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ON THE MAXIMUM TERM OF MA AND MAX-AR MODELS
WITH MARGINS IN ANDERSON'S CLASS¹⁾

Мы рассматриваем некоторые целочисленные стационарные модели типа MA или max-AR и изучаем предельное распределение максимального члена после соответствующей нормировки. В частности, рассматриваются маргинальные распределения, которые не принадлежат области притяжения какого-либо экстремального max-устойчивого закона, но имеют квазиустойчивое предельное поведение в смысле Андерсона [2] и поэтому принадлежат области притяжения некоторого макс-полуустойчивого закона. Примерами таких распределений являются отрицательно-биномиальное и логарифмическое распределения, которые широко используются для моделирования реальных данных.

Проверяя соответствующие условия зависимости, мы получаем предельное распределение максимального члена для нескольких рассматриваемых моделей. Исследование мотивировано анализом экстремального поведения целочисленных данных, требующим специфического моделирования временных рядов.

Ключевые слова и фразы: целочисленные стационарные последовательности, экстремальный индекс, биномиальное прореживание.

1. Introduction. During the last two decades many efforts have been made in order to model integer-valued variate time series since traditional methods frequently render inadequate. Integer-valued and in particular count data arise in a variety of contexts, and examples include the number of accidents in a manufacturing plant each month, the number of calls in a telephone network noted during a time interval, the number of night-guests

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in hotels. Such data may also arise from the discretization of continuous variate time series as for instance the life span of humans which may be registered in years. For a review of different models for integer-valued data see the recent work by McKenzie [16].

When the extremes of discrete time series are of interest several drawbacks emerge since many integer-valued distributions do not belong to the domain of attraction of any extreme value distribution. Examples include the binomial, negative binomial, and Poisson distributions. Nevertheless, there has been some interest in this topic and several authors have already studied the extremal behavior of some integer-valued time series models. These include a paper by Serfozo [19] which considers queue lengths in $M/G/1$ and $GI/M/1$ systems, two papers by McCormick and Park [14] and [15] which consider AR negative binomial processes and queue lengths of $M/M/s$ systems, a paper by Hall [12] considering a max-AR process, and two other papers by Hall [10] and [8] considering sequences within a generalized class of integer-valued MA models when driven by independent and identically distributed (i.i.d.) heavy tailed or exponential type tailed innovations.

For the purpose of this work we shall consider the following three different classes of count data models.

1. An integer-valued MA model defined in [10]:

$$X_n = \sum_{j=-\infty}^{\infty} \beta_j \circ Z_{n-j},$$

where $\{Z_n\}$ is an i.i.d. sequence of discrete random variables and $\circ$ denotes binomial thinning.

2. A MA model of order q proposed by McKenzie [17]:

$$X_n = \begin{cases} \beta_0 \circ Z_n & \text{with probability } b_0, \\ \beta_0 \circ Z_n + \beta_1 \circ Z_{n-1} & \text{with probability } b_1, \\ \dots & \\ \beta_0 \circ Z_n + \dots + \beta_{q-1} \circ Z_{n-q+1} & \text{with probability } b_{q-1}, \\ \beta_0 \circ Z_n + \dots + \beta_{q-1} \circ Z_{n-q+1} + Z_{n-q} & \text{with probability } b_q, \end{cases}$$

where $\{Z_n\}$ is an integer valued i.i.d. sequence of geometric random variables, $\circ$ denotes binomial thinning, and the mixing coefficients b_i , $i = 0, 1, \dots, q$, are conveniently chosen.

3. A max-AR model

$$X_n = \max\{X_{n-2}, Y_n\} - k,$$

where $\{Y_n\}$ is an i.i.d. sequence of integer-valued random variables and $k \in \mathbb{N}$.

In all cases considered, the marginal distribution of the process and the distribution of the innovations do not belong to the domain of attraction of

any extreme value distribution but are included in a class of distributions considered by Anderson [2] for which the maxima in an i.i.d. setting possesses a quasi-stable limiting behavior. Anderson proved that an integer-valued distribution function F , with infinite right endpoint, satisfies

$$\lim_{n \rightarrow +\infty} \frac{1 - F(n-1)}{1 - F(n)} = r \quad \text{with } r \text{ in }]1, +\infty[, \quad (1)$$

if and only if

$$\begin{cases} \limsup_{n \rightarrow \infty} F^n(x + b_n) \leq \exp(-r^{-x}), \\ \limsup_{n \rightarrow \infty} F^n(x + b_n) \geq \exp(-r^{-(x-1)}) \end{cases}$$

for any real x and b_n conveniently chosen. We shall say that a distribution belongs to Anderson's class if it satisfies (1).

In an attempt to overcome the presence of limiting bounds instead of a well-defined limiting distribution, Temido [20] proved that (1) is necessary and sufficient for the existence of a nondecreasing positive integer sequence $\{k_n\}$ satisfying

$$\lim_{n \rightarrow +\infty} \frac{k_{n+1}}{k_n} = r, \quad r \in]1, +\infty[, \quad (2)$$

and of a real sequence $\{u_n\}$ such that $k_n(1 - F(u_n)) \rightarrow \tau > 0$ as $n \rightarrow \infty$ for some $\tau > 0$. So, if instead of looking at the maximum term of the first n observations we look at the maximum term of the first k_n observations, where k_n satisfies (2), we can obtain a well-defined limiting distribution for the maximum term.

In this paper we are interested in obtaining well-defined limiting distributions for the maximum term of each of the stationary models given above. In order to obtain the desired results we shall use appropriate dependence conditions similar to $D(u_n)$ and $D^{(m)}(u_n)$ defined in [13] and [5], respectively.

The rest of the paper is organized as follows. Section 2 provides a background description of basic theoretical results related to max-stable and max-semistable laws. Section 3 contains results that allow us to study the extremal behavior of dependent data under a max-semistable setup. In the remaining sections we obtain the limiting distribution of the maximum term of the three models described above.

