pi}(C, \theta_{-1}) = \inf \left\{ \eta_0 + \tau(\theta_0 - \theta_{-1}) S_0 : (\eta, \theta) \in \mathcal{A} \text{ and dominates } C \right\}.$$

We refer to [6] and [2] for a practical use of the dual representation.

Let us introduce the set $\mathcal{P}(\lambda, \mu)$ consisting of all pairs (Q, ρ) satisfying:

(P1) Q is a probability measure equivalent to P with density $dQ/dP \in L^q$;

(P2) $\rho = \{\rho_t, t = 0, \dots, T\}$ is an $\mathbb{F}$ -adapted process with values in the interval $[1 - \mu, 1 + \lambda]$;

(P3) the process $\rho S = \{\rho_t S_t, t = 0, \dots, T\}$ is a martingale under Q .

We then have the following theorem.

Theorem. Suppose that $\mathcal{P}(\lambda - \varepsilon, \mu - \varepsilon) \neq \emptyset$ for some $0 < \varepsilon \leq \min\{\lambda, \mu\}$. Then for any $(C, \theta_{-1}) \in \mathcal{L}_S^p \times \mathbb{R}$, the super-replication cost of C given the initial holding θ_{-1} is given by

$$\hat{\pi}(C, \theta_{-1}) = \sup_{(Q, \rho) \in \mathcal{P}(\lambda, \mu)} \mathbf{E}^Q \left[C^0 + \rho_T (C^1 - \theta_{-1}) S_T \right].$$

The set $\mathcal{P}(\lambda, \mu)$ involved in the Theorem plays the same role in our model than the set of equivalent martingale measures in a financial market. It is proved in [3] that the condition $\mathcal{P}(\lambda, \mu) \neq \emptyset$ is an equivalent characterization of the no free lunch condition which is stronger than the usual notion of no arbitrage. Therefore our condition $\mathcal{P}(\lambda - \varepsilon, \mu - \varepsilon) \neq \emptyset$ is stronger than the no free lunch condition in our financial market and can be interpreted as a no free lunch condition in a financial market with smaller transaction costs.

The dual representation of the Theorem has been first established by Jouini and Kallal [3] under the weaker assumption $\mathcal{P}(\lambda, \mu) \neq \emptyset$ (i.e., no arbitrage) but with a different, and somewhat unnatural, definition of the super-replication cost²⁾. In the case of a finite probability space, Kusuoka [7] proved the dual representation of the Theorem under the assumption $\mathcal{P}(\lambda, \mu) \neq \emptyset$. In the following proposition, we shall prove that Kusuoka's assumption is not weaker than ours under a nondegeneracy condition on the process $\{S_t, t = 0, \dots, T\}$.

Proposition. Let Ω be finite and the conditional variance $\mathbf{D}[S_t | \mathcal{F}_{t-1}] \neq 0$ for all $t = 1, \dots, T$. Suppose that $\mathcal{P}(\lambda, \mu) \neq \emptyset$. Then there exists ε , with $0 < \varepsilon \leq \min\{\lambda, \mu\}$ such that $\mathcal{P}(\lambda - \varepsilon, \mu - \varepsilon) \neq \emptyset$.

The proof of the Proposition is given in Appendix. Notice also that the dual representation of the Theorem has been established by Cvitanic and Karatzas [1] in a continuous time framework with diffusion price process. In the special case, where our price process S is a discrete record from a continuous time diffusion process as in [1], we clearly have $\mathcal{P}(0, 0) \neq \emptyset$ which is a stronger requirement than our condition. Let us mention that while we were finishing the paper, we learned that Kabanov [5] extended the results of Cvitanic and Karatzas [1] to the general semi-martingale case which includes our discrete-time framework. But his dual representation formula is established under the same (stronger) assumption than in [1].

²⁾ Namely, the lower-semicontinuous envelope of the $\hat{\pi}(C, \theta_{-1})$ with respect to the C variable.

