Math-Net.Ru

Общероссийский математический портал

P. Grandits, On martingale measures for stochastic processes with independent increments, *Теория вероятн. и ее примен.*, 1999, том 44, выпуск 1, 87–100

DOI: <https://doi.org/10.4213/tvp599>

Использование Общероссийского математического портала Math-Net.Ru подразумевает, что вы прочитали и согласны с пользовательским соглашением

<http://www.mathnet.ru/rus/agreement>

Параметры загрузки:

IP: 54.172.240.79

26 апреля 2023 г., 11:29:01

© 1999 г.

GRANDITS P.*

ON MARTINGALE MEASURES
FOR STOCHASTIC PROCESSES
WITH INDEPENDENT INCREMENTS¹⁾

В статье рассматривается специальный семимартингал X с независимыми приращениями и доказывается существование и эквивалентность локальной мартингальной меры $\mathbf{P}^H$ для X , которая минимизирует процесс Хеллингера, в предположении, что эквивалентная локально мартингальная мера вообще существует. Это сделано при условии полунепрерывности слева и ограниченности скачков процесса X . Также исследуется связь между хорошо известной минимальной мартингальной мерой $\mathbf{P}^{\min}$ и $\mathbf{P}^H$. Показано, что в некотором смысле $\mathbf{P}^{\min}$ есть аппроксимация для $\mathbf{P}^H$.

Ключевые слова и фразы: процессы с независимыми приращениями, эквивалентная локальная мартингальная мера, минимальная мартингальная мера, процесс Хеллингера.

1. Introduction. A basic method in Mathematical Finance is to replace the original probability measure by an equivalent martingale measure, sometimes called a risk-neutral measure. This measure is used for pricing and hedging given contingent claims (e.g. options, futures etc.). The method is also used in actuarial mathematics and is related to the several premium calculation principles (cf. [3]).

In contrast to the situation of the classical Black–Scholes option pricing formula, where the market is assumed to be complete (every contingent claim is hedgeable and there exists only one equivalent martingale measure), one is confronted with the problem of choosing the appropriate martingale measure in the incomplete case. One possibility is to consider the minimal martingale measure $\mathbf{P}^{\min}$, which was introduced by Föllmer and Schweizer in [5] and which coincides with the variance-optimal measure $\mathbf{P}^{\text{opt}}$ in the case of independent increments considered below [11]. Besides the advantage that its density process has a very simple form, it further locally minimizes the quadratic risk of hedging [5]. Difficulties arise, if one considers processes

*Institut für Statistik, Universität Wien, Brünnerstraße 72, A-1210 Wien, Austria.

¹⁾ Supported by «Fonds zur Förderung der wissenschaftlichen Forschung in Österreich», Project № P10035.

with jumps. In this situation $\mathbf{P}^{\min}$ sometimes exists only as signed measure [2], and we get serious problems with the economic interpretation, because one should use not only probability measures, but even measures, which are equivalent to the original measure $\mathbf{P}$ for pricing contingent claims [8]. In general, results, which hold for continuous as well as for discontinuous processes are not so numerous. This is a pity because of the importance of discontinuous processes in applications (jump-diffusion processes).

So it would be nice to have results of the form: If there exists an equivalent local martingale measure (ELMM) for the process X (general necessary and sufficient conditions have been given by Delbaen and Schachermayer in terms of no-arbitrage concepts [4]), then a measure $\mathbf{P}^*$, which is characterized by minimizing a certain quantity, is also equivalent. The tool we apply is the theory of Hellinger processes [9], which is also used by Kabanov and Kramkov to prove some results on asymptotic arbitrage [10]. It turns out that the quantity, which should be minimized is the terminal value of the Hellinger process.

Our paper deals with processes with independent (but not stationary) increments (PII) and our main result will be: Let X be a special semimartingale with independent increments, quasileftcontinuous, with uniformly bounded jumps, obeying a further technical condition called (D) in the sequel. *If there exists an ELMM for X , then the martingale measure $\mathbf{P}^H$, which minimizes the terminal value of the Hellinger process, exists and is equivalent to the original measure.* We also give some examples for the failure of this result, if either the assumption of uniformly bounded jumps or the condition (D) are not satisfied. The restriction of quasileftcontinuity is made in order to avoid too technical proofs. We are convinced that with more effort the result can be proved without this last assumption. Nevertheless the quasileftcontinuous case very often appears in applications.

The paper is organized as follows. In Section 2 we give some preliminary results used later on. In Section 3 we prove the existence and equivalence of $\mathbf{P}^H$. In Section 4 we give a characterization of $\mathbf{P}^{\min}$ as an approximation of $\mathbf{P}^H$. We finally give some examples in Section 5, which should clear up the difference between $\mathbf{P}^{\min}$ and $\mathbf{P}^H$, and which show the effect of dropping some of the assumptions in our main theorem.

2. Preliminaries. We need the following notation and results from the theory of semimartingales. A more detailed description is given in the monograph of Jacod and Shiryaev [9], and we refer to that book for eventually unexplained notation. In the whole paper the following assumptions are used (unless something else is stated).

Assumption 1. *Let X be a real special semimartingale on the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^\infty, \mathbf{P})$, which has independent increments, which is quasileftcontinuous and has uniformly bounded jumps. In addition we use*

the canonical setting of [9] III.2.13, and for simplicity we assume $X_0 = 0$ and that $\mathcal{F}_0$ is trivial.

