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IS THERE A PREDICTABLE CRITERION
FOR MUTUAL SINGULARITY OF
TWO PROBABILITY MEASURES
ON A FILTERED SPACE?

Вопрос о нахождении предсказуемых критериев абсолютной непрерывности и взаимной сингулярности двух процессов плотности на фильтрованном вероятностном пространстве широко изучен, например, в монографии Ж. Жакода и А. Н. Ширяева [9]. Тогда как проблема абсолютной непрерывности представлена там в полной общности, в том, что касается взаимной сингулярности, остается открытым одно техническое затруднение [9, с. 210 англ. изд.]: «Мы не знаем, возможно ли вывести *предсказуемый* критерий (необходимые и достаточные условия) для $P'_T \perp P_T, \dots$ ». Оказывается, что на этот вопрос, поставленный в [9], который мы выбрали также для заглавия нашей статьи, есть два ответа: с негативной стороны мы приводим легкий пример, показывающий, что в общем случае ответом является «не существует», даже когда мы используем довольно широкую интерпретацию понятия «предсказуемый критерий». Трудность вызвана тем, что процесс плотности вероятностной меры P по отношению к другой мере P' может внезапно скакнуть в нуль.

С положительной стороны, мы можем охарактеризовать множество, где мера P' становится сингулярной относительно P , — при условии, что это происходит не внезапно, а непрерывным образом, — как множество, где расходится процесс Хеллингера, что несомненно есть «предсказуемый критерий». Эта теорема обобщает результаты, приведенные в книге [9].

Ключевые слова и фразы: непрерывность и сингулярность вероятностных мер, процессы Хеллингера, стохастические интегралы, моменты остановки.

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1. Introduction. We adopt the notation of [9], which means that we are given two fixed probability measures P, P' on a filtered space $(\Omega, (\mathcal{F}_t)_{t \in \mathbf{R}^+}, \mathcal{F})$ with right continuous filtration and $\mathcal{F} = \vee_t \mathcal{F}_t$, $Q = (P + P')/2$, z and z' denote the density processes of P and P' , relative to Q . We define the process $(Y_t)_{t \geq 0}$ by $Y_t = \sqrt{z_t z'_t}$ and let $h = h(\frac{1}{2})$ denote the Hellinger process of order $\frac{1}{2}$, i.e., the predictable increasing process h such that $h_0 = 0$ and

$$M = Y + Y_- \bullet h \quad (1)$$

is a Q -martingale. For a stopping time T we denote by P_T and P'_T the restrictions of P and P' to $\mathcal{F}_T$. We shall investigate the following question, which has been considered by several authors in the past, and which is a main theme in the book [9]:

1. Problem. *Under which conditions can we assert that $P'_T \ll P_T$ or $P'_T \perp P_T$? More generally, can we find a Hahn-decomposition of Ω into two sets, such that P'_T is absolutely continuous (respectively singular) with respect to P_T on these sets?*

The answer to this question should be in terms of a «predictable criterion». By this concept we mean that the answer should be in terms of the values of a predictable process, such as the Hellinger process h , evaluated at time T .

Before proceeding to answering this question we pause for some remarks: In order to avoid irrelevant complications at $t = 0$ we suppose that $P_0 \sim P'_0$. We define the stopping time S as the first moment when either z or z' vanishes,

$$S = \inf\{t: z_t = 0 \text{ or } z'_t = 0\}. \quad (2)$$

Noting that (1) determines the Hellinger process h only up to time S , we define h to be constant after S , i.e., we consider the «Hellinger process in the strict sense» in the terminology of [9]. We also introduce the Hellinger process $h(0)_t$ of order 0 as the compensator of the process $\mathbb{I}_{S, \infty}$.

Now we review the known results: there is a very satisfactory answer to our problem as regards the question of absolute continuity:

2. Theorem ([9, Theorem IV 2.6]). *In the above setting we have, for every stopping time T ,*

$$P'_T \ll P_T \iff P'(G_T) = 1,$$

where $G_T = \{h_T < \infty\} \cap \{h(0)_T = 0\}$.