2. Max-semistable laws. The class of max-stable laws, also known as extreme value distributions, arises as the possible limiting distributions for the linearized maxima of n i.i.d. random variables, as $n \rightarrow \infty$. If instead of considering n random variables we consider k_n random variables, where $\{k_n\}$ is a nondecreasing positive integer-valued sequence satisfying

$$\lim_{n \rightarrow +\infty} \frac{k_{n+1}}{k_n} = r, \quad r \in [1, +\infty[, \quad (3)$$

then we obtain a larger class of possible limiting distributions known as the max-semistable (MSS) distributions. The class of MSS distributions was introduced by Pancheva [18]. This class includes nondegenerate limiting distributions for the maxima of i.i.d. random variables with either discrete distributions or multi-modal continuous distributions which are not extreme value distributions.

Following Grinevich [7] we will say that a distribution function G on $\mathbf{R}$ is MSS if there are reals $r > 1$, $\gamma > 0$, and β such that

$$G(x) = G^r(x\gamma^{-1} + \beta), \quad x \in \mathbf{R}, \quad (4)$$

or equivalently, if there are a sequence of i.i.d. random variables with distribution function F and two real sequences $\{a_n > 0\}$ and $\{b_n\}$ for which

$$\lim_{n \rightarrow +\infty} F^{k_n}(xa_n^{-1} + b_n) = G(x), \quad (5)$$

for each continuity point of G , with $\{k_n\}$ satisfying (3).

Analytically, the MSS laws can be written as follows.

Theorem 2.1 (see [7]). 1. For $\gamma > 0$ and $\beta = t(1 - 1/\gamma)$ with $t \in \mathbf{R}$ the solutions of (4) are the following distributions:

$$\begin{aligned} G_1(x) &= \exp\{-(t-x)^\alpha \nu(\ln(t-x))\}, & x \in]-\infty, t[, \\ G_2(x) &= \exp\{-(x-t)^{-\alpha} \nu(\ln(x-t))\}, & x \in]t, +\infty[, \end{aligned}$$

where $\alpha = |\ln r / \log \gamma|$ and the ν are periodic positive and bounded functions with period $p = |\log \gamma|$.

2. For $\gamma = 1$ and $\beta > 0$ the solutions of (4) are of the form

$$G_3(x) = \exp\{e^{-\alpha x} \nu(x)\}, \quad x \in \mathbf{R},$$

where $\alpha = (\ln r)/\beta$, and the ν are periodic positive and bounded functions with period β .

The general definition of the MSS laws is given in [6] and [7]. Canto e Castro, de Haan, and Temido [4] give a characterization of the MSS laws simpler than the one given in [6]. Necessary and sufficient conditions on F such that (4) holds are also given in [4].

If limit (5) holds we shall say that F belongs to the domain of attraction of G .

Considering discrete random variables, Temido [20] proved the following theorem.

Theorem 2.2 (see [20]). For any discrete distribution function F defined over the set $\{x_n\}$, a necessary and sufficient condition for the existence of a nondecreasing positive integer sequence satisfying (3) and of a real sequence $\{u_n\}$ such that $F^{k_n}(u_n)$ has a nondegenerate limit is given by

$$\lim_{n \rightarrow +\infty} \frac{1 - F(x_{n-1})}{1 - F(x_n)} = r. \quad (6)$$

This theorem allows us to prove the previously mentioned result by which any distribution function in Anderson's class, satisfying (1), can only belong to the domain of attraction of a MSS distribution. Furthermore, the limit lies in the discrete Gumbel family, G_3 , as stated in the following theorem.

Theorem 2.3 (see [21]). *Let F be an integer-valued distribution function with infinite right endpoint. If there are sequences $\{k_n\}$, $\{a_n\}$, and $\{b_n\}$ such that*

$$\lim_{n \rightarrow +\infty} F^{k_n}(xa_n^{-1} + b_n) = G(x),$$

then G is of the form

$$G(x) = \exp(-\beta r^{-[x]}), \quad x \in \mathbf{R}, \quad \beta > 0,$$

if and only if

$$\lim_{n \rightarrow +\infty} \frac{1 - F(n)}{1 - F(n+1)} = r > 1.$$

As already stated before, unlike any extreme value distribution a MSS distribution may have several modes or may be discrete. Examples of distribution functions which do not belong to the domain of attraction of any extreme value distribution but belong to the domain of attraction of a max-semistable law are the negative binomial, $NB(n, p)$, and von Mises distribution given by $F(x) = 1 - \exp(-x - \sin x)$ for $x \geq 0$. They belong to the domain of attraction of $G(x) = \exp(-p^{[x]})$ and $G(x) = \exp(-\exp(-x - \sin x))$, with real x , respectively.

3. Dependent sequences. Temido and Canto e Castro [22] consider all stationary sequences, $\{X_n\}$, satisfying a new dependence restriction, $D_{k_n}(u_n)$, which generalizes Leadbetter's condition $D(u_n)$. They prove that when $\{X_n\}$ satisfies $D_{k_n}(u_n)$ the limiting behavior of the maximum $M_{k_n} = \max\{X_1, \dots, X_{k_n}\}$ can be inferred from the limiting behavior of the maximum, $\widehat{M}_{k_n}$, of the associated independent sequence, $\{\widehat{X}_n\}$, using an extension of the well known extremal index. Thus, the corresponding classical result and Leadbetter's extremal types theorem are generalized.

Definition 3.1 (see [22]). Let $\{k_n\}$ be a nondecreasing sequence of positive integers. The sequence of random variables $\{X_n\}$ satisfies condition $D_{k_n}(u_n)$ if for any integers $1 \leq i_1 < \dots < i_p < j_1 < \dots < j_q \leq k_n$, for which $j_1 - i_p > l_n$, we have

$$\left| \mathbf{P}\{X_{i_1} \leq u_n, \dots, X_{i_p} \leq u_n, X_{j_1} \leq u_n, \dots, X_{j_q} \leq u_n\} - \mathbf{P}\{X_{i_1} \leq u_n, \dots, X_{i_p} \leq u_n\} \mathbf{P}\{X_{j_1} \leq u_n, \dots, X_{j_q} \leq u_n\} \right| \leq \alpha_{n, l_n},$$

where $\lim_{n \rightarrow +\infty} \alpha_{n, l_n} = 0$ for some sequence $l_n = o_n(k_n)$.