One can regard X as discounted stock price in finance or as aggregate claim process in insurance mathematics (cf. for example [3] or [6]). The generalization to a d -dimensional process is straightforward, but in order to simplify notation, we restrict to the one-dimensional case. The characteristics of X , here denoted as (B, C, ν) , are deterministic, and its Doob–Meyer decomposition is given by $X = M + B$. Because of the uniform boundedness of jumps we can choose $h(x) = x$ as «truncation function». We further use the notation $H \cdot X$ for stochastic integration and $Y * \mu_t$ for the integral $\int_0^t \int_{\mathbf{R}} Y(\omega, s, x) \mu(\omega, ds, dx)$, where μ is a random measure on $[0, \infty) \times \mathbf{R}$ and Y is a $(\mathcal{P} \otimes \mathcal{B})$ -measurable function ($\mathcal{P}$ is the predictable σ -field). The random jump measure of the process X is defined as $\mu^X(\omega, dt, dx) = \sum_{s, \{\Delta X_s \neq 0\}} \varepsilon_{(s, \Delta X_s)}(dt, dx)$, where ε denotes as usual the Dirac measure. Our assumptions imply that we can find a version of the characteristics (B, C, ν) , which is of the form (Proposition II.2.9 in [9])

$$B = b \cdot A, \quad C = c \cdot A, \quad \nu(dt, dx) = dA_t K_t(dx),$$

where A, b, c are functions on $\mathbf{R}_+$, A is continuous and nondecreasing and c nonnegative. $K_t(dx)$ denotes a positive measure with compact support (we have bounded jumps!) on $\mathbf{R}$, which fulfills $\int x^2 K_t(dx) < \infty$. We write $\mathbf{P}' \ll^{\text{loc}} \mathbf{P}$, if $\mathbf{P}'_{/\mathcal{F}_t} \ll \mathbf{P}_{/\mathcal{F}_t}$ for all $t \in \mathbf{R}_+$ and X^c for the continuous martingale part of X . The following definitions are used below

$$\begin{aligned} \mathcal{M}^{\text{loc}}(X) &= \{\mathbf{P}' \mid \mathbf{P}' \ll^{\text{loc}} \mathbf{P}, X \text{ is a } \mathbf{P}'\text{-local martingale}\}, \\ \mathcal{M}(X) &= \{\mathbf{P}' \mid \mathbf{P}' \ll \mathbf{P}, X \text{ is a } \mathbf{P}'\text{-local martingale}\}, \\ \mathcal{M}^{e, \text{loc}}(X) &= \{\mathbf{P}' \mid \mathbf{P}' \sim^{\text{loc}} \mathbf{P}, X \text{ is a } \mathbf{P}'\text{-local martingale}\}, \\ \mathcal{M}^e(X) &= \{\mathbf{P}' \mid \mathbf{P}' \sim \mathbf{P}, X \text{ is a } \mathbf{P}'\text{-local martingale}\}, \end{aligned}$$

and we need the following results.

Lemma 1. *Let X be as in Assumption 1. In addition assume that $\mathbf{P}' \ll^{\text{loc}} \mathbf{P}$. Then the density process Z of $\mathbf{P}'$ relative to $\mathbf{P}$ is given by*

$$Z_t = \mathcal{E}(N)_t \quad \text{if } t \in \Delta, \quad Z_t = 0 \quad \text{if } t \notin \Delta.$$

Furthermore a version of the Hellinger process of order $\frac{1}{2}$ is given by

$$h^0\left(\frac{1}{2}, \mathbf{P}, \mathbf{P}'\right)_t = \frac{1}{8} \beta^2 c \cdot A_t + \frac{1}{2} (1 - \sqrt{Y})^2 * \nu_t.$$

In the formulation of this lemma the following abbreviations are used: $N = \beta \cdot X^c + (Y - 1) * (\mu^X - \nu)$, where β and Y stem from the Girsanov theorem for semimartingales (cf. Theorem III.3.24 in [9]), which implies a

transformation of the characteristics $B' = B + c\beta \cdot A + x(Y - 1) * \nu$, $C' = C$, $\nu' = Y\nu$. Furthermore we set

$$\Delta = \bigcup_{n=1}^{\infty} 0, T_n], \quad T_n = \inf \left(t: Z_t < \frac{1}{n} \right).$$

P r o o f. The lemma directly follows from IV.3.32, III.5.19, III.4.34 and IV.3.35 of [9], if we bear in mind that $a_t = \nu(\{t\} \times \mathbf{R}) = 0 = a'_t$ holds in the quasileftcontinuous case.

Lemma 2. *Let X be as in Assumption 1 and let $\mathbf{P}' \ll^{\text{loc}} \mathbf{P}$. Then*

$$\mathbf{P}' \ll \mathbf{P} \iff \mathbf{P}' \left(h^0 \left(\frac{1}{2} \right)_{\infty} < \infty \right) = 1.$$

P r o o f. This is Corollary IV.2.8 of [9].

Lemma 3. *Let X be as in Assumption 1 and let $\mathbf{P}' \ll^{\text{loc}} \mathbf{P}$. In addition assume that X is a process with independent increments with respect to $\mathbf{P}'$. A necessary and sufficient condition for $\mathbf{P}' \sim \mathbf{P}$ is the simultaneous validity of*

$$\mathbf{P}' \sim^{\text{loc}} \mathbf{P}, \quad h^0 \left(\frac{1}{2} \right)_{\infty} < \infty, \quad Y > 0.$$

P r o o f. Noticing that for $Y > 0$ the roles of $\mathbf{P}'$ and $\mathbf{P}$ are interchangeable, we can infer the result from Theorem IV.4.32 in [9].