2. Proof of the Theorem. In order to establish the required result, we provide some useful lemmas.

Lemma 1. Let $(Q, \rho) \in \mathcal{P}(\lambda, \mu)$. Then, for any self-financing trading strategy $(\eta, \theta) \in \mathcal{A}$, the process $\{\eta_t + \rho_t \theta_t S_t, t = 0, \dots, T-1\}$ is a supermartingale under Q .

P r o o f. The proof is similar to that of Proposition 3.1 in [6].

Next define the set:

$$K = \{(U, V) \in \mathcal{L}_S^p: \exists (\eta, \theta) \in \mathcal{A}, \eta_0 + \tau(\theta_0) S_0 \leq 0 \text{ and } (\eta, \theta) \text{ dominates } (U, V)\}.$$

The elements of K are interpreted as contingent claims which can be dominated by some self-financing strategy with nonpositive initial cost.

Lemma 2. Under the conditions of the Theorem the set K is closed in $\mathcal{L}_S^p$.

P r o o f. Consider a sequence $(U^n, V^n)_n$ of elements of K converging to (U, V) in $\mathcal{L}_S^p$. Then it converges to (U, V) almost surely (possibly along a subsequence). Moreover, there exists a sequence $(\hat{\eta}^n, \hat{\theta}^n)_n \in \mathcal{A}^N$, i.e.,

$$-\delta_t^n := \hat{\eta}_t^n - \hat{\eta}_{t-1}^n + \tau(\hat{\theta}_t^n - \hat{\theta}_{t-1}^n) S_t \leq 0 \quad P\text{-a.s.} \quad (2)$$

with nonpositive initial cost, i.e.,

$$-\delta_0^n := \hat{\eta}_0^n + \tau(\hat{\theta}_0^n) S_0 \leq 0 \quad (3)$$

and dominating (U^n, V^n) , i.e.,

$$\hat{\eta}_{T-1}^n - U^n - \tau(V^n - \hat{\theta}_{T-1}^n) S_T \geq 0 \quad P\text{-a.s.} \quad (4)$$

First notice that the strategy (η^n, θ^n) defined by $\theta^n = \hat{\theta}^n$ and $\eta_t^n = \hat{\eta}_t^n + \sum_{i=0}^t \delta_i^n$, $t = 0, \dots, T-1$, is a self-financing trading strategy with:

$$\eta_0^n + \tau(\theta_0^n) S_0 = 0, \quad (5)$$

$$\eta_t^n - \eta_{t-1}^n + \tau(\theta_t^n - \theta_{t-1}^n) S_t = 0 \quad P\text{-a.s.} \quad (6)$$

which also dominates (U^n, V^n) , i.e.,

$$\eta_{T-1}^n - U^n - \tau(V^n - \theta_{T-1}^n) S_T \geq 0 \quad P\text{-a.s.} \quad (7)$$

Now, let $(Q^*, \rho^*) \in \mathcal{P}(\lambda - \varepsilon, \mu - \varepsilon)$. Then, the convergence of (U^n, V^n) towards (U, V) in $\mathcal{L}_S^p$ implies that: $U^n \rightarrow U$ and $V^n S_T \rightarrow V S_T$ in $L^1(Q^*)$, by the Hölder inequality. In order to prove the required result, we intend to show the existence of a self-financing strategy $(\eta, \theta) \in \mathcal{A}$ with nonpositive initial cost and dominating (U, V) .

In the sequel, we denote by $\mathcal{H}^1(Q^*)$ the set of $\mathbb{F}$ -adapted processes $\{Y_t, t = 0, \dots, T-1\}$ such that $Y_t \in L^1(Q^*)$ for any t . The linear space $\mathcal{H}^1(Q^*)$ is naturally endowed with the product norm.