Note that the set G_T is defined in terms of the values of the two predictable processes h and $h(0)$ at time T . To compare this result with the situation of mutual singularity below we make the following (trivial) reformulation: denote by H the predictable, increasing, $[0, \infty]$ -valued process $H_t = h_t + \infty \cdot h(0)_t$ (where $\infty \cdot 0 = 0$). Then

$$P'_T \ll P_T \iff P'(H_T < \infty) = 1.$$

For the question of mutual singularity (or, more generally, for the question of the Hahn-decomposition) the situation is more subtle, the difficulty arising from the fact that z_t may jump to zero as we shall presently see. Of course, we can always get rid of this difficulty by shifting it into the assumptions of a theorem: we thus obtain the subsequent result which directly follows from [9, Lemma IV 2.12 a), b) and d)]:

3. Theorem. *In the above setting suppose in addition that*

$$P'_T[S \leq T \text{ and } z_{S-} > z_S = 0] = 0.$$

Then the restriction of P'_T to $\{h_T < \infty\}$ is absolutely continuous with respect to P_T while the restriction of P'_T to $\{h_T = \infty\}$ is singular with respect to P_T .

In particular $P'_T \perp P_T \iff P'(h_T = \infty) = 1$.

Our main task in this note is to analyze what we still can say when we do not use the simplifying assumption that z_t is not allowed to jump to zero; unfortunately there is no hope for a complete analogue to Theorem 2, as the following elementary example, which will be constructed in Section 3 below, shows.

4. Example. There is a filtered space $(\Omega, (\mathcal{F}_t)_{t=0}^2, \mathcal{F})$ equipped with two probability measures P and P' with the following property:

There is no $[0, \infty]$ -valued predictable process H such that, for every stopping time T , $P'_T \perp P_T \iff P'(H_T = \infty) = 1$.

In fact Ω may be chosen to consist only of 4 elements.

Despite this discouraging example we can formulate an interesting positive result, where we shift the problem, that z_t may jump to zero from the assumption (as in Theorem 3 above) into the conclusion of the theorem:

5. Theorem. *Under the assumption that $P_0 \sim P'_0$ we have, for every stopping time T ,*

$$\{S \leq T, z_{S-} = 0\} = \{h_T = \infty\} \quad P'\text{-a.s.} \quad (3)$$

The set on the left-hand side may be interpreted as the set where P'_T is singular with respect to P_T , but in such a way that, as $t \nearrow S \leq T$, this singularity was not obtained by a «sudden jump», but rather in a continuous way. The assertion of the theorem is that — even in the presence of jumps of z_t to zero — it is precisely this set which is characterized by the divergence of the Hellinger process.

R e m a r k. We tried to formulate Theorem 5 in a manner that suits best for comparison with the results in [9], but there are other ways to state it. Actually, the assertion of Theorem 5 is equivalent to the assertion

$$\{z_{S-} = 0\} = \{h_S = \infty\} \quad P'\text{-a.s.} \quad (3')$$

To verify (3) $\Rightarrow$ (3'), we take $T := S$, and (3') $\Rightarrow$ (3) follows from

$$\begin{aligned}\{S \leq T, z_{S-} = 0\} &= \{S \leq T, h_S = \infty\} = \{S \leq T, h_T = \infty\} \\ &= \{h_T = \infty\} \quad P'\text{-a.s.},\end{aligned}$$

where we used the constancy of h from time S on and the fact that $h_T < \infty$ for $T < S$, P' -a.s., which follows e.g. from [9, Theorem IV 1.18].

Theorem 5 and Example 4 answer the question raised in [9] right after Corollary IV 2.8 and also sharpen the assertions of [9, Lemma IV 2.12]. Our proof is quite different from the methodology used in [9]: it uses a close monitoring of those paths of z_t , for which $z_{S-} = 0$, and an extension of the Borel–Cantelli lemma due to P. Lévy. Although the proof is elementary it is somewhat laborious and technical.