Theorem 3.1 (see [22]). Let $\{k_n\}$ be a nondecreasing positive integer sequence satisfying (3). Let $\{X_n\}$ be a stationary sequence, and let $\{a_n > 0\}$ and $\{b_n\}$ be real sequences such that $\limsup_{n \rightarrow +\infty} k_n(1 - F(x/a_n + b_n)) < +\infty$ and $\mathbf{P}\{a_n(M_{k_n} - b_n) \leq x\}$ converges to $G(x)$ for each continuity point of the nondegenerate distribution function G . If $\{X_n\}$ satisfies $D_{k_n}(x/a_n + b_n)$, for all x in $\mathbf{R}$, then G belongs to one of the three classes of max-semistable laws.

Moreover, Temido [20] introduced suitable adaptations of the local dependence conditions $D'(u_n)$ of [13] and $D^{(m)}(u_n)$ of [5], denoted by $D'_{k_n}(u_n)$ and by $D^{(m)}_{k_n}(u_n)$, respectively.

Definition 3.2 (see [20]). Let $\{k_n\}$ be a nondecreasing positive integer sequence and $\{u_n\}$ a real sequence. The sequence $\{X_n\}$ satisfies

1) condition $D^{(m)}_{k_n}(u_n)$, $k \geq 1$, if it satisfies $D_{k_n}(u_n)$ and if there are positive integer sequences $\{s_n\}$ and $\{l_n\}$ such that

$$\begin{aligned} \lim_{n \rightarrow +\infty} s_n &= +\infty, & \lim_{n \rightarrow +\infty} \frac{k_n}{s_n} &= +\infty, \\ \lim_{n \rightarrow +\infty} s_n \alpha_{n, l_n} &= 0, & \lim_{n \rightarrow +\infty} \frac{l_n s_n}{k_n} &= 0, \end{aligned} \quad (7)$$

and

$$\lim_{n \rightarrow +\infty} k_n \mathbf{P}\{X_1 > u_n \geq M_{2,m}, M_{m+1,r_n} > u_n\} = 0, \quad (8)$$

where $r_n = \lfloor \frac{k_n}{s_n} \rfloor$, $M_{ij} = \max\{X_k, k = i, \dots, j\}$ for $i < j$ and $M_{ij} = -\infty$ if $i \geq j$;

2) condition $D'_{k_n}(u_n)$ if it satisfies $D_{k_n}(u_n)$, there are positive integer sequences $\{s_n\}$ and $\{l_n\}$ satisfying (7), and $\lim_{n \rightarrow +\infty} k_n \sum_{j=2}^{r_n} \mathbf{P}\{X_1 > u_n, X_j > u_n\} = 0$.

Note that condition $D'_{k_n}(u_n)$ implies condition $D^{(1)}_{k_n}(u_n)$.

In what follows we consider $r_n := \lfloor k_n/s_n \rfloor$.

First note that for an i.i.d. sequence $\{X_n\}$ and for any nondecreasing positive integer sequence $\{k_n\}$, $\lim_{n \rightarrow \infty} k_n(1 - F(u_n)) = \tau$ is equivalent to $\lim_{n \rightarrow \infty} F^{k_n}(u_n) = e^{-\tau}$. Thus, in the sequel we deal with levels $u_n = u_n(\tau, k_n)$ satisfying these limits.

Now define

$$\Gamma(F, k_n) = \left\{ \tau > 0: \exists \{u_n\}: \lim_{n \rightarrow +\infty} k_n(1 - F(u_n)) = \tau \right\}.$$

We note that if F is discrete and $\lim_{n \rightarrow \infty} k_{n+1}/k_n = r > 1$, then $\Gamma(F, k_n)$ is not necessarily equal to $]0, \infty[$. Indeed, if for some $\tau > 0$, $u_n(\tau, k_n)$ exists, then for another $\tau' > 0$ the existence of levels $u_n(\tau', k_n)$ is equivalent to the existence of an integer m such that $\tau = r^m \tau'$.

We now observe that the limiting distribution of the maximum term of stationary sequences which satisfy condition $D'_{k_n}(u_n)$ is equal to the one

of the i.i.d. associated sequence. Namely, if for all τ in $\Gamma(F, k_n)$ the stationary sequence $\{X_n\}$ satisfies condition $D'_{k_n}(u_n)$ than $\lim_{n \rightarrow \infty} \mathbf{P}\{M_{k_n} \leq u_n(\tau, k_n)\} = e^{-\tau}$.

Furthermore, in [21] it is proved that if a stationary sequence $\{X_n\}$ satisfies condition $D_{k_n}(u_n(\tau, k_n))$ for all τ in $\Gamma(F, k_n)$ and $\mathbf{P}\{M_{k_n} \leq u_n(\tau, k_n)\}$ converges for some τ in $\Gamma(F, k_n)$, then there exists a constant θ in $[0, 1]$ such that

$$\lim_{n \rightarrow +\infty} \mathbf{P}\{M_{k_n} \leq u_n(\tau, k_n)\} = e^{-\theta\tau},$$

for all such τ . In order to name this constant θ we give the following definition.

Definition 3.3 (see [22]). We say that $\{X_n\}$ has an extremal index θ , with θ in $[0, 1]$, if there exists a nondecreasing positive integer sequence $\{k_n\}$ satisfying (3) such that, for all $\tau \in \Gamma(F, k_n)$, we have $\lim_{n \rightarrow +\infty} \mathbf{P}\{M_{k_n} \leq u_n(\tau, k_n)\} = e^{-\theta\tau}$.

We may now obtain the limiting distribution of the maximum term of stationary sequences which satisfy condition $D_{k_n}^{(m)}(u_n)$, for some $m \in \mathbf{N}$.