3. Minimizing the Hellinger process. This section is devoted to the proof of our main result, namely that, if there exists an equivalent local martingale measure (ELMM) for our process X ($\mathcal{M}^e(X) \neq \emptyset$), then the local martingale measure $\mathbf{P}^H$, which minimizes the terminal value of the Hellinger process $h^0(\frac{1}{2})$, exists and is equivalent to the original measure. The difficulty is not to show that $\mathbf{P}^H$ is an equivalent measure, if it exists, but rather to show its existence, i.e., to show that the infimum is really attained. So what we have to do is to solve an extremal problem with constraints. Fortunately h^0 as well as the constraint are convex. Therefore the existence of a solution of the Euler–Lagrange equations is not only necessary, but also sufficient for a global minimizer [7]. As first step we prove

Proposition 1. *Let X be as in Assumption 1 and let $\mathcal{M}^{e,\text{loc}}(X) \neq \emptyset$. Furthermore we assume that the condition $dA(\{K_t[\underline{x}, \underline{x} + \varepsilon] = o(\varepsilon^2) \text{ or } K_t(\bar{x} - \varepsilon, \bar{x}] = o(\varepsilon^2)\}) = 0$, which we call (D) in the sequel, is valid ($\underline{x}$ and $\bar{x}$ stand for $\min(\kappa) = \min(\text{support}(K))$, respectively for $\max(\kappa)$). Then the extremal problem*

$$\min h^0 \left(\frac{1}{2}, \mathbf{P}, \mathbf{P}' \right)_{\infty},$$

$$\mathbf{P}' \in \mathcal{M}^{\text{loc}}(X) \text{ and } X \text{ is a PII w.r.t. } \mathbf{P}'$$

has a unique solution $\mathbf{P}^H$, if

$$\begin{aligned} & \{ \mathbf{P}' \mid \mathbf{P}' \in \mathcal{M}^{\text{loc}}(X) \text{ and } X \text{ is a PII w.r.t. } \mathbf{P}' \} \\ & \cap \left\{ \mathbf{P}' \mid h^0 \left(\frac{1}{2}, \mathbf{P}, \mathbf{P}' \right)_{\infty} < \infty \right\} \neq \emptyset. \end{aligned}$$

Moreover $\mathbf{P}^H \in \mathcal{M}^{e, \text{loc}}(X)$.

In addition, if

$$\begin{aligned} & \{\mathbf{P}' \mid \mathbf{P}' \in \mathcal{M}^{\text{loc}}(X) \text{ and } X \text{ is a PII w.r.t. } \mathbf{P}'\} \\ & \cap \left\{ \mathbf{P}' \mid h^0\left(\frac{1}{2}, \mathbf{P}, \mathbf{P}'\right)_\infty < \infty \right\} = \emptyset, \end{aligned}$$

the measure $\mathbf{P}^H$ constructed in the proof is locally equivalent, but of course not a unique minimizer of the problem under consideration.

(D) is a condition, which controls the decrease of mass at the boundary of the state space.

P r o o f. Remembering that X is the canonical process, we note that the extremal problem is equivalent to

$$\begin{aligned} \min_{Y, \beta} h^0\left(\frac{1}{2}, Y, \beta\right)_\infty &= \min_{Y, \beta} \frac{1}{8} \beta^2 c \cdot A + \frac{1}{2} (\sqrt{Y} - 1)^2 * \nu, \\ 0 &= B + c\beta \cdot A + x(Y - 1) * \nu, \\ \beta(t), Y(t, x) &\text{ are deterministic functions.} \end{aligned}$$

Considering a fixed t yields dA_t -a.e. the extremal problem

$$\min_{Y, \beta} \frac{1}{8} \beta^2 c + \frac{1}{2} \int (\sqrt{Y} - 1)^2 K(dx), \quad 0 = b + c\beta + \int x(Y - 1) K(dx). \quad (1)$$

In the sequel we abbreviate $\beta = \beta(t)$ and $Y(x) = Y(x, t)$ for one fixed t . Bearing in mind that our problem is convex, we have to show that

$$\langle DL, \phi \rangle = 0 \quad \forall \phi \in C^\infty(\kappa) \times \mathbf{R} \times \mathbf{R}$$

($C^\infty(\kappa)$ is dense in $L^1(\kappa)$) has a solution.

$$L(Y(x), \beta, \lambda) = \frac{1}{8} \beta^2 c + \frac{1}{2} \int (\sqrt{Y} - 1)^2 K(dx) - \lambda \left(b + c\beta + \int x(Y - 1) K(dx) \right)$$

is the Lagrangian of the problem, D denotes the Frechet derivative, $\langle \cdot, \cdot \rangle$ stands for the evaluation of the linear functional DL and $\kappa = \text{support}(K)$, which we assumed to be compact. Explicitly we have the Euler-Lagrange equations

$$\begin{aligned} \frac{1}{2} - \frac{1}{2\sqrt{Y(x)}} - \lambda x &= 0 \quad K\text{-a.e.}, & \frac{1}{4} \beta c - \lambda c &= 0, \\ b + c\beta + \int x(Y - 1) K(dx) &= 0. \end{aligned} \quad (2)$$

C a s e 1: $c > 0$. We assume that $\underline{x} < 0 < \bar{x}$ holds, because the calculations for the other cases are even easier. A simple computation leads to

$$\begin{aligned} \lambda &= \frac{\beta}{4}, & \beta &= -\frac{b}{c} - \frac{1}{c} \int x(Y - 1) K(dx), \\ \frac{1}{\sqrt{Y(x)}} &= 1 + \left(\frac{b}{2c} + \frac{1}{2c} \int s(Y - 1) K(ds) \right) x. \end{aligned} \quad (3)$$

Before squaring the last equation, we have to ensure that a positive solution exists (e.g. in the case $\kappa = \mathbf{R}$, where we shall not prove existence, squaring simply the third equation would give us a wrong result, because there are always $x \in \kappa$, which make the right-hand side negative, if $Y \neq 1$ (see also Section 5)). Assuming $x \neq 0$ ($Y(0) = 1$ without loss of generality because $K(0) = 0$), we multiply equation (3) for $x = \bar{x}$ by $-x/\bar{x}$ and add it to equation (3), which leads to

$$\frac{1}{\sqrt{Y(x)}} = \frac{x}{\bar{x}} \left(\frac{1}{\sqrt{Y(\bar{x})}} - \left(1 - \frac{\bar{x}}{x} \right) \right), \quad (4)$$

giving us $Y(x)$ as function of $Y(\bar{x})$, provided that the right-hand side is positive for all $x \in \kappa$, which is ensured by

