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THE INVARIANCE PRINCIPLE FOR SOME CLASS OF MARKOV CHAINS¹⁾

Работа посвящена принципу инвариантности для однородных марковских цепей с конечным пространством состояний. Доказательство основано на сравнении асимптотического поведения сумм случайных величин, построенных по марковским цепям, имеющим одну и ту же переходную матрицу (за один шаг) и разные начальные распределения.

Ключевые слова и фразы: принцип инвариантности, марковская цепь с конечным пространством состояний.

1. Introduction. Let $\{X_n, n \geq 1\}$ be a sequence of integer-valued Markov chain with stationary transition probabilities

$$p_{lj} = \mathbf{P}\{X_{k+1} = j \mid X_k = l\} \quad (1.1)$$

for all $k = 1, 2, \dots$, and $l, j \in S = \{-m, \dots, -1, 0, 1, \dots, m\}$, and the initial probabilities $\pi_j = \mathbf{P}\{X_1 = j\}$, $j \in S$. Denote $S_n = X_1 + \dots + X_n$. If the sequence is stationary with

$$p_j := \mathbf{P}\{X_k = j\} = \pi_j \quad (1.2)$$

for all $k = 1, 2, \dots$, then $\{X_n, n \geq 1\}$ is φ -mixing with mixing coefficients $C\rho^n$, $n = 1, 2, \dots$, for some $C > 0$ and $0 < \rho < 1$ (cf. [3]). Consequently, at this time, the weak invariance principle holds, in other words, putting

$$Y_n(t, \omega) = \frac{S_{[nt]}(\omega) - \mathbf{E}S_{[nt]}(\omega)}{\sqrt{\mathbf{D}S_n(\omega)}}, \quad 0 \leq t \leq 1, \quad (1.3)$$

we have

$$Y_n \Rightarrow W \quad \text{as } n \rightarrow \infty, \quad (1.4)$$

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where $\{W(t), 0 \leq t \leq 1\}$ is a standard Wiener process on $(D, \mathcal{D})$ with Skorokhod topology (see, e.g., [2, Section 20]). It follows without difficulty that

$$\begin{aligned} \mathbf{E}X_1 &= \sum_{j=-m}^m j p_j =: a, \\ \mathbf{E}(X_1 - a)^2 &= \sum_{j=-m}^m j^2 p_j - a^2, \\ \mathbf{E}(X_1 - a)(X_k - a) &= \sum_{j_1, \dots, j_k \in S} j_1 j_k p_{j_1} p_{j_1 j_2} \cdots p_{j_{k-1} j_k} - a^2, \quad k \geq 2, \end{aligned} \quad (1.5)$$

and hence

$$\begin{aligned} \mathbf{E}S_n &= na, \\ \frac{1}{n} \mathbf{D}S_n - \mathbf{E}(X_1 - a)^2 + 2 \sum_{k=2}^{\infty} \mathbf{E}(X_1 - a)(X_n - a) &=: \sigma^2. \end{aligned} \quad (1.6)$$

We always assume $\sigma > 0$. Therefore, we can redefine

$$Z_n(t, \omega) = \frac{S_{[nt]}(\omega) - [nt]a}{\sigma\sqrt{n}}, \quad 0 \leq t \leq 1, \quad (1.7)$$

and rewrite (1.4) as

$$Z_n \Rightarrow W \quad \text{as } n \rightarrow \infty. \quad (1.8)$$

A natural question is what can we say when the sequence is nonstationary, i.e., the probability $\mathbf{P}\{X_k = j\}$ depends on k . This paper aims at extending the invariance principle (1.8) to the nonstationary situation.

2. Main results. Denote the transition probability matrix by $P = (p_{lj})$. The n -step transition probability matrix is $P^{(n)} := (p_{lj}(n)) = P^n$. Suppose that for some $n > 0$, all elements of $P^{(n)}$ are positive, it is well known that there exist positive limits

$$p_j = \lim_{n \rightarrow \infty} p_{lj}(n) \quad (2.1)$$

for all l and j , and further, there exist C and ρ ($0 < \rho < 1$) such that

$$\max_{l,j} |p_{lj}(n) - p_j| \leq C \rho^n \quad (2.2)$$

(cf., e.g., [3]). The numbers p_j , $j \in S$, form a stationary probability distribution.