Theorem 3.2 (see [20]). *Let $\{k_n\}$ be a nondecreasing positive integer sequence satisfying (3) and $\{X_n\}$ a stationary sequence of random variables. Suppose that for some positive integer m and for all τ in $\Gamma(F, k_n)$ condition $D_{k_n}^{(m)}(u_n(\tau, k_n))$ holds. Then $\{X_n\}$ has extremal index θ if and only if $\lim_{n \rightarrow +\infty} \mathbf{P}(M_{2,m} \leq u_n(\tau, k_n) \mid X_1 > u_n(\tau, k_n)) = \theta$.*

4. The integer-valued MA model. Consider the integer-valued MA model obtained as a discrete analogue of the conventional MA model and defined by

$$X_n = \sum_{j=-\infty}^{\infty} \beta_j \circ Z_{n-j}, \quad (9)$$

where $\{Z_n\}$ is an i.i.d. sequence of integer-valued random variables with distribution F_Z and $\circ$ denotes binomial thinning, $\beta \circ Z = \sum_{s=1}^Z B_s(\beta)$, $\beta \in [0, 1]$, where $\{B_s(\beta)\}$ is an i.i.d. sequence of Bernoulli random variables satisfying $\mathbf{P}\{B_s(\beta) = 1\} = \beta$.

Because thinning is a random operation, the equality given above defines a rich class of models by allowing different dependence structures on the thinning operations. In particular, we consider the general class of models consisting of all stationary sequences defined by (9), for which the vector of terms $(\dots, \beta_{-1} \circ Z_k, \beta_0 \circ Z_k, \beta_1 \circ Z_k, \dots)$ has some fixed dependence structure induced by the thinning operation for every k , and all other thinning relations are of independence. For specific examples of models within this class, see [3].

Hall [10] studied the extremal behavior of this model when driven by heavy tailed innovations. More recently, Hall [8] considered margins in Anderson's class given by

$$1 - F_Z(x) \sim K[x]^\xi(1 + \lambda)^{-[x]} \quad \text{as } x \rightarrow +\infty, \quad \lambda > 0. \quad (10)$$

In order to guarantee almost sure convergence of the infinite series (9) the coefficients β_i shall satisfy

$$\beta_i = O(|i|^{-\delta}), \quad i \rightarrow \pm\infty, \quad (11)$$

for some $\delta > 2$.

The marginal distribution of $\{X_n\}$ is related to the distribution of the innovations by the following theorem obtained in [7].

Theorem 4.1 (see [8]). *Let $\beta_{\max} = \max_i \{\beta_i\}$ and let $\Upsilon = \{i_1, \dots, i_k\}$, $i_1 < \dots < i_k$, be the set of indices such that $\beta_i = \beta_{\max}$. Then*

$$\mathbf{P}\left\{\sum_{i=-\infty}^{+\infty} \beta_i \circ Z_{-i} > n\right\} \sim \check{K} n^\xi (1 + \check{\lambda})^{-n} \quad \text{as } n \rightarrow \infty, \quad (12)$$

for $\xi \neq -1$, where $\check{\lambda} = \lambda/\beta_{\max}$,

$$\begin{aligned} \check{\xi} &= \begin{cases} k\xi + k - 1 & \text{if } \xi > -1, \\ \xi & \text{if } \xi < -1, \end{cases} & K^* &= \beta_{\max} K \left(\frac{1 + \lambda}{\lambda + \beta_{\max}} \right)^{\xi+1} \\ \check{K} &= \begin{cases} \check{\lambda}^{k-1} K^* k \frac{(\Gamma(\xi+1))^k}{\Gamma(k(\xi+1))} \mathbf{E}(1 + \check{\lambda})^{\sum_{i \notin \Upsilon} \beta_i \circ Z_{-i}} & \text{if } \xi > -1, \\ k K^* (\mathbf{E}(1 + \check{\lambda}))^{k-1} \mathbf{E}(1 + \check{\lambda})^{\sum_{i \notin \Upsilon} \beta_i \circ Z_{-i}} & \text{if } \xi < -1. \end{cases} \end{aligned}$$

By proving that $\{X_n\}$ satisfies both conditions $D(u_n)$ and $D'(u_n)$ in [13], Hall [8] obtained the following bounds for the limiting distribution of the maximum term, $M_n = \max\{X_1, \dots, X_n\}$, as long as the sequence of constants $\{\beta_i\}$ has a unique maximum:

$$\begin{cases} \limsup_{n \rightarrow +\infty} \mathbf{P}\{M_n \leq x + b_n\} \leq \exp(-(1 + \check{\lambda})^{-x}), \\ \liminf_{n \rightarrow +\infty} \mathbf{P}\{M_n \leq x + b_n\} \geq \exp(-(1 + \check{\lambda})^{-(x-1)}), \end{cases}$$

where $b_n = (\ln(1 + \check{\lambda}))^{-1}(\ln n + \check{\xi} \ln \ln n + \ln \check{K})$.

Using the results of the previous section we shall now prove that $\{X_n\}$ satisfies conditions $D_{k_n}(u_n)$ together with $D'_{k_n}(u_n)$ and obtain the limiting max-semistable distribution of the maximum term.

Theorem 4.2. *Let $\{X_n\}$ be a stationary sequence defined by (9) with marginal distribution F satisfying (10) with $\xi \neq -1$, and with coefficients β_i*

satisfying (11) and having a unique maximal element $\beta_{\max}$. Then there exist a nondecreasing positive integer sequence $\{k_n\}$ satisfying (3) and a real sequence $\{b_n\}$ such that

$$\lim_{n \rightarrow +\infty} \mathbf{P}\{M_{k_n} \leq x + b_n\} = \exp(-(1 + \check{\lambda})^{-[x]}).$$

For the proof we shall need the following auxiliary results.

Lemma 4.1. *Let $\{k_n\}$ be a nondecreasing positive integer sequence satisfying (3) and let $\{X_n\}$ be a stationary sequence defined by (9). Suppose that there is a real sequence $\{b_n\}$ such that*

$$\lim_{n \rightarrow +\infty} \mathbf{P}\{\widehat{M}_{k_n} \leq x + b_n\} = G(x),$$

for some nondegenerate distribution G . If there exist sequences $\{\varepsilon_n\}$ and $\{l_n\}$ such that for $l_n = o(k_n)$ the following relations hold:

$$\lim_{n \rightarrow +\infty} k_n \mathbf{P}\left\{\sum_{i=l_n}^{+\infty} \beta_i \circ Z_{-i} > \varepsilon_n\right\} = 0, \quad (13)$$

$$\lim_{n \rightarrow +\infty} k_n \mathbf{P}\left\{\sum_{i=-\infty}^{-l_n} \beta_i \circ Z_{-i} > \varepsilon_n\right\} = 0, \quad (14)$$

then condition $D_{k_n}(x + b_n)$ holds for $\{X_n\}$.