$$\frac{1}{\sqrt{Y(\bar{x})}} < 1 - \frac{\bar{x}}{\underline{x}}. \quad (5)$$

So what remains is to prove that, using equation (4), equation (3) for $x = \bar{x}$ has a solution obeying (5). To simplify notation let us introduce the abbreviations $p(x) = x/\bar{x}$, $q(x) = 1 - x/\bar{x}$ and $z(x) = 1/\sqrt{Y}$, which yields instead of (4)

$$Y(x) = \frac{1}{(z(\bar{x})p(x) + q(x))^2}.$$

Plugging this into equation (3) for $x = \bar{x}$ leads to

$$z(\bar{x}) = 1 + \left(\frac{b}{2c} + \frac{1}{2c} \int \left(\frac{x}{(z(\bar{x})p(x) + q(x))^2} - x \right) K(dx) \right) \bar{x} = f(z(\bar{x})).$$

Viewing both sides as function of $z(\bar{x})$, it is easy to see that there are no singularities of the integrand for $z(\bar{x}) \in (0, 1 - \bar{x}/\underline{x}) = I$, that $f(z(\bar{x}))$ is continuous in I (note: $\int x^2 K(dx) < \infty$ holds) and finally that $f(0+) = +\infty$, respectively $f((1 - \bar{x}/\underline{x})-) = -\infty$, because of our condition (D), and we infer the existence of a solution $z(\bar{x}) \in I$, finishing the case $c > 0$.

C a s e 2: $c_t = 0$ ($b_t > 0$ without loss of generality). Now the Euler-Lagrange equations are given by

$$\frac{1}{2} - \frac{1}{2\sqrt{Y}} - \lambda x = 0, \quad b + \int x(Y - 1) K(dx) = 0.$$

C a s e 2.a: $\underline{x} < 0 < \bar{x}$. We infer $Y(x) = (1 - 2\lambda x)^{-2}$ from the first of the Euler-Lagrange equations, provided that $\lambda \in (1/(2\underline{x}), 1/(2\bar{x}))$. So we have to prove that

$$g(\lambda) = b + \int \left(\frac{x}{(1 - 2\lambda x)^2} - x \right) K(dx) = 0$$

has a solution $\lambda \in (1/(2\underline{x}), 1/(2\bar{x}))$. Our condition (D) yields $\lim_{\lambda \rightarrow 1/(2\underline{x})} g(\lambda) = -\infty$, respectively $\lim_{\lambda \rightarrow 1/(2\bar{x})} g(\lambda) = +\infty$, completing Case 2.a, because g is continuous in $(1/(2\underline{x}), 1/(2\bar{x}))$.

C a s e 2.b: $\underline{x} \leq \bar{x} \leq 0$ (but not $\underline{x} = \bar{x} = 0$). As above we have to show that

$$g(\lambda) = b + \int \left(\frac{x}{(1 - 2\lambda x)^2} - x \right) K(dx) = 0$$

has a solution $\lambda \in (1/(2\underline{x}), +\infty)$. Since the integral tends to $-\infty$ for $\lambda \rightarrow 1/(2\underline{x})$ because of condition (D), and since $g(0) = b > 0$, we can deduce again the existence of a $\lambda \in (1/(2\underline{x}), 0)$ with $g(\lambda) = 0$.

C a s e 2.c: $0 \leq \underline{x} \leq \bar{x}$ (but not $\underline{x} = \bar{x} = 0$). First of all note that by assumption we have $\mathcal{M}^{e, \text{loc}}(X) \neq \emptyset$, which yields $b < \int x K(dx)$, because otherwise the constraint $b + \int x(Y - 1) K(dx) = 0$ can not be satisfied. Again we have to prove the existence of a solution $\lambda \in (-\infty, 1/(2\underline{x}))$ for

$$g(\lambda) = b - \int \left(x - \frac{x}{(1 - 2\lambda x)^2} \right) K(dx) = 0.$$

We note that for fixed λ the integral is finite, because $\int x^2 K(dx) < \infty$ (cf. Section 2) holds. By monotone convergence we can infer that the integral tends to $\int x K(dx)$ for $\lambda \rightarrow -\infty$. Therefore and by our assumption $g(-\infty) = b - \int x K(dx) < 0$ holds. Noticing that we have again $g(0) = b > 0$ completes our proof.

The next theorem reveals that the measure constructed in Proposition 1 is already in $\mathcal{M}^e(X)$ under natural assumptions.

Theorem 1. *Let X be as in Assumption 1, let the condition (D) hold and assume*

$$\mathcal{M}^e(X) \cap \{\mathbf{P}' \in \mathcal{M}^{\text{loc}}(X) \mid X \text{ is a PII w.r.t. } \mathbf{P}'\} \neq \emptyset.$$

Then the measure $\mathbf{P}^H$ constructed in Proposition 1 fulfills $\mathbf{P}^H \in \mathcal{M}^e(X)$.

P r o o f. The theorem follows directly from Lemma 3.

In applications (e.g. actuarial mathematics) Theorem 1 fits quite well, because there changes of measure are considered, so that the process X stays even within a certain class, e.g. the class of compound Poisson processes, with respect to the new measure (cf. [3] for an economic interpretation). Nevertheless one can of course ask: if there exists no equivalent local martingale measure for X so that X remains an PII with respect to the new measure, is there an equivalent local martingale measure at all? The answer is «no», as is shown by the subsequent theorem.