1st step. We first prove that $\{(\eta_t^n, \theta_t^n S_t), t = 0, \dots, T-1\}$ is bounded in $\mathcal{H}^1(Q^*) \times \mathcal{H}^1(Q^*)$ uniformly in n .³⁾ Define:

$$X_t^n = \mathbb{E}_t^{Q^*} [U^n - \tau(-V^n) S_T] \quad (8)$$

which is bounded in $L^1(Q^*)$ from the convergence of $(U^n)_n$ and $(V^n S_T)_n$ in $L^1(Q^*)$. We also introduce:

$$I_t^n = \sum_{u=0}^{t-1} \theta_u^n (S_{u+1} - S_u).$$

By the self-financing condition (6), we have:

$$\begin{aligned} \eta_t^n &= \eta_0^n - \sum_{u=0}^{t-1} \tau(\theta_{u+1}^n - \theta_u^n) S_{u+1} \leq \eta_0^n + r\theta_0^n S_0 - r\theta_t^n S_t + rI_t^n \\ &\leq -r\theta_t^n S_t + rI_t^n, \quad \forall r \in [1 - \mu, 1 + \lambda], \end{aligned} \quad (9)$$

³⁾ This part of the proof is inspired from Soner, Shreve and Cvitanic [8] and Cvitanic and Karatzas [1] who deal with a continuous time model with Brownian filtration.

where we used the Remark and (5). From the sublinearity of τ , (7) implies:

$$\eta_{T-1}^n - \tau(-\theta_{T-1}^n) S_T \geq U^n - \tau(-V^n) S_T \quad P\text{-a.s.}$$

Now, from the Remark and the Lemma applied to the pair $(Q^*, \rho^*) \in \mathcal{P}(\lambda - \varepsilon, \mu - \varepsilon)$, we get:

$$\eta_t^n + \rho_t^* \theta_t^n S_t \geq X_t^n \quad P\text{-a.s.} \quad (10)$$

which implies, by plugging (9) (with $r = 1 + \lambda$ and $r = 1 - \mu$) in (10):

$$\begin{aligned} \theta_t^n S_t &\leq \frac{1 + \lambda}{1 + \lambda - \rho_t^*} I_t^n - \frac{1}{1 + \lambda - \rho_t^*} X_t^n, \\ \theta_t^n S_t &\geq \frac{\mu - 1}{\rho_t^* - (1 - \mu)} I_t^n + \frac{1}{\rho_t^* - (1 - \mu)} X_t^n, \end{aligned}$$

and therefore, since $\rho_t^* \in [1 - \mu + \varepsilon; 1 + \lambda - \varepsilon]$: $|\theta_t^n S_t| \leq \varepsilon^{-1}((2 + \lambda)|I_t^n| + 2|X_t^n|)$. Thus, in order to show that $(\theta_t^n S_t)_n$ is bounded in $L^1(Q^*)$ for all $t = 0, \dots, T - 1$, it remains to prove that I_t^n is bounded in $L^1(Q^*)$. To see this, notice that $I_t^n = I_{t-1}^n + \theta_{t-1}^n(S_t - S_{t-1})$ and therefore: $\mathbf{E}^{Q^*}|I_t^n| \leq \mathbf{E}^{Q^*}|I_{t-1}^n| + \mathbf{E}^{Q^*}|\theta_{t-1}^n S_{t-1}| + \mathbf{E}^{Q^*}|\theta_{t-1}^n S_t|$. Moreover, $\mathbf{E}^{Q^*}|\theta_{t-1}^n S_t| \leq \mathbf{E}^{Q^*}|\theta_{t-1}^n \rho_t^* S_t|/(1 - \mu)$, and using the martingale property of $\rho^* S$ under Q^* , we get

$$\mathbf{E}^{Q^*}|\theta_{t-1}^n S_t| \leq \frac{1}{1 - \mu} \mathbf{E}^{Q^*}|\theta_{t-1}^n \rho_{t-1}^* S_{t-1}| \leq \frac{1 + \lambda}{1 - \mu} \mathbf{E}^{Q^*}|\theta_{t-1}^n S_{t-1}|, \quad (11)$$