P r o o f. Let $1 \leq i_1 < \dots < i_p < j_1 < \dots < j_q \leq k_n$ be integers such that $j_1 - i_p \geq 2l_n$. Use the following notation:

$$\begin{aligned} X_i^* &= (X_{i_1}, \dots, X_{i_p}), & X_j^* &= (X_{j_1}, \dots, X_{j_q}), \\ X_i^{*'} &= (X'_{i_1}, \dots, X'_{i_p}), & X_j^{*''} &= (X''_{j_1}, \dots, X''_{j_q}), \end{aligned}$$

where $X'_j = \sum_{i=-\infty}^{l_n-1} \beta_i \circ Z_{i+j}$ and $X''_j = \sum_{i=-l_n+1}^{\infty} \beta_i \circ Z_{i+j}$. Consider $u_n = x + b_n$. Since $X_i^{*'}$ and $X_j^{*''}$ are independent, we have

$$\begin{aligned} \mathbf{P}\{X_i^* \leq u_n, X_j^* \leq u_n\} &\leq \mathbf{P}\{X_i^{*'} \leq u_n + \varepsilon_n, X_j^{*''} \leq u_n + \varepsilon_n\} + \mathbf{P}\{M'_{k_n} > \varepsilon_n\} \\ &+ \mathbf{P}\{M''_{k_n} > \varepsilon_n\} \leq \mathbf{P}\{X_i^* \leq u_n + 2\varepsilon_n\} \mathbf{P}\{X_j^* \leq u_n + 2\varepsilon_n\} \\ &+ 2\mathbf{P}\{M'_{k_n} > \varepsilon_n\} + 2\mathbf{P}\{M''_{k_n} > \varepsilon_n\}, \end{aligned}$$

where $M'_{k_n} = \max\{X_0 - X'_0, X_1 - X'_1, \dots\}$ and $M''_{k_n} = \max\{X_0 - X''_0, X_1 - X''_1, \dots\}$. Therefore

$$\begin{aligned} &\left| \mathbf{P}\{X_i^* \leq u_n, X_j^* \leq u_n\} - \mathbf{P}\{X_i^* \leq u_n\} \mathbf{P}\{X_j^* \leq u_n\} \right| \\ &\leq \mathbf{P}\{X_i^* \leq u_n\} \mathbf{P}\{X_j^* \leq u_n\} + \sum_{j=1}^{k_n} \mathbf{P}\{u_n < X_j \leq u_n + 2\varepsilon_n\} \\ &\quad + 2\mathbf{P}\{M'_{k_n} > \varepsilon_n\} + 2\mathbf{P}\{M''_{k_n} > \varepsilon_n\} - \mathbf{P}\{X_i^* \leq u_n\} \mathbf{P}\{X_j^* \leq u_n\} \end{aligned}$$

$$\begin{aligned}
&\leq k_n \mathbf{P}\{u_n < X_0 < u_n + 2\varepsilon_n\} + 2k_n \mathbf{P}\{X_0 - X'_0 > \varepsilon_n\} \\
&\quad + 2k_n \mathbf{P}\{X_0 - X''_0 > \varepsilon_n\} = k_n \mathbf{P}\{u_n < X_0 < u_n + 2\varepsilon_n\} \\
&\quad + 2k_n \mathbf{P}\left\{\sum_{i=l_n}^{\infty} \beta_i \circ Z_{-i} > \varepsilon_n\right\} + 2k_n \mathbf{P}\left\{\sum_{i=-\infty}^{-l_n} \beta_i \circ Z_{-i} > \varepsilon_n\right\} \\
&= k_n \mathbf{P}\{X_0 > u_n + 2\varepsilon_n\} - k_n \mathbf{P}\{X_0 > u_n\} + o(1) \\
&= -\ln G(x) + \ln G(x) + o(1) = o(1).
\end{aligned}$$

To conclude the proof it suffices to use the reverse inequality.

Lemma 4.2. Let $\beta_i = O(|i|^{-\delta})$ as $i \rightarrow \pm\infty$, for some $\delta > 1$ and suppose conditions (13) and (14) hold for $\{X_n\}$. If there exists $\{u_n\}$, $\{\gamma_n\}$, $\gamma_n < k_n/s_n$, and $\varepsilon > 0$ such that

$$\lim_{n \rightarrow +\infty} k_n \sum_{j=1}^{2\gamma_n} \mathbf{P}\{X_0 + X_j > 2u_n\} = 0, \quad (15)$$

$$\lim_{n \rightarrow +\infty} \frac{k_n^2}{s_n} \mathbf{P}\left\{\sum_{i=\gamma_n+1}^{\infty} \beta_i \circ Z_{-i} > \varepsilon\right\} = 0, \quad (16)$$

$$\lim_{n \rightarrow +\infty} \frac{k_n^2}{s_n} \mathbf{P}\left\{\sum_{i=-\infty}^{-\gamma_n-1} \beta_i \circ Z_{-i} > \varepsilon\right\} = 0, \quad (17)$$

then condition $D'_{k_n}(u_n)$ holds for $\{X_n\}$.