Theorem 2. *Let X be as in Proposition 1 and assume additionally $\mathcal{M}^e(X) \neq \emptyset$, then $\mathbf{P}^H$ exists and is equivalent to $\mathbf{P}$.*

P r o o f. As existence is already clear by Proposition 1, we have only to show equivalence. Assuming the contrary, we get $h^0(\frac{1}{2}, \mathbf{P}, \mathbf{P}^H)_\infty = \infty$

(local equivalence follows from Proposition 1), but on the other hand there has to exist a measure $\mathbf{P}' \in \mathcal{M}^e(X)$, and *a fortiori* $\mathbf{P}' \ll \mathbf{P}$ has to hold. This implies by Lemma 2

$$h^0\left(\frac{1}{2}, \mathbf{P}, \mathbf{P}'\right)_\infty < \infty = h^0\left(\frac{1}{2}, \mathbf{P}, \mathbf{P}^H\right)_\infty \quad \mathbf{P}'\text{-a.s.}$$

But this is prohibited by the Proposition 2 below.

In order to formulate Proposition 2 we may define a very strong concept of minimality, adapted to the need of Theorem 2.

Definition 1. We say that $\tilde{\mathbf{P}}$ is a solution of

$$\min_{\mathbf{P}' \in \mathcal{M}^{\text{loc}}(X)} h^0\left(\frac{1}{2}, \mathbf{P}, \mathbf{P}'\right)_\infty,$$

if there exists no pair $(F, \mathbf{P}')$, where F is a set in $\mathcal{F}_\infty$ and $\mathbf{P}'$ a measure in $\mathcal{M}^{\text{loc}}(X)$ with $\mathbf{P}'(F) > 0$, so that $h^0(\frac{1}{2}, \mathbf{P}, \mathbf{P}')_\infty < h^0(\frac{1}{2}, \mathbf{P}, \tilde{\mathbf{P}})_\infty$ holds on F .

We can now formulate

Proposition 2. Let X be as in Proposition 1. Then $\mathbf{P}^H$ is a solution of $\min_{\mathbf{P}' \in \mathcal{M}^{\text{loc}}(X)} h^0(\frac{1}{2}, \mathbf{P}, \mathbf{P}')_\infty$.

Proof. Assuming the contrary yields by the definition above the existence of a measure $\mathbf{P}'$ in $\mathcal{M}^{\text{loc}}(X)$ and a set $F \in \mathcal{F}_\infty$ with $\mathbf{P}'(F) > 0$ and $h^0(\frac{1}{2}, \mathbf{P}, \mathbf{P}')_\infty < h^0(\frac{1}{2}, \mathbf{P}, \mathbf{P}^H)_\infty$ on F . Denoting the $(\mathcal{P} \otimes \mathcal{B})$ -measurable function, respectively the predictable process corresponding to $\mathbf{P}'$ by Y' , respectively β' and using the definition

$$G_t = \left\{ \omega \mid \frac{1}{8}(\beta'_t)^2 c + \frac{1}{2} \int (\sqrt{Y'_t} - 1)^2 K(dx) < \frac{1}{8}(\beta_t^H)^2 c + \frac{1}{2} \int (\sqrt{Y_t^H} - 1)^2 K(dx) \right\} \in \mathcal{F}_t,$$

we may define a set $\mathcal{T} = \{t \mid \mathbf{P}'(G_t) > 0\}$. $\mathcal{T}$ has strictly positive dA_t -measure by Fubini. Noting that

$$0 = b + c\beta'_t + \int x(Y'_t - 1) K(dx)$$

holds on sets $H_t \in \mathcal{F}_t$ with $\mathbf{P}'(H_t) = 1$ for dA_t -almost all t , we may define

$$Y^h(t, x) = \begin{cases} Y'(t, x, \bar{\omega}) & \text{for } t \in \mathcal{T}, \\ Y^H(t, x) & \text{otherwise,} \end{cases}$$

respectively

$$\beta^h(t) = \begin{cases} \beta'(t, \bar{\omega}) & \text{for } t \in \mathcal{T}, \\ \beta^H(t) & \text{otherwise} \end{cases}$$

for some $\bar{\omega} \in G_t \cap H_t$. Obviously

$$0 = b + c\beta_t^h + \int x(Y_t^h - 1) K(dx) \quad dA_t\text{-a.s.},$$

$$h^0\left(\frac{1}{2}, \mathbf{P}, \mathbf{P}^h\right)_\infty < h^0\left(\frac{1}{2}, \mathbf{P}, \mathbf{P}^H\right)_\infty$$

holds, which is a contradiction to the fact that (Y^H, β^H) is a solution of the extremal problem of Proposition 1.

R e m a r k. We note that the assumption of uniformly bounded jumps and the condition (D) are only sufficient for the validity of Theorem 2. Nevertheless in Section 5 we give examples for the failure of the theorem, if these conditions do not hold.

After proving the existence of a minimizer, we can now give the explicit form of the solution. We only deal with the case $c > 0$, leaving $c = 0$ as an easy exercise.

Corollary 1. *The solution of the extremal problem (1) is given by*

$$Y^H(x) = \frac{1}{(1 + \rho x)^2}, \quad \beta^H = -\frac{b}{c} - \frac{1}{c} \int (Y^H - 1) x K(dx),$$

where ρ is fixed by

$$\rho = \frac{b}{2c} + \frac{1}{2c} \int \left(\frac{x}{(1 + \rho x)^2} - x \right) K(dx). \quad (6)$$

P r o o f. From (2) we conclude

$$\begin{aligned} \beta^H &= -\frac{b}{c} - \frac{1}{c} \int (Y^H - 1) x K(dx), \\ \frac{1}{\sqrt{Y^H(x)}} &= 1 + \left(\frac{b}{2c} + \frac{1}{2c} \int (Y^H - 1) s K(ds) \right) x. \end{aligned}$$

Because we have already proved the existence of a strict positive solution, we can now square the second equation, yielding the result.

R e m a r k. The density process Z^H is given by (Section 2) $Z^H = \mathcal{E}(N)$ with $N = \beta^H X^c + (Y^H - 1) * (\mu^X - \nu)$.