The paper is organized as follows. In Section 2 we give the proof of Theorem 5 and in Section 3 we construct Example 4.

2. Proof of Theorem 5. The inclusion $\{0 < S \leq T, z_{S-} = 0\} \supseteq \{h_T = \infty\}$, P' -a.s. is proved in [9, Lemma IV 2.12 a]. The reverse inclusion can be deduced from [9, Lemma IV 2.12 b+d] in the case, where z is not allowed to jump to 0 up to time T , i.e., under the assumption of Theorem 3

$$P'_T[0 < S \leq T, z_{S-} > z_S = 0] = 0. \quad (4)$$

It remains to prove

$$\{0 < S \leq T, z_{S-} = 0\} \subseteq \{h_T = \infty\} \quad P'\text{-a.s.} \quad (5)$$

without assuming (4). As our proof will rely heavily on the fact that the local martingale M given by (1) is a Q -martingale, we are aiming at Q -almost sure results. So the first thing to do is replace (5) by

$$\{0 < S \leq T, z_{S-} = 0 \text{ or } z'_{S-} = 0\} \subseteq \{h_T = \infty\} \quad Q\text{-a.s.}, \quad (5')$$

which is indeed equivalent to (5). We need the following lemma:

Lemma 1. *Let $(A_t)_{t \in [0, \infty[}$ be an adapted increasing process on a filtered space $(\Omega, (\mathcal{F}_t), \mathcal{F}, \mathbf{P})$, $(W_n)_{n \in \mathbf{N}}$ an increasing sequence of stopping times, such that for all $n \in \mathbf{N}$*

$$\mathbf{P}[W_{n-1} < \infty, \mathbf{P}[A_{W_n} - A_{W_{n-1}} \geq \alpha \mid \mathcal{F}_{W_{n-1}}] < \alpha] \leq 2^{-n}, \quad (6)$$

for some $\alpha > 0$. Then

$$\bigcap_{n=1}^{\infty} \{W_n < \infty\} \subseteq \{A_{\infty} = \infty\} \quad \mathbf{P}\text{-a.s.} \quad (7)$$

P r o o f. Let $B_n = \{W_n < \infty\}$, $B = \bigcap_{n=1}^{\infty} \{W_n < \infty\}$, $E_n = \{A_{W_n} - A_{W_{n-1}} \geq \alpha\}$ and $\xi_n = \mathbf{P}[E_n \mid \mathcal{F}_{W_{n-1}}]$. If $\mathbf{P}[B] = 0$, (7) is trivially satisfied, so we assume $\mathbf{P}[B] = a > 0$. Since $B_n \searrow B$, we have

$$\mathbf{P}[\xi_n < \alpha \mid B] \leq \frac{\mathbf{P}[\xi_n < \alpha, B_n]}{\mathbf{P}[B]} \leq a^{-1} 2^{-n}.$$

The Borel–Cantelli lemma yields $\mathbf{P}[\xi_n \geq \alpha \text{ i.o.} \mid B] = 1$, therefore $B \subseteq \{\sum_{k=1}^n \xi_k \rightarrow \infty\}$ $\mathbf{P}$ -a.s. By Levy’s extension of the Borel–Cantelli lemmas (cf. [12, p. 518] or [13, Theorem 12.15]) $\{\sum_{k=1}^{\infty} \xi_k = \infty\} = \{\sum_{k=1}^{\infty} \mathbb{I}_{E_k} = \infty\}$ $\mathbf{P}$ -a.s., and the observation $A_{W_n} \geq \alpha \sum_{k=1}^n \mathbb{I}_{E_k}$ completes the proof.