P r o o f. Since $\mathbf{P}\{X_0 > u_n, X_j > u_n\} \leq \mathbf{P}\{X_0 + X_j > 2u_n\}$ relation (15) implies

$$\lim_{n \rightarrow +\infty} k_n \sum_{j=1}^{2\gamma_n} \mathbf{P}\{X_0 > u_n, X_j > u_n\} = 0.$$

Let $X'_0 = \sum_{i=-\infty}^{\gamma_n} \beta_i \circ Z_{-i}$ and $X''_j = \sum_{i=-\gamma_n}^{+\infty} \beta_i \circ Z_{-i+j}$ and note that X'_0 and X''_j are independent for $j > 2\gamma_n$. Now

$$\begin{aligned}
&\mathbf{P}\{X_0 > u_n, X_j > u_n\} \leq \mathbf{P}\{X'_0 > u_n - \varepsilon\} \mathbf{P}\{X''_j > u_n - \varepsilon\} \\
&\quad + \mathbf{P}\left\{\sum_{i=\gamma_n+1}^{\infty} \beta_i \circ Z_{-i} > \varepsilon\right\} + \mathbf{P}\left\{\sum_{i=-\infty}^{-\gamma_n-1} \beta_i \circ Z_{-i+j} > \varepsilon\right\}.
\end{aligned}$$

Using stationarity and taking $u'_n = u_n - \varepsilon$, we obtain

$$\begin{aligned}
&k_n \sum_{j=2\gamma_n+1}^{r_n} \mathbf{P}\{X_0 > u_n, X_j > u_n\} \\
&\leq \frac{k_n^2}{s_n} \mathbf{P}\left\{\sum_{i=-\infty}^{\gamma_n} \beta_i \circ Z_{-i} > u'_n\right\} \mathbf{P}\left\{\sum_{i=-\gamma_n}^{+\infty} \beta_i \circ Z_{-i} > u'_n\right\} \\
&\quad + \frac{k_n^2}{s_n} \mathbf{P}\left\{\sum_{i=\gamma_n+1}^{\infty} \beta_i \circ Z_{-i} > \varepsilon\right\} + \frac{k_n^2}{s_n} \mathbf{P}\left\{\sum_{i=-\infty}^{-\gamma_n-1} \beta_i \circ Z_{-i} > \varepsilon\right\},
\end{aligned}$$

and noticing that

$$\begin{aligned} & \mathbf{P}\left\{\sum_{i=-\infty}^{\gamma_n} \beta_i \circ Z_{-i} > u'_n\right\} \mathbf{P}\left\{\sum_{i=\gamma_n+1}^{+\infty} \beta_i \circ Z_{-i} > -\varepsilon\right\} \\ &= \mathbf{P}\left\{\sum_{i=-\infty}^{\gamma_n} \beta_i \circ Z_{-i} > u'_n, \sum_{i=\gamma_n+1}^{+\infty} \beta_i \circ Z_{-i} > -\varepsilon\right\} \\ &\leq \mathbf{P}\{X_0 > u'_n - \varepsilon\} = O(k_n^{-1}), \end{aligned}$$

we obtain

$$\mathbf{P}\left\{\sum_{i=-\gamma_n}^{+\infty} \beta_i \circ Z_{-i} > u'_n\right\} = O(k_n^{-1}).$$

Similarly

$$\mathbf{P}\left\{\sum_{i=-\infty}^{\gamma_n} \beta_i \circ Z_{-i} > u'_n\right\} = O(k_n^{-1}).$$

Hence

$$\begin{aligned} & \frac{k_n^2}{s_n} \mathbf{P}\left\{\sum_{i=-\infty}^{\gamma_n} \beta_i \circ Z_{-i} > u'_n\right\} \mathbf{P}\left\{\sum_{i=-\gamma_n}^{+\infty} \beta_i \circ Z_{-i} > u'_n\right\} \\ &= \frac{k_n}{s_n} k_n O(k_n^{-1}) O(k_n^{-1}) = \frac{1}{s_n} O(1) \rightarrow 0, \quad n \rightarrow +\infty. \end{aligned}$$

Lemma 4.2 is proved.

P r o o f o f T h e o r e m 4.2. Consider $u_n = x + b_n$. To prove that condition $D_{k_n}(u_n)$ holds for $\{X_n\}$ we apply Lemma 4.1. First note that

$$1 - F_X(x + b_n) \sim \check{K} [x + b_n]^{\check{\xi}} (1 + \check{\lambda})^{-[u_n]}$$

so that using $k_n = [\check{K}^{-1}(1 + \check{\lambda})^{b_n} b_n^{-\check{\xi}}]$ we obtain

$$\lim_{n \rightarrow +\infty} k_n (1 - F_X(u_n)) = (1 + \check{\lambda})^{-[x]}.$$

To prove (13) and (14) we use the Markov inequality to obtain

$$k_n \mathbf{P}\left\{\sum_{i=l_n}^{+\infty} \beta_i \circ Z_{-i} > \varepsilon_n\right\} \leq k_n \frac{\mathbf{E}(\sum_{i=l_n}^{+\infty} \beta_i \circ Z_{-i})^2}{\varepsilon_n^2}.$$

Moreover,

$$\begin{aligned} & \mathbf{E}\left(\sum_{i=l_n}^{+\infty} \beta_i \circ Z_{-i}\right)^2 = \mathbf{D}\left(\sum_{i=l_n}^{+\infty} \beta_i \circ Z_{-i}\right) + \left(\mathbf{E}\left(\sum_{i=l_n}^{+\infty} \beta_i \circ Z_{-i} > \varepsilon_n\right)\right)^2 \\ &= (\sigma^2 - \mu) \sum_{i=l_n}^{+\infty} \beta_i^2 + \mu \sum_{i=l_n}^{+\infty} \beta_i + \mu^2 \left(\sum_{i=l_n}^{+\infty} \beta_i\right)^2 \leq C \sum_{i=l_n}^{+\infty} \frac{1}{i^\delta} \end{aligned}$$

attending to (11). Using $\varepsilon_n = k_n^{1-\delta/2}$ and noticing that (12) implies $\sum_{i=l_n}^{\infty} \beta_i = O(l_n^{1-\delta})$ we obtain

$$k_n \mathbf{P} \left\{ \sum_{i=l_n}^{+\infty} \beta_i \circ Z_{-i} > \varepsilon_n \right\} \leq C \frac{k_n}{\varepsilon_n^2} \sum_{i=l_n}^{+\infty} \beta_i \leq C_1 k_n^{\delta-1} l_n^{1-\delta} = o(1).$$

Now to prove that condition $D'_{k_n}(u_n)$ holds for $\{X_n\}$ we apply Lemma 4.2.

First note that by [8] $\mathbf{E}[(1+h)^{X_0+X_j}]$ is uniformly bounded in $j \geq 1$, for all $h \in]0, -1 + \sqrt{1 + \lambda/\tilde{\beta}}[$.