where we used again the martingale property of $\rho^* S$ under Q^* . This provides:

$$\begin{aligned} \mathbf{E}^{Q^*}|I_t^n| &\leq \mathbf{E}^{Q^*}|I_{t-1}^n| + \left(1 + \frac{1 + \lambda}{1 - \mu}\right) \mathbf{E}^{Q^*}|\theta_{t-1}^n S_{t-1}| \\ &\leq \mathbf{E}^{Q^*}|I_{t-1}^n| + \frac{2 + \lambda}{(1 - \mu)\varepsilon} \left\{ (2 + \lambda) \mathbf{E}^{Q^*}|I_{t-1}^n| + 2 \mathbf{E}^{Q^*}|X_{t-1}^n| \right\}, \end{aligned}$$

and therefore the boundedness of I_t^n in $L^1(Q^*)$ follows from a simple induction argument (recall that $I_0^n = 0$).

Finally, since the sequence $(\theta_t^n S_t)_n$ is bounded in $L^1(Q^*)$ for any $t = 0, \dots, T - 1$, we deduce immediately from (5), (6) and (11) that the sequence $(\eta_t^n)_n$ is also bounded in $L^1(Q^*)$ for any $t = 0, \dots, T - 1$.

2 n d s t e p. From the boundedness of the processes $(\eta^n)_n$ and $(\theta^n S)_n$ in $\mathcal{H}^1(Q^*)$, we deduce by the Komlós theorem (see [4, Theorem 7.3, p. 205])⁴⁾ the existence of two $\mathbb{F}$ -adapted processes η and θ and subsequences (η^{n_k}) , (θ^{n_k}) such that: $\tilde{\eta}^n \rightarrow \eta$ and $\tilde{\theta}^n \rightarrow \theta$ P -a.s., where $\tilde{\eta}^n = (1/n) \sum_{k=1}^n \eta^{n_k}$ and $\tilde{\theta}^n = (1/n) \sum_{k=1}^n \theta^{n_k}$. By the convexity of the function τ and sending n to infinity, equations (5) and (6) imply

$$\eta_t - \eta_{t-1} + \tau(\theta_t - \theta_{t-1}) S_t \leq 0 \quad P\text{-a.s.}$$

and $\eta_0 + \tau(\theta_0) S_0 \leq 0$. Now, define $\tilde{U}^n = (1/n) \sum_{k=1}^n U^{n_k}$ and $\tilde{V}^n = (1/n) \sum_{k=1}^n V^{n_k}$. By Césaro's theorem, $(\tilde{U}^n, \tilde{V}^n)$ converge almost surely to (U, V) . By the convexity of τ and sending n to infinity, the domination relation (7) imply $\eta_{T-1} - U - \tau(V - \theta_{T-1}) S_T \geq 0$ P -a.s. which proves that $(U, V) \in K$. Lemma 2 is proved.

For the proof of the Theorem, we need to introduce the set $\mathcal{L}_{1/S}^q$ consisting of all $\mathcal{F}_T$ -measurable random variables $Z = (Z^0, Z^1)$ such that $Z^0 \in L^q$ and $Z^1/S_T \in L^q$. $\mathcal{L}_{1/S}^q$ is the dual space of $\mathcal{L}_S^p$ when endowed with the norm $\|Z\|_{\mathcal{L}_{1/S}^q} = \|Z^0\|_q + \|Z^1/S_T\|_q$.