4. A property of the minimal martingale measure. Several properties of the minimal martingale measure $\mathbf{P}^{\min}$ have been considered in [5] and [11]. The aim of this section is to investigate the position of $\mathbf{P}^{\min}$ in our setting. We shall prove that $\mathbf{P}^{\min}$ minimizes a functional, which can be seen as an approximation of $h^0(\frac{1}{2})_\infty$. In this section we drop the assumption of quasileftcontinuity, because the technical difficulties are not so big as they would have been in Section 3. Hence we get in analogy to Lemma 1.

Lemma 4. *For X satisfying the conditions of Assumption 1, but without quasileftcontinuity, and for $\mathbf{P}' \in \mathcal{M}^{\text{loc}}(X)$, $h^0(\frac{1}{2})$ is of the form*

$$h^0\left(\frac{1}{2}, \mathbf{P}, \mathbf{P}'\right)_t = \frac{1}{8} \beta^2 c \cdot A_t + \frac{1}{2} (\sqrt{Y} - 1)^2 * \nu_t + \frac{1}{2} \sum_{s \leq t} (\sqrt{1 - a_s} - \sqrt{1 - a'_s})^2.$$

In addition $Z = \mathcal{E}(N)$ on Δ , $Z = 0$ otherwise with

$$N = \beta \cdot X^c + \left(Y - 1 + \frac{\hat{Y} - a}{1 - a} 1_{\{a < 1\}} \right) * (\mu^X - \nu). \quad (7)$$

Here Δ is defined as after Lemma 1, and Z denotes the density process of $\mathbf{P}'$ with respect to $\mathbf{P}$. We also use the notation

$$a_t = \nu(\{t\} \times \mathbf{R}), \quad a'_t = \nu'(\{t\} \times \mathbf{R}) = \hat{Y} = \int Y(t, x) \nu(\{t\} \times dx).$$

P r o o f. The lemma directly follows from IV.3.35, III.5.19, III.4.34 and IV.3.32 in [9].

Taylor expansion around $(Y, a') = (1, a)$ yields the functional

$$h^{0,a}\left(\frac{1}{2}, \mathbf{P}, \mathbf{P}'\right)_t = \frac{1}{8} \beta^2 c \cdot A_t + \frac{1}{8} (Y - 1)^2 * \nu_t + \frac{1}{8} \sum_{s \leq t} \frac{(a'_s - a_s)^2}{1 - a_s} 1_{\{a_s < 1\}}. \quad (8)$$

As we want to study the minimal martingale measure, we adopt now the setting of M. Schweizer in [11].

Definition 2. Let X be as in Assumption 1, but not necessarily quasileftcontinuous. Additionally let the local martingale part M of the Doob–Meyer decomposition fulfill $M \in \mathcal{H}_{\text{loc}}^2$ (cf. [9, I.1.45] for the notation), and let $B \ll \langle M \rangle$ with density process λ_t . We call the local martingale measure $\mathbf{P}^{\min}$ defined by

$$\frac{d\mathbf{P}^{\min}}{d\mathbf{P}} \Big|_{\mathcal{F}_t} = Z_t^{\min} = \mathcal{E}(-\lambda \cdot M)_t$$

the minimal local martingale measure, if it exists as a measure (and not only as a signed measure).

Our assertion is

Theorem 3. *If $\mathbf{P}^{\min}$ exists in the sense of Definition 2, and the corresponding density is strictly positive, then $\mathbf{P}^{\min}$ is a minimizer of $h_t^{0,a}$ among all local martingale measures with strictly positive densities in $\mathcal{H}_{\text{loc}}^2$.*

P r o o f. By Theorem 1 of [11] every strictly positive local martingale measure with density Z' in $\mathcal{H}_{\text{loc}}^2$ can be written as $Z' = \mathcal{E}(N) = \mathcal{E}(-\lambda \cdot M + L)$, where $L \in \mathcal{H}_{\text{loc}}^2$ is strongly orthogonal to M . Therefore $\langle N \rangle_t = \langle \lambda \cdot M \rangle_t + \langle L \rangle_t$, which means that the minimal measure is a minimizer of the process

$\langle N \rangle$ in the class of strictly positive local martingale measures, where N is defined by $Z = \mathcal{E}(N)$, and Z is the density process of $\mathbf{P}'$ with respect to $\mathbf{P}$.

It remains to prove that $\langle N \rangle_t = 8h^{0,a}(\frac{1}{2}, \mathbf{P}, \mathbf{P}')_t$, and we use equations (7) and (8) to show this. As calculations are even simpler if $a_s = 1$, we stick to the case $a_s < 1$ for all $s \leq t$, and get

$$\begin{aligned}\langle N \rangle_t &= \beta^2 \cdot \langle X^c \rangle_t + \left\langle \left(Y - 1 + \frac{\hat{Y} - a}{1 - a} \right) * (\mu^X - \nu) \right\rangle_t \\ &= \beta^2 c \cdot A_t + \langle W * (\mu^X - \nu) \rangle_t \\ &= \beta^2 c \cdot A_t + (W - \widehat{W})^2 * \nu_t + \sum_{s \leq t} (1 - a_s) (\widehat{W}_s)^2,\end{aligned}$$

where we have used II.1.34 of [9] and the definition $W = (Y - 1 + (\hat{Y} - a)/(1 - a))$. A simple calculation gives $\widehat{W}_s = (\hat{Y}_s - a_s)/(1 - a_s)$, and plugging this in the expression above yields

$$\begin{aligned}\langle N \rangle_t &= \beta^2 c \cdot A_t + (Y - 1)^2 * \nu_t + \sum_{s \leq t} (1 - a_s) \frac{(\hat{Y}_s - a_s)^2}{(1 - a_s)^2} \\ &= \beta^2 c \cdot A_t + (Y - 1)^2 * \nu_t + \sum_{s \leq t} \frac{(\hat{Y}_s - a_s)^2}{(1 - a_s)} = 8h^{0,a}\left(\frac{1}{2}, \mathbf{P}, \mathbf{P}'\right)_t,\end{aligned}$$

which concludes our proof.