R e m a r k. Given $\alpha > 0$ and a nonnegative random variable X , defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, we point out the following simple implications $\mathbf{E}[X \wedge 2] \geq 3\alpha \implies \mathbf{P}[X \wedge 2 \geq \alpha] \geq \alpha \implies \mathbf{E}[X \wedge 2] \geq \alpha^2$, which justify replacing hypothesis (6) by the equivalent hypothesis

$$\mathbf{P}\left[W_{n-1} < \infty, \mathbf{E}[(A_{W_n} - A_{W_{n-1}}) \wedge 2 \mid \mathcal{F}_{W_{n-1}}] < \alpha'\right] \leq 2^{-n}, \quad \text{for some } \alpha' > 0. \quad (6')$$

We now proceed to the proof of Theorem 5. First we define an increasing sequence of stopping times $(U_n)_{n \in \mathbf{N}}$ by

$$U_0 = 0, \quad U_n = \inf \left\{ t: U_{n-1} \leq t \leq T, z_t \neq 0, z'_t \neq 0, \right. \\ \left. \left\{ \frac{z_t}{z_{U_{n-1}}}, \frac{z'_t}{z'_{U_{n-1}}} \right\} \not\subset \left] \frac{1}{2}, 2 \right[\right\}, \quad n \geq 1,$$

where $\inf \emptyset := \infty$. This definition ensures, that

$$0 \leq Y_{\tau-} \leq 2Y_{U_{n-1}} \quad Q\text{-a.s. for stopping times } \tau \in]U_{n-1}, U_n] \quad (8)$$

and that on the set $\{U_n < \infty\}$ we have

$$\frac{z_{U_n}}{z_{U_{n-1}}} \leq \frac{1}{2} \quad \text{or} \quad \frac{z_{U_n}}{z_{U_{n-1}}} \geq 2 \quad \text{or} \quad \frac{2 - z_{U_n}}{2 - z_{U_{n-1}}} \leq \frac{1}{2} \quad \text{or} \quad \frac{2 - z_{U_n}}{2 - z_{U_{n-1}}} \geq 2.$$

Therefore $z_{U_n} \in C(z_{U_{n-1}})$ Q -a.s. on the set $\{U_n < \infty\}$, where

$$C(x) := \left] 0, \frac{x}{2} \vee 2x - 2 \right] \cup \left[2x \wedge 1 + \frac{x}{2}, 2 \right]. \quad (9)$$

If $Q[\bigcap_{n=1}^{\infty} \{U_n < \infty\}] > 0$, then for any $\mu_0 > 0$

$$Q(z_{U_n} \in]0, \mu_0[\cup]2 - \mu_0, 2[\mid U_n < \infty) \longrightarrow 1, \quad \text{as } n \rightarrow \infty, \quad (10)$$

since on the set $\bigcap_{n=1}^{\infty} \{U_n < \infty\}$ the sequence $(z_{U_n})_{n \in \mathbf{N}}$ converges Q -a.s. to a random variable, which takes values in the two element set $\{0, 2\}$ (other

values are not possible by the definition of $(U_n)_{n \in \mathbb{N}}$, and since $Q[U_n < \infty] \longrightarrow Q[\cap_{n=1}^{\infty} \{U_n < \infty\}]$ as $n \rightarrow \infty$. In particular, we have

$$\{0 < S \leq T, z_{S-} = 0 \text{ or } z'_{S-} = 0\} = \bigcap_{n=1}^{\infty} \{U_n < \infty\} \quad Q\text{-a.s.}$$

so (5') and thus Theorem 5 will follow from Lemma 1, if we can establish the truth of the subsequent Lemma 2.

Lemma 2. *There exists $\gamma > 0$ such that*

$$Q[U_{n-1} < \infty, \mathbf{E}[(h_{U_n} - h_{U_{n-1}}) \wedge 2 \mid \mathcal{F}_{U_{n-1}}] < \gamma^2] \longrightarrow 0$$

as $n \rightarrow \infty$.