P r o o f o f t h e T h e o r e m. Since $\mathcal{P}(\lambda - \varepsilon, \mu - \varepsilon) \neq \emptyset$, it is clear that we also have $\mathcal{P}(\lambda, \mu) \neq \emptyset$. The proof of the inequality

$$\hat{\pi}(C, \theta_{-1}) \geq \sup_{(Q, \rho) \in \mathcal{P}(\lambda, \mu)} \mathbf{E}^Q \left[C^0 + \rho_T (C^1 - \theta_{-1}) S_T \right]$$

⁴⁾ We are indebted to Yu. Kabanov for mentioning us the Komlós theorem.

is similar to that of Proposition 3.2 in [6]. Therefore, we only prove the converse inequality. Take an arbitrary $b < \hat{\pi}(C, \theta_{-1})$ and consider the convex cone K as well as the singleton $A = \{(C^0 - b, C^1 - \theta_{-1})\}$. From Lemma 2, K is closed and by the definition of $\hat{\pi}(C, \theta_{-1})$ it is easily seen that $K \cap A = \emptyset$. Therefore by the Hahn-Banach Separation Theorem and the Riesz Representation Theorem, there exists a pair $(Z^0, Z^1) \in \mathcal{L}_{1/S}^q \setminus \{0\}$ such that:

$$\mathbf{E}[Z^0 U + Z^1 V] \leq 0 < \mathbf{E}[Z^0(C^0 - b) + Z^1(C^1 - \theta_{-1})] \quad \forall (U, V) \in K. \quad (12)$$

We first claim that $Z^0 \geq 0$ P-a.s. To see this consider the pair $(-1_{\{Z^0 < 0\}}, 0)$ which lies in K , since it is dominated by the null strategy, and apply the left-hand-side inequality of (12). Next, consider an arbitrary element $(\hat{Q}, \hat{p}) \in \mathcal{P}(\lambda, \mu)$ and define $\hat{Z}^0 := d\hat{Q}/dP$, $\hat{Z}^1 := \hat{Z}^0 \hat{p}_T S_T$ and the sequences

$$Z^{n,0} = \frac{1}{n} \hat{Z}^0 + \left(1 - \frac{1}{n}\right) Z^0, \quad Z^{n,1} = \frac{1}{n} \hat{Z}^1 + \left(1 - \frac{1}{n}\right) Z^1.$$

Finally, we define the pair (Q^n, ρ^n) by:

$$\frac{dQ^n}{dP} = Z^{n,0} \quad \text{and} \quad \rho_t^n = \frac{\mathbf{E}_t(Z^{n,1})}{S_t \mathbf{E}_t(Z^{n,0})}, \quad t = 0, \dots, T.$$

We now use the following lemma whose proof is carried out later.

Lemma 3. For all $n > 0$, we have $(Q^n, \rho^n) \in \mathcal{P}(\lambda, \mu)$ and

$$\lim_{n \rightarrow \infty} \mathbf{E}^{Q^n} [C^0 + \rho_T^n S_T (C^1 - \theta_{-1})] = \mathbf{E} [Z^0 C^0 + Z^1 (C^1 - \theta_{-1})].$$

Now, by the right-hand-side inequality of (12), we have: $b < \mathbf{E}^Q [C^0 + \rho_T (C^1 - \theta_{-1}) S_T]$. Using the limit result of Lemma 3, this proves that, for some sufficiently large n , we have

$$b \leq \mathbf{E}^{Q^n} [C^0 + \rho_T^n (C^1 - \theta_{-1}) S_T] \leq \sup_{(Q, \rho) \in \mathcal{P}(\lambda, \mu)} \mathbf{E}^Q [C^0 + \rho_T (C^1 - \theta_{-1}) S_T];$$

the required result follows from the arbitrariness of $b < \hat{\pi}(C, \theta_{-1})$.

Proof of Lemma 3. (i) We first prove that $(Q^n, \rho^n) \in \mathcal{P}(\lambda, \mu)$. Since $\hat{Z}^0 > 0$ and $Z^0 \geq 0$ P-a.s., the measure Q^n satisfies condition (P1). It is easily checked that (Q^n, ρ^n) satisfies (P3) by definition. It remains to prove that (Q^n, ρ^n) satisfies (P2). Fix an arbitrary nonnegative bounded $\mathcal{F}_t$ -measurable random variable ξ and consider the self-financing strategy:

$$\begin{aligned} \eta_s &= \theta_s = 0 \quad \text{for } s = 0, \dots, t-1, \\ \eta_s &= -(1+\lambda)\xi S_t \quad \text{and } \theta_s = \xi \text{ for } s = t, \dots, T-1. \end{aligned}$$