5. Examples. Example 1. Our first example shows the difference between $\mathbf{P}^H$ and $\mathbf{P}^{\min}$, if the process X is a combination of Brownian motion and a standard Poisson process. Let $X_t = N_t + W_t + 2t$, where W is a standard Brownian motion and N a standard Poisson process, $t \in [0, 1]$. The Doob–Meyer decomposition is given by $X = M + B$, with $M = N - t + W$ and $B = 3t$. An easy computation yields $\langle M, M \rangle = 2t$, and we get for the minimal measure

$$Z_t^{\min} = \mathcal{E}\left(-\frac{3}{2}(N_t - t + W_t)\right) = \exp\left\{-\frac{3}{2}(N_t - t + W_t) - \frac{9}{4}t\right\} \prod_{s \leq t} \left(1 - \frac{3}{2} \Delta N_s\right),$$

which is obviously only a signed measure.

For the calculation of $\mathbf{P}^H$ we need the continuous martingale part $X^c = W$, which yields $C = t$. We set $A = t$ and conclude that $c = 1$, respectively $b = 3$, holds. Furthermore our compensated jump measure is given by $\nu(dt, dx) = dt \otimes \varepsilon_1(dx)$ and consequently $K(dx) = \varepsilon_1(dx)$ has to hold. Using this and the abbreviation $\hat{Y}^H = Y^H(t, 1)$ in (6), we can infer $\rho = \frac{3}{2} + \frac{1}{2}(1/(1 + \rho)^2 - 1)$ or

$$\frac{1}{2} \hat{Y}^H - \frac{1}{\sqrt{\hat{Y}^H}} = -2, \quad (9)$$

which has the unique solution $(\hat{Y}^H, \beta^H) \approx (0.2242, -2.2242)$. Finally the density process is given by

$$Z^H = \mathcal{E}(N) \quad \text{with } N \approx -2.2242 W - 0.7758(N - t),$$

which is of course positive and yields a measure $\mathbf{P}^H$, which is equivalent to the original measure.

R e m a r k. Considering $\mathbf{P}^{\min}$ instead of $\mathbf{P}^H$ would change equation (9) to $2\hat{Y}^{\min} = -1$. In contrast of (9) the left hand side viewed as function of $\hat{Y}^{\min}$ is now not a bijective function from $\mathbf{R}_+$ onto $\mathbf{R}$. Hence we can not expect a positive solution $\hat{Y}^{\min}$ for all possible right hand sides.

E x a m p l e 2. Our second example shows that Theorem 2 will not hold, if we drop our assumption of uniformly bounded jumps. We consider a process X , which has only one two-sided unbounded jump at time $t = 1$. In that case the process is not quasileftcontinuous, but we believe that with a little more effort one could construct a quasileftcontinuous example. But in order to keep technicalities as little as possible and to see the idea more clearly, we remain with this very simple example.

Let

$$X_t = \frac{\omega - 0.25}{\sqrt{1 - 4\omega^2}} 1_{[1, \infty)}(t) \quad \text{with } \omega \in \Omega = \left(-\frac{1}{2}, \frac{1}{2}\right)$$

and let $\mathbf{P}$ be the Lebesgue measure on Ω . Our claim is that — in contrast to above — now the minimum of $h^0(\frac{1}{2})$ is not attained. The expression for $h^0(\frac{1}{2}, \mathbf{P}, \mathbf{P}')$ is given by $\mathbf{E}_{\mathbf{P}}(1 - \sqrt{Z})$ (cf. IV.1.69 in [9]), where $Z = d\mathbf{P}'/d\mathbf{P}$ holds. Now the extremal problem is given by

$$\min_{\mathbf{P}'} \int_{\Omega} (1 - \sqrt{Z}) d\mathbf{P}, \quad \int_{\Omega} Z d\mathbf{P} = 1, \quad \int_{\Omega} X Z d\mathbf{P} = 0.$$

Our assertion is $\forall \varepsilon > 0 \exists Z^\varepsilon \in \mathcal{M}^\varepsilon: h^0(\frac{1}{2}, Z^\varepsilon) < \varepsilon$. Of course h^0 is strictly positive for all Z 's obeying the constraints.

Let us define a family of density processes via

$$Z^{\delta, \sigma}(\omega) = \frac{1 - \delta^{1-\sigma}}{1 - \delta} 1_{(-1/2, 1/2-\delta)} + \delta^{-\sigma} 1_{[1/2-\delta, 1/2)}, \quad \delta \in (0, 1), \sigma \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

Obviously $\int_{\Omega} Z^{\delta, \sigma} d\mathbf{P} = 1 \forall \delta \in (0, 1), \forall \sigma \in [\frac{1}{4}, \frac{3}{4}]$ holds, and an elementary but lengthy computation yields

$$\int_{\Omega} X Z^{\delta, \sigma} d\mathbf{P} = -\frac{\pi}{8} + \frac{1}{4} \delta^{1/2-\sigma} + O(\delta^{1-\sigma}),$$

where O is a Landau symbol as usual. It is easy to see that we have $\int_{\Omega} X Z^{\delta, 1/4} d\mathbf{P} < 0$ and $\int_{\Omega} X Z^{\delta, 3/4} d\mathbf{P} > 0$ for δ small enough, and our conclusion is

$$\exists \sigma_0(\delta) \in \left(\frac{1}{4}, \frac{3}{4}\right): \int_{\Omega} X Z^{\delta, \sigma_0} d\mathbf{P} = 0 \quad \text{for } \delta \text{ small enough,}$$

because of continuity in σ . So both constraints are realized by Z^{δ, σ_0} , and one easily computes

$$\int_{\Omega} \sqrt{Z^{\delta, \sigma_0}} d\mathbf{P} > 1 - \frac{1}{2} \delta^{1/4} + o(\delta^{1/4}).$$

Choosing $\delta = (2\varepsilon)^4$ completes our proof.