To prove (15) we use Bernstein's inequality

$$\mathbf{P}\{X_0 + X_j > 2u_n\} \leq \mathbf{E}[(1+h)^{X_0+X_j}](1+h)^{-2u_n} \leq C(1+h)^{-2u_n}.$$

Take h such that $1 + \tilde{\lambda} < (1+h)^2 < 1 + \lambda/\tilde{\beta}$. Next choose η in $]0, 1[$ and take $\gamma_n = (k_n/s_n)^\eta$. Thus, we get

$$\begin{aligned} k_n \sum_{i=1}^{2\gamma_n} \mathbf{P}\{X_0 + X_j > 2u_n\} &\leq 2k_n \gamma_n O((1+h)^{-2b_n}) \\ &\leq C_1 (1+\tilde{\lambda})^{b_n} b_n^{-\xi} \frac{(1+\tilde{\lambda})^{b_n \eta} b_n^{-\xi \eta}}{s_n^\eta} (1+h)^{-2b_n} \rightarrow 0, \quad n \rightarrow +\infty. \end{aligned}$$

To prove (16) take $\varepsilon > 1$ and $h_n = (k_n^2/s_n)^{\theta/\varepsilon}$ for some $\theta \in]1, \varepsilon[$ and use again Bernstein's inequality to obtain

$$\begin{aligned} \frac{k_n^2}{s_n} \mathbf{P} \left\{ \sum_{i=\gamma_n+1}^{\infty} \beta_i \circ Z_{-i} > \varepsilon \right\} &\leq \frac{k_n^2}{s_n} \frac{\mathbf{E}[(1+h_n) \sum_{i=\gamma_n+1}^{\infty} \beta_i \circ Z_{-i}]}{(1+h_n)^\varepsilon} \\ &\leq C \frac{k_n^2}{s_n} \frac{1}{h_n^\varepsilon} = C_1 \frac{k_n^{2(1-\theta)}}{s_n^{1-\theta}} \rightarrow 0, \quad n \rightarrow \infty, \end{aligned}$$

since $\mathbf{E}(1+h_n) \sum_{i=\gamma_n+1}^{\infty} \beta_i \circ Z_{-i}$ is finite for $\theta < \varepsilon$. The proof of (17) is similar.

5. McKenzie's MA model. Now consider the MA model of order q proposed by McKenzie [17],

$$\begin{aligned} X_n &= \begin{cases} \beta_0 \circ Z_n & \text{with probability } b_0, \\ \beta_0 \circ Z_n + \beta_1 \circ Z_{n-1} & \text{with probability } b_1, \\ \dots & \\ \beta_0 \circ Z_n + \dots + \beta_{q-1} \circ Z_{n-q+1} & \text{with probability } b_{q-1}, \\ \beta_0 \circ Z_n + \dots + \beta_{q-1} \circ Z_{n-q+1} + Z_{n-q} & \text{with probability } b_q, \end{cases} \quad (18) \\ b_i &= \begin{cases} \beta_0 & \text{if } i = 0, \\ (1-\beta_0) \dots (1-\beta_{i-1}) \beta_i & \text{if } q-1 \geq i \geq 1, \\ (1-\beta_0) \dots (1-\beta_{q-1}) & \text{if } i = q, \end{cases} \end{aligned}$$

where $\{Z_n\}$ is an integer valued i.i.d. sequence and all the thinning operations are independent. All the thinning operations involved in X_n and X_m , $n \neq m$, are also independent.

This model has the particular characteristic that if Z_n is geometric with parameter p , that is, [9] $P\{X = x\} = (1 - p)p^x$, $x = 0, 1, 2, \dots$, then $\{X_n\}$ forms a stationary sequence with the same marginal distribution.

For this model Hall [9] has proved the following result.

Theorem 5.1 (see [9]). *Let $\{X_n\}$ be a stationary sequence defined by (18). Then, for all $x \in \mathbf{R}$, and considering $b_n = -(\ln n)/\ln p - 1$, we have*

$$\begin{cases} \limsup_{n \rightarrow \infty} \mathbf{P}\{M_n \leq x + b_n\} \leq e^{-p^x}, \\ \liminf_{n \rightarrow \infty} \mathbf{P}\{M_n \leq x + b_n\} \geq e^{-p^{(x-1)}}. \end{cases}$$

Using the results of Section 3 we shall now obtain the following theorem, where F denotes the distribution function of the geometric distribution with parameter p .

Theorem 5.2. *Let $\{X_n\}$ be a stationary sequence defined by (18). Define $u_n = x + n - 1$ and $k_n = \lfloor p^{-n} \rfloor$. Then*

$$\lim_{n \rightarrow +\infty} k_n(1 - F(n - 1 + x)) = p^{\lfloor x \rfloor}$$

and

$$\lim_{n \rightarrow +\infty} \mathbf{P}\{M_{k_n} \leq x + n - 1\} = \exp(-p^{\lfloor x \rfloor}), \quad x \in \mathbf{R} \setminus \mathbf{Z}.$$

P r o o f. In order to obtain the limiting distribution of the maximum term M_{k_n} we shall prove that condition $D'_{k_n}(u_n)$ hold for $\{X_n\}$. Since the model is q -dependent, condition $D_{k_n}(u_n)$ trivially holds.