Then it is clear that $(\eta_{T-1}, \theta_{T-1}) \in K$ (recall that $S_t \in L^p$) and, therefore, applying the left-hand-side inequality of (12) we get $\mathbf{E}[Z^1 \xi] \leq \mathbf{E}[Z^0(1+\lambda)\xi S_t]$. By a similar argument, we get $\mathbf{E}[Z^1 \xi] \geq \mathbf{E}[Z^0(1-\mu)\xi S_t]$. By the arbitrariness of ξ , this provides: $(1-\mu) S_t \mathbf{E}_t[Z^0] \leq \mathbf{E}_t[Z^1] \leq (1+\lambda) S_t \mathbf{E}_t[Z^0]$ for all $t = 0, \dots, T-1$. Considering the contingent claims $(-(1+\lambda)\xi S_T, \xi) \in K$ and $((1-\mu)\xi S_T, -\xi) \in K$ for arbitrary $\mathcal{F}_T$ -measurable bounded and nonnegative ξ (since they are dominated by the null strategy), we see that the last inequalities hold also for T . Now, since $(\hat{Q}, \hat{p})$ satisfies (P2), it is easily checked that $(\hat{Z}^0, \hat{Z}^1)$ also satisfies the last inequalities and therefore: $(1-\mu) S_t \mathbf{E}_t[Z^{n,0}] \leq \mathbf{E}_t[Z^{n,1}] \leq (1+\lambda) S_t \mathbf{E}_t[Z^{n,0}]$, $t = 0, \dots, T$, which proves that ρ^n satisfies (P2).

(ii) To prove the limit result of the lemma, it suffices to notice that

$$\begin{aligned} & \mathbf{E}^{Q^n} [C^0 + \rho_T^n S_T (C^1 - \theta_{-1})] - \mathbf{E} [Z^0 C^0 + Z^1 (C^1 - \theta_{-1})] \\ &= \frac{1}{n} \mathbf{E} [(\hat{Z}^0 - Z^0) C^0 + (\hat{Z}^1 - Z^1) (C^1 - \theta_{-1})]. \end{aligned}$$

3. Appendix: proof of the Proposition. Let Ω be finite and assume that there exists $(Q, \rho) \in \mathcal{P}(\lambda, \mu)$. We intend to prove that there exists ε , with $0 < \varepsilon \leq \min\{\lambda, \mu\}$ such that $\mathcal{P}(\lambda - \varepsilon, \mu - \varepsilon) \neq \emptyset$.

1st step. Fix some t in $\{0, \dots, T-1\}$ and let σ_t be any atom of $\mathcal{F}_t$; to simplify notation, we denote $(\rho^0, S^0) = (\rho_t(\sigma_t), S_t(\sigma_t))$ and $(\rho^1, S^1), \dots, (\rho^n, S^n)$ the values of those processes ρ and S at successors of σ_t . We assume without loss of generality that $S^1 \geq \dots \geq S^n > 0$. We will also denote by $q^1, \dots, q^n$ the transition probabilities between σ_t and its successors induced by the probability measure Q . By definition of set $\mathcal{P}(\lambda, \mu)$, we have:

$$\rho^0 S^0 = \sum_{i=1}^n q^i \rho^i S^i, \quad 1 - \mu \leq \rho^i \leq 1 + \lambda \quad \text{and} \quad q^i > 0 \text{ for all } i = 0, \dots, n. \quad (13)$$

(i) We first prove that it is possible to transform $(\rho^0, \dots, \rho^n) \in [1 - \mu; 1 + \lambda]^{n+1}$ into $(\tilde{\rho}^0, \dots, \tilde{\rho}^n) \in [1 - \mu + \varepsilon; 1 + \lambda - \varepsilon] \times [1 - \mu; 1 + \lambda]^n$, for some $\varepsilon > 0$, in such a way that equation (13) remains valid. To see this, notice that there are two cases.