Note that in our example the process X has p -th moments for $p < 2$. It would be interesting to investigate the possibility of constructing counterexamples with higher moments.

Example 3. By this example we want to show that a too slow decrease of the transition kernel K on the boundary of the state space causes the same troubles as in Example 2, namely that the minimum in our extremal problem is not attained.

Consider a fixed time t , where $c_t = 0$ holds, let the transition kernel be given by $K(dx) = f(x) dx$, where dx denotes Lebesgue measure on $(-\frac{1}{2}, \frac{1}{2})$, let f be

$$f(x) = \left(x + \frac{1}{2}\right)^2 1_{(-1/2, 0]}(x) + \frac{23}{12} 1_{(0, -1/2]}(x),$$

and assume further that $b_t = \frac{45}{192}$ holds. The values of c_t and b_t are chosen to keep the computation easier. This yields the extremal problem

$$\min_Y \frac{1}{2} \int (\sqrt{Y} - 1)^2 f(x) dx, \quad \int xY f(x) dx = 0,$$

respectively the Euler–Lagrange equation

$$\frac{1}{2} - \frac{1}{2\sqrt{Y}} - \lambda x = 0.$$

If this equation is solvable, the solution must be of the form $Y = (1 - 2\lambda x)^2$, and the constraint gives us

$$\int_{-1/2}^0 \frac{x(x + \frac{1}{2})^2}{(1 - 2\lambda x)^2} dx + \frac{23}{12} \int_0^{1/2} \frac{x}{(1 - 2\lambda x)^2} dx = h(\lambda) = 0.$$

In order to keep $h(\lambda)$ finite, λ must vary in the interval $[-1, 1)$. An easy monotony argument reveals that $h(\lambda) > 0$ in $[-1, 1)$ and therefore the minimum is again not attained.

In order to see better, why the minimum is not attained in this extremal problem, we construct a minimal sequence, by demanding the additional constraint $Y \leq T_n \in \mathbf{R}_+$, solve this minimal problem and let T_n tend to ∞ . If $\{Y_n\}$ is not K -uniformly integrable, we are not able to extract a sequence $\{Y_{n_k}\}$ converging weakly in $L^1((-\frac{1}{2}, \frac{1}{2}), K)$ (Dunford–Pettis theorem) and the minimum is not attained.

The Kuhn–Tucker conditions for the new extremal problems are

$$\frac{1}{2} - \frac{1}{2\sqrt{Y_n}} - \lambda x + \mu(x) = 0, \quad \int \mu(x)(Y_n - T_n) f(x) dx = 0,$$

$$\mu(x) \geq 0, \quad \int x Y_n f(x) dx = 0, \quad Y_n \leq T_n.$$

A lengthy but elementary computation yields the result

$$Y_n = T_n 1_{A_n}(x) + 1_{(-1/2, 1/2) \setminus A_n}(x) \frac{1}{(1 - 2\lambda_n x)^2},$$

where $A_n = \{x \mid x \in (-\frac{1}{2}, -\frac{1}{2} + \rho/\sqrt[3]{T_n})\}$,

$$\lambda_n = -1 - 2\rho \frac{1}{\sqrt[3]{T_n}} + o\left(\frac{1}{\sqrt[3]{T_n}}\right), \quad \rho = \sqrt[3]{\frac{46 \ln 2 - 26}{16}} + o(1).$$

As is easily seen the sequence $\{Y_n\}$ is not K -uniformly integrable. Taking the same values for c_t, b_t and $K_t(dx)$ on a set $R \subset \mathbf{R}_+$ with $dA_t(R) > 0$ yields an example where the minimum of $h^0(\frac{1}{2})$ is not attained.

Acknowledgement. I would like to thank W. Schachermayer for many helpful discussions and an anonymous referee for suggesting Theorem 2 and a clearer proof of Theorem 3.

REFERENCES

1. Ansel J. P., Stricker C. Lois de martingale, densités et decomposition de Föllmer–Schweizer. — Ann. Inst. H. Poincaré, 1992, v. 28, p. 375–392.
2. Ansel J. P., Stricker C. Unicité et existence de la loi minimale. — Lecture Notes in Math., 1993, v. 1557, p. 22–29.
3. Delbaen F., Haezendonck J. A martingale approach to premium calculation principles in an arbitrage free market. — Insurance: Math. Econ., 1989, v. 8, p. 269–277.
4. Delbaen F., Schachermayer W. A general version of the fundamental theorem of asset pricing. — Math. Ann., 1994, v. 300, № 3, p. 463–520.
5. Föllmer H., Schweizer M. Hedging of contingent claims under incomplete information. — In: Applied Stochastics Monographs, (Stochastics Monographs, v. 5). Ed. by M. H. A. Davis and R. J. Elliot. London: Gordon and Breach, 1991, p. 389–414.
6. Gerber H. U. An Introduction to Mathematical Risk Theory. Huebner Foundation Monograph Series № 8.
7. Girsanov I. V. Lectures on Mathematical Theory of Extremum Problems. Berlin–Heidelberg: Springer-Verlag, 1972. 136 p.
8. Harrison J. M., Pliska S. R. Martingales and stochastic integrals in the theory of continuous trading. — Stochastic Process. Appl., 1981, v. 11, № 3, p. 215–260.
9. Жакоб Ж., Ширяев А. Н. Предельные теоремы для случайных процессов. М.: Физматлит, 1994. Т. 1, 544 с.; Т. 2, 368 с.
10. Кабанов Ю. М., Крамков Д. О. Большие финансовые рынки: асимптотический арбитраж и континуальность. — Теория вероятн. и ее примен., 1994, т. 39, в. 1, с. 222–228.
11. Schweizer M. On the minimal martingale measure and the Föllmer–Schweizer decomposition. — Stochastic Anal. Appl., 1995, v. 13, № 5, p. 573–599.
12. Struwe M. Variational Methods. Berlin: Springer-Verlag, 1990.

Поступила в редакцию
15.IX.1998