P r o o f. Suppose to the contrary that, for any $0 < \gamma < 1$, there is a constant $\beta = \beta(\gamma) > 0$ and for infinitely many n (depending on γ) an $\mathcal{F}_{U_{n-1}}$ -measurable set $E_{n-1} = E_{n-1}(\gamma) \subset \{U_{n-1} < \infty\}$, such that $Q[E_{n-1}] \geq \beta$ and on E_{n-1} we have

$$Q[h_{U_n} - h_{U_{n-1}} \geq \gamma \mid \mathcal{F}_{U_{n-1}}] < \gamma. \quad (11)$$

For given small $\mu_0 > 0$ (to be chosen later), we can and will assume by passing again to a subsequence and invoking (10), that

$$z_{U_{n-1}} \in]0, \mu_0[\cup]2 - \mu_0, 2[\quad (12)$$

on an $\mathcal{F}_{U_{n-1}}$ -measurable subset of E_{n-1} of Q -probability at least $\beta/2$, which we again denote by E_{n-1} .

We then define a sequence of stopping times $(V_n)_{n \in \mathbb{N}}$ by $V_n = \inf\{t: U_{n-1} \leq t, h_t - h_{U_{n-1}} \geq \gamma\}$, so that (11) implies

$$\mathbf{E}[\mathbb{1}_{\{V_n < U_n\}} \mid \mathcal{F}_{U_{n-1}}] < \gamma \quad Q\text{-a.s. on } E_{n-1}. \quad (13)$$

The definition of $(V_n)_{n \in \mathbb{N}}$ ensures, that

$$h_{V_n} - h_{U_{n-1}} < 2 \quad Q\text{-a.s.}, \quad (14)$$

since the jumps of h are bounded by 1 (see, e.g. [9, IV 1.30]).

In order to get grip of h , we employ the Doob-Meyer decomposition of the supermartingale Y given by (1) $Y = M - Y_- \bullet h$, where h is the Hellinger process of order $\frac{1}{2}$ in the strict sense and M is a uniformly integrable martingale (cf. [9, IV 1.18]). Taking the difference $Y_t - Y_{U_{n-1}}$ and dividing by $Y_{U_{n-1}}$, we formally arrive at

$$\frac{Y_t}{Y_{U_{n-1}}} = 1 + \frac{1}{Y_{U_{n-1}}} (M_t - M_{U_{n-1}}) - \left[\left(\frac{Y_-}{Y_{U_{n-1}}} \bullet h \right)_t - \left(\frac{Y_-}{Y_{U_{n-1}}} \bullet h \right)_{U_{n-1}} \right].$$

This can be looked upon as the Doob–Meyer decomposition of the supermartingale

$$Y_t^{(n)} := \mathbb{I}_{\{t \leq U_{n-1}\}} + \mathbb{I}_{\{t > U_{n-1}\}} \frac{Y_t}{Y_{U_{n-1}}}$$

on the set $\{t > U_{n-1}\}$, which we rewrite as $Y^{(n)} = M^{(n)} - Y_-^{(n)} \bullet h^{(n)}$, with $M_t^{(n)} := 1 + Y_{U_{n-1}}^{-1} (M_t - M_{U_{n-1}}) \mathbb{I}_{\{t > U_{n-1}\}}$ and $h_t^{(n)} := (h_t - h_{U_{n-1}}) \mathbb{I}_{\{t > U_{n-1}\}}$. The martingale $M^{(n)}$ is again uniformly integrable and starts at 1. In the sequel all expectations are taken with respect to Q . We are going to derive a contradiction in computing expectations at time $U_n \wedge V_n$ conditional on $\mathcal{F}_{U_{n-1}}$:

$$\mathbf{E}[Y_{U_n \wedge V_n}^{(n)} | \mathcal{F}_{U_{n-1}}] = \mathbf{E}[M_{U_n \wedge V_n}^{(n)} | \mathcal{F}_{U_{n-1}}] - \mathbf{E}[Y_-^{(n)} \bullet h_{U_n \wedge V_n}^{(n)} | \mathcal{F}_{U_{n-1}}].$$

From the Corollary below (which is a kind of «uniformly strict» Jensen inequality for the concave function $f(x) = \sqrt{x(2-x)}$) it follows that for properly chosen μ_0 (cf. (12)) there exists $\varepsilon > 0$ such that on the set E_{n-1} we have Q -a.s.