1) Suppose that there exist i_1 and i_2 in $\{1, \dots, n\}$ such that $\rho^{i_1} S^{i_1} \neq \rho^{i_2} S^{i_2}$, then the result is immediate by a slight modification of q^{i_1} and q^{i_2} ; recall that q^i lies in the open interval $(0, 1)$.

2) Otherwise, from the nondegeneracy condition $D[S_{t+1} | \mathcal{F}_t] > 0$ P-a.s., we must have $\rho^1 < \rho^n$. This implies that $\rho^1 < 1 + \lambda$ and $\rho^n > 1 - \mu$. It is then possible to increase (respectively, decrease) ρ^0 by increasing ρ^1 (respectively, decreasing ρ^n).

(ii) Next, we prove that it is possible to transform $(\rho^1, \dots, \rho^n) \in [1 - \mu; 1 + \lambda]^n$ into $(\tilde{\rho}^1, \dots, \tilde{\rho}^n) \in [1 - \mu + \varepsilon; 1 + \lambda - \varepsilon]^n$, keeping fixed ρ^0 , for some $\varepsilon > 0$, in such a way that equation (13) remains valid. To see this, notice that there are again two cases.

1) Suppose that $\rho^i = 1 + \lambda$ for all $i = 1, \dots, n$. Then it is sufficient to increase q^1 and to decrease q^n . The case $\rho^i = 1 - \mu$ for all $i = 1, \dots, n$ is solved by a similar argument.

2) Otherwise, there exist i_1 and i_2 such that $\rho^{i_1} > \rho^{i_2}$. Then we can decrease ρ^{i_1} and increase ρ^{i_2} . Therefore, for any $\rho^i \in \{1 - \mu; 1 + \lambda\}$, it is possible to change its value by changing either ρ^{i_1} or ρ^{i_2} , keeping fixed ρ^0 .

2nd step. We now argue by forward induction. At time $t = 0$, we apply the transformation described in (i) of the first step, and then the second transformation as in (ii) of the first step. Next, for all $t = 1, \dots, T-1$, apply transformation (ii) of the first step to any atom. Notice that each transformation defines some $\varepsilon_i > 0$. Since Ω is finite, the number of transformations is necessarily finite. Therefore, the required result is obtained for the minimum value of all ε_i .

REFERENCES

1. Cvitanić J., Karatzas I. Hedging and portfolio optimization under transaction costs: A martingale approach. — *Math. Finance*, 1996, v. 6, p. 133–166.
2. Cvitanić J., Pham H., Touzi N. A closed-form solution to the problem of super-replication under transaction costs. — *Finance and Stochastics*, 1999, v. 3, № 1, p. 35–54.
3. Jouini E., Kallal H. Martingales and arbitrage in securities markets with transaction costs. — *J. Econ. Theory*, 1995, v. 66, p. 178–197.
4. Hall P., Heyde C. C. *Martingale Limit Theory and its Application*. New York: Academic Press, 1980, 308 p.
5. Kabanov Yu. M. Hedging and liquidation under transaction costs in currency markets. — *Finance and Stochastics*, 1999, v. 3, № 2, p. 237–248.
6. Koehl P.-F., Pham H., Touzi N. Hedging in discrete-time under transaction costs and continuous-time limit. — *J. Appl. Probab.*, 1999, v. 36, p. 163–178.
7. Kusuoka S. Limit theorem on option replication with transaction costs. — *Ann. Appl. Probab.*, 1995, v. 5, p. 198–221.
8. Soner H., Shreve S., Cvitanić J. There is no nontrivial hedging portfolio for option pricing with transaction costs. — *Ann. Appl. Probab.*, 1995, v. 5, p. 327–355.

Поступила в редакцию
27.IX.1998