Now, for condition $D'_{k_n}(u_n)$ we first use the q -dependence of the process to write

$$\begin{aligned} & k_n \sum_{j=2}^{r_n} \mathbf{P}\{X_1 > u_n, X_j > u_n\} \\ &= k_n \sum_{j=2}^{q+1} \mathbf{P}\{X_1 > u_n, X_j > u_n\} + k_n \sum_{j=q+2}^{r_n} \mathbf{P}\{X_1 > u_n, X_j > u_n\} \\ &= k_n \sum_{j=2}^{q+1} \mathbf{P}\{X_1 > u_n, X_j > u_n\} + o(1). \end{aligned}$$

In [9] it is proved that there exists $\alpha > p^{-1/2}$ such that for $2 \leq j \leq q+1$

$$\mathbf{P}\{X_1 > u_n, X_j > u_n\} \leq \frac{C}{\alpha^{\lfloor 2u_n \rfloor}}.$$

Therefore

$$\begin{aligned} k_n \sum_{j=2}^{r_n} \mathbf{P}\{X_1 > u_n, X_j > u_n\} &\leq k_n \sum_{j=2}^{q+1} \frac{C}{\alpha^{[2u_n]}} \\ &\leq C_1 p^{-n} \alpha^{-[2x+2n-2]} = o(1), \quad n \rightarrow \infty. \end{aligned}$$

6. A max-AR model of second order. Consider the second order max-AR stationary sequence defined by

$$X_n = \max\{X_{n-2}, Y_n\} - k, \quad (19)$$

where $\{Y_n\}$ is an i.i.d. sequence of integer-valued random variables with distribution function F and $k \in \mathbf{N}$. We also assume that X_0 and X_1 are independents with distribution function H and $\{Y_n\}$ is independent of X_0 and of X_1 .

Similarly to the results of Alpuim [1] we prove that $\{X_n\}$ has stationary distribution H if and only if $F(x) = H(x - k)/H(x)$, that is,

$$H(x) = \prod_{i=1}^{+\infty} F(x + ik).$$

Moreover H is nondegenerate if and only if there exists a real x_0 such that $F(x_0 + k) > 0$ and $0 < \sum_{i=1}^{+\infty} (1 - F(x_0 + ik)) < +\infty$.

For this model, Hall [11] proved that $\{X_n\}$ satisfies condition $D^{(3)}(u_n)$ in [5], where $\{u_n\}$ satisfies $1 - H(u_n) = O(1/n)$, but the extremal index of Leadbetter is zero whenever it exists. Consequently, M_n and $\widehat{M}_n$, the maximum related to the i.i.d. associated sequence, cannot both have nondegenerated limiting distributions based on the same normalizing constants.

However, considering H with infinite right endpoint and satisfying (1), Hall proved that there exists a real sequence $\{b_n\}$ such that

$$\begin{cases} \limsup_{n \rightarrow +\infty} \mathbf{P}\{M_n \leq x + b_n\} \leq \exp(-\theta r^{-x}), \\ \liminf_{n \rightarrow +\infty} \mathbf{P}\{M_n \leq x + b_n\} \geq \exp(-\theta r^{-(x-1)}), \end{cases} \quad (20)$$

where $\theta = 1 - r^{-k}$.

Suppose H satisfies the same assumptions and take a nondecreasing and positive integer-valued sequence satisfying (2). Consider $u_n := u_n(\tau, k_n)$ for all τ in $\Gamma(H, k_n)$ and the positive integer sequence $\{s_n\}$ satisfying (7).

In [20] it is proved that $\{X_n\}$ satisfies condition $D_{k_n}(u_n)$.

Now we prove that

$$\lim_{n \rightarrow +\infty} k_n \sum_{j=4}^{r_n} \mathbf{P}\{X_1 > u_n, X_2 \leq u_n, X_3 \leq u_n, X_j > u_n\} = 0$$

which implies condition $D_{k_n}^{(3)}(u_n)$. Indeed,

$$\begin{aligned} & k_n \sum_{j=4}^{r_n} \mathbf{P}\{X_1 > u_n, X_2 \leq u_n, X_3 \leq u_n, X_j > u_n\} \\ &= k_n \sum_{j=5, j \text{ odd}}^{r_n} \mathbf{P}\{X_1 > u_n, X_3 \leq u_n, X_j > u_n\} \mathbf{P}\{X_2 \leq u_n\} \\ &+ k_n \sum_{j=4, j \text{ even}}^{r_n} \mathbf{P}\{X_1 > u_n, X_3 \leq u_n\} \mathbf{P}\{X_2 \leq u_n, X_j > u_n\}, \quad (21) \end{aligned}$$

where the last sum does not exceed

$$k_n r_n \mathbf{P}\{X_1 > u_n\} \mathbf{P}\{X_4 > u_n\} = o(1), \quad n \rightarrow +\infty.$$

For the first sum of the right-hand side of (21), taking into account that given $\{X_3 \leq u_n\}$, the events $\{X_1 > u_n\}$ and $\{X_j > u_n\}$ are independent, we obtain

$$\begin{aligned} & k_n \sum_{j=5, j \text{ odd}}^{r_n} \mathbf{P}\{X_1 > u_n, X_3 \leq u_n, X_j > u_n\} \mathbf{P}\{X_2 \leq u_n\} \\ & \leq k_n \sum_{j=5, j \text{ odd}}^{r_n} \mathbf{P}(X_1 > u_n \mid X_3 \leq u_n) \mathbf{P}(X_j > u_n \mid X_3 \leq u_n). \quad (22) \end{aligned}$$

Due to $\mathbf{P}(X_1 > u_n \mid X_3 \leq u_n) = 1 - F(u_n + k) \leq 1 - H(u_n)$ and to

$$\mathbf{P}(X_j > u_n \mid X_3 \leq u_n) = 1 - \prod_{i=1}^{(j-3)/2} F(u_n + ik) \leq 1 - H(u_n)$$

the right-hand side of (22) is bounded by

$$k_n r_n (1 - H(u_n))^2 = o(1), \quad n \rightarrow +\infty.$$

Then condition $D_{k_n}^{(3)}(u_n)$ is satisfied. Moreover, for the extremal index we have

$$\begin{aligned} \theta &= \lim_{n \rightarrow +\infty} \mathbf{P}(X_2 \leq u_n, X_3 \leq u_n \mid X_1 > u_n) = \lim_{n \rightarrow +\infty} 1 - \frac{1 - H(u_n + k)}{1 - H(u_n)} \\ &= 1 - \frac{r^{-[x+k]}}{r^{-[x]}} = 1 - r^{-k}. \end{aligned}$$

Thus

$$\lim_{n \rightarrow +\infty} \mathbf{P}\{M_{k_n} \leq x a_n^{-1} + b_n\} = \exp(-(1 - r^{-k}) r^{-[x]}), \quad x \in \mathbf{R} \setminus \mathbf{Z}.$$

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