$$\mathbf{E}[Y_{U_n}^{(n)} | \mathcal{F}_{U_{n-1}}] \leq 1 - \varepsilon + \varepsilon Q[U_n = \infty | \mathcal{F}_{U_{n-1}}].$$

This yields

$$\begin{aligned} \mathbf{E}[Y_{U_n \wedge V_n}^{(n)} | \mathcal{F}_{U_{n-1}}] &= \mathbf{E}[Y_{U_n}^{(n)} | \mathcal{F}_{U_{n-1}}] - \mathbf{E}[Y_{U_n}^{(n)} \mathbb{I}_{\{V_n < U_n\}} | \mathcal{F}_{U_{n-1}}] \\ &\quad + \mathbf{E}[Y_{V_n}^{(n)} \mathbb{I}_{\{V_n < U_n\}} | \mathcal{F}_{U_{n-1}}] \\ &\leq 1 - \varepsilon + \varepsilon Q[U_n = \infty | \mathcal{F}_{U_{n-1}}] + 2\gamma, \end{aligned}$$

where the estimate of the third term employs (8) and (13).

Furthermore (8), (13) and (14) yield

$$\begin{aligned} \mathbf{E}[(Y_-^{(n)} \bullet h)_{U_n \wedge V_n} - (Y_-^{(n)} \bullet h)_{U_{n-1}} | \mathcal{F}_{U_{n-1}}] &\leq 2\mathbf{E}[(h_{U_n \wedge V_n} - h_{U_{n-1}}) | \mathcal{F}_{U_{n-1}}] \\ &= 2\mathbf{E}[(h_{V_n} - h_{U_{n-1}}) \mathbb{I}_{\{V_n < U_n\}} | \mathcal{F}_{U_{n-1}}] \\ &\quad + 2\mathbf{E}[(h_{U_n} - h_{U_{n-1}}) \mathbb{I}_{\{V_n \geq U_n\}} | \mathcal{F}_{U_{n-1}}] \leq 4\gamma + 2\gamma \end{aligned}$$

also Q -a.s. on the set E_{n-1} . Thus we obtain Q -a.s. on the set E_{n-1}

$$1 - \varepsilon + \varepsilon Q[U_n = \infty | \mathcal{F}_{U_{n-1}}] \geq 1 - 8\gamma,$$

which, if γ is small enough ($\gamma = \varepsilon/10$ will do), can only be true for finitely many n , since $Q[U_n = \infty, U_{n-1} < \infty | \mathcal{F}_{U_{n-1}}] \rightarrow 0$ in Q -probability. This is the desired contradiction.

In order to have a means of discussing the problem of replacing h in Theorem 5 by certain Hellinger-like processes (we will address this problem

in more detail in the remark after the Corollary), we introduce some classes of concave functions, and prove a lemma, which is a «uniformly strict» version of Jensen's inequality for one of these classes of functions.

D e f i n i t i o n. Let $\mathbf{F}_0$ be the class of concave functions $f: [0, 2] \rightarrow \mathbf{R}^+$, satisfying $f(0) = 0$ and

$$1 - \frac{a}{2} - 2^{-b} > 0, \quad 1 + b - 2^a > 0, \quad (15)$$

where we denote

$$a = \limsup_{x \rightarrow 0} \frac{x f'(x)}{f(x)}, \quad b = \liminf_{x \rightarrow 0} \frac{x f'(x)}{f(x)}. \quad (16)$$

Let moreover $\mathbf{F}_2 = \{f(\cdot): f(2\cdot) \in \mathbf{F}_0\}$ and $\mathbf{F} = \mathbf{F}_0 \cap \mathbf{F}_2$.

Note, that these classes are not empty. The set of admissible values of (a, b) in (15) is in fact a subset of $]0, 1[^2$ containing the set $\{(a, b): 0 < a = b < 1\}$. Typical elements of $\mathbf{F}$ are the functions $f(x) = x^{\alpha_0}(2 - x)^{\alpha_2}$, for any pair $0 < \alpha_0, \alpha_2 < 1$.

Lemma 3. *Let $f \in \mathbf{F}_0$. Then there exist $\varepsilon_f > 0$ and $\mu_f > 0$ such that for all $\mu \in]0, \mu_f]$ the inequality*

$$\frac{\mathbf{E}[f(X)]}{f(\mu)} \leq 1 - \varepsilon_f p$$

holds for any random variable X ranging in $[0, 2]$ and satisfying $\mathbf{E} X = \mu$, $\mathbf{P}[X \in D(\mu)] = p > 0$, where $D(\mu) =]0, 2[\setminus]\mu/2, 2\mu[$.

P r o o f. Since f is concave, the affine function $\ell(x) = f(\mu) + f'(\mu)(x - \mu)$ satisfies $f(x) \leq \ell(x)$ and $f(\mu) = \ell(\mu)$. On the set $D(\mu)$ we even have $f(x) \leq \ell(x) - m(\mu)$ with $m(\mu) = \min(\ell(\mu/2) - f(\mu/2), \ell(2\mu) - f(2\mu))$. Therefore

$$\begin{aligned} \mathbf{E}[f(X)] &\leq \mathbf{E}[\ell(X) - m(\mu) \mathbb{I}_{\{X \in D(\mu)\}}] = \ell(\mu) - m(\mu) \mathbf{P}[X \in D(\mu)] \\ &= f(\mu) - m(\mu) p. \end{aligned}$$

It remains to show that $m(\mu)/f(\mu)$ is bounded away from 0 uniformly as $\mu \rightarrow 0$. By (16) we have $\mu f'(\mu) \leq a f(\mu)(1 + o(1))$ and $f'(\mu)/f(\mu) \geq (b/\mu)(1 + o(1))$. Integrating the latter inequality from $\mu/2$ to μ yields $\ln(f(\mu)/f(\mu/2)) \geq b \ln 2(1 + o(1))$ and thus $f(\mu/2) \leq 2^{-b} f(\mu)(1 + o(1))$. Therefore, as $\mu \rightarrow 0$,

$$\ell\left(\frac{\mu}{2}\right) - f\left(\frac{\mu}{2}\right) = f(\mu) - \frac{\mu}{2} f'(\mu) - f\left(\frac{\mu}{2}\right) \geq \left(1 - \frac{a}{2} - 2^{-b} + o(1)\right) f(\mu),$$

and similarly $\ell(2\mu) - f(2\mu) \geq (1 + b - 2^a + o(1)) f(\mu)$. We have derived

$$\frac{m(\mu)}{f(\mu)} \geq \min\left(1 - \frac{a}{2} - 2^{-b}, 1 + b - 2^a\right) + o(1).$$

Choosing now $\varepsilon_f = \frac{1}{2} \min(1 - a/2 - 2^{-b}, 1 + b - 2^a)$ and μ_f such that the function $g(x)$ implied by the symbol $o(1)$ in the last inequality satisfies $|g(x)| \leq \varepsilon_f$ for $x \in]0, \mu_f]$ makes the proof complete.

Corollary. *There exist $\varepsilon > 0$ and $\mu_0 > 0$ such that for all $\mu \in]0, \mu_0] \cup [2 - \mu_0, 2[$ and for any random variable X ranging in $[0, 2]$ and satisfying $\mathbf{E} X = \mu$, $\mathbf{P}[X \in C(\mu)] = p > 0$, where $C(\mu)$ is given by (9), we have*

$$\mathbf{E} \left[\sqrt{\frac{X(2-X)}{\mu(2-\mu)}} \right] \leq 1 - \varepsilon p.$$

P r o o f. This follows from Lemma 3 and the fact that $f(2-x) = f(x)$.

R e m a r k. For a given concave function f we can define a supermartingale $Y_t = f(z_t) = M_t + A_t$, and ask, if the increasing process H defined via $Y = M - Y_- \bullet H$ and required to be 0 at 0 and constant after S (cf. (2)), can replace the Hellinger process h of order $\frac{1}{2}$ in Theorem 5, i.e., satisfies

$$\{Y_{S-} = 0\} = \{H_S = \infty\} \quad Q\text{-a.s.} \quad (17)$$

for any Q -martingale $(z_t)_{t=0}^\infty$ with $z_0 = 1$ and $0 \leq z_t \leq 2$. Note that the Hellinger process $h(\frac{1}{2})$ (respectively $h(\alpha)$, for $0 < \alpha < 1$) corresponds to the choice $f(z) = z^\alpha(2-z)^{1-\alpha}$.

One obvious condition, f must satisfy, is $f(0) = f(2) = 0$. Otherwise H_t , given by $\int_0^t dA_s/Y_{s-}$, would be finite Q -a.s. on at least one of the sets $\{z_{S-} = 0\}$ resp. $\{z_{S-} = 2\}$.

Another necessary condition for (17) is $f'(0) = f'(2) = \infty$. To see this, take $z_t = 1 + B_{t \wedge S}$, where (B_t) is a standard Brownian motion with respect to Q and $S = \inf\{t \geq 0: |B_t| = 1\}$, and take $f(x) = 1 - |x - 1|$, which satisfies $f'(0) = 1$. The Tanaka formula then reveals that $H_t = L_{t \wedge S}$, where L denotes the local time at 0 of (B_t) . Now $S < \infty$ Q -a.s. and therefore also $H_S < \infty$ Q -a.s., but $Y_{S-} = f(z_{S-})0$ Q -a.s.

The methods of our paper suffice to prove the $\subseteq$ -part of (17) for concave functions f belonging to the class $\mathbf{F}$. In particular equations (8) and (14) remain true for $f \in \mathbf{F}$. The $\supseteq$ -part of (17) for $f \in \mathbf{F}$ can be proved as in [9, Lemma IV 2.12 a]. Thus, in (17), H can in particular be one of the Hellinger processes of order α , for $0 < \alpha < 1$. However, there are concave functions satisfying $f(0) = f(2) = 0$ and $f'(0) = f'(2) = \infty$, but not contained in $\mathbf{F}$, such as $f(x) = x(2-x) \ln(x/e) \ln((2-x)/e)$ and $f(x) = (\ln(x/(2e)) \ln((2-x)/(2e)))^{-1}$.

We do not know whether for general concave functions f the conditions

$$f(0) = f(2) = 0, \quad f'(0) = f'(2) = \infty$$

are also sufficient for (17) to hold, and leave this question for future research.

3. Example 4. Here we write down the example referred to in Section 1. Let $(\Omega, (\mathcal{F}_t)_{t=0}^2, \mathcal{F})$ and two probability measures P, P' be given by

$$\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}, \quad \mathcal{F}_0 = \{\emptyset, \Omega\},$$

$$\mathcal{F}_1 = \sigma(\{\omega_1\}, \{\omega_2\}, \{\omega_3, \omega_4\}), \quad \mathcal{F}_2 = \mathcal{F} = \sigma(\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}),$$

$$P(\omega_1) = 2P(\omega_3) = \frac{2}{3}, \quad P'(\omega_2) = 2P'(\omega_4) = \frac{2}{3}.$$

In this case we have $S = \mathbb{I}_{\{\omega_1, \omega_2\}} + 2 \cdot \mathbb{I}_{\{\omega_3, \omega_4\}}$.

Claim. *There is no $[0, \infty]$ -valued predictable process H such that, for every stopping time T , $P'_T \perp P_T \iff P'(H_T = \infty) = 1$.*

P r o o f. Assume that there is. Then $H_1 \equiv \text{const} < \infty$, P' -a.s., since $\mathcal{F}_0$ is trivial and $P'_1 \not\perp P_1$. On the other hand $H_S \equiv \infty$, P' -a.s., since $P_S \perp P'_S$. Now on the set $\{S = 1\}$, which has positive P' -measure, we have conflicting definitions of H_1 . This contradicts our assumption.

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