Theorem. Let $\{X_n, n \geq 1\}$ be a Markov chain with stationary transition probabilities (1.1) and initial probabilities $\{\pi_j\}$. Suppose that all elements of $(p_{lj})^n$ are positive for some $n > 0$. Then the invariance principle (1.8) holds, where a and σ are defined by (1.5) and (1.6), $\{p_j\}$ in (1.5) is defined by (2.1).

P r o o f. Let D' be the topological dual of D , and $\mathbf{P}_p$ and $\mathbf{E}_p$ the probability and expectation corresponding to the other initial distribution $\{p_i, i \in S\}$. From the invariance principle (1.8), the difference of characteristic functionals

$$|\mathbf{E}e^{if(W)} - \mathbf{E}_p e^{if(Z_n)}| \longrightarrow 0 \quad \text{as } n \rightarrow \infty \quad (2.3)$$

for any $f \in D'$. On the other hand, putting $n' = [\log n]$ and $p_0 = \min_{j \in S} p_j > 0$ and noting (2.2), we have

$$\begin{aligned} |\mathbf{E}_p e^{if(Z_n)} - \mathbf{E} e^{if(Z_n)}| &\leq \left| \mathbf{E}_p \exp \left\{ \frac{i}{\sigma\sqrt{n}} \sum_{k=1}^{n'} f(X_k - a) \right\} - 1 \right| \\ &\quad + \left| \mathbf{E} \exp \left\{ \frac{i}{\sigma\sqrt{n}} \sum_{k=1}^{n'} f(X_k - a) \right\} - 1 \right| + \left| (\mathbf{E}_p - \mathbf{E}) \exp \left\{ \frac{it}{\sigma\sqrt{n}} \sum_{k=n'+1}^n f(X_k - a) \right\} \right| \\ &\leq \frac{2n'}{\sigma\sqrt{n}} \max_{j \in S} |f(j - a)| + \left| \sum_{j, j_{n'+1}, j_n \in S} (p_{j_{n'+1}} - \pi_j p_{jj_{n'+1}}(n')) p_{j_{n'+1}j_n}(n - n' - 1) \right| \\ &\leq \frac{2n'}{\sigma\sqrt{n}} \max_{j \in S} |f(j - a)| + p_0^{-1} C \rho^{n'} \longrightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (2.4)$$

It follows from (2.3) and (2.4) that it suffices for showing (1.8) to prove the uniform tightness of $\{Z_n\}$ (cf., e.g., [1]). By Theorem 8.4 (adapted to D) in [2], the uniform tightness condition will follow if we prove that, for any $\varepsilon > 0$, there exist a $\lambda > \sigma$ and an integer N such that $n \geq N$ implies

$$\mathbf{P} \left\{ \max_{j \leq n} \left| \sum_{l=k+1}^{k+j} X_l - ja \right| \geq \lambda \sqrt{n} \right\} \leq \frac{\varepsilon}{\lambda^2} \quad \text{for all } k. \quad (2.5)$$

Since $\{X_n, n \geq 1\}$ is stationary and φ -mixing with coefficients $C\rho^n$ in $\mathbf{P}_p$ and $X_n, n \geq 1$, are bounded by m , we have uniform integrability as follows:

$$\begin{aligned} & \sup_{k \geq 1} \sup_{n \geq 1} \mathbf{E}_p \left\{ \frac{1}{n} \left(\sum_{l=k+1}^{k+n} X_l - na \right)^2 I \left(\frac{1}{n} \left(\sum_{l=k+1}^{k+n} X_l - na \right)^2 > A \right) \right\} \\ & \leq \sup_{k \geq 1} \sup_{n \geq 1} \frac{1}{A^2} \mathbf{E}_p \left\{ \frac{1}{n^2} \left(\sum_{l=k+1}^{k+n} X_l - na \right)^4 \right\} \\ & \leq \sup_{k \geq 1} \sup_{n \geq 1} \frac{M}{A^2} (m + |a|)^4 \longrightarrow 0 \quad \text{as } A \rightarrow \infty, \end{aligned} \quad (2.6)$$

where $M > 0$ is a constant (see, e.g., [2, Section 20, Lemma 4]). Hence, by (2.6), for given $\varepsilon > 0$ we can choose λ such that

$$\mathbf{P}_p \left\{ \left| \sum_{l=k+1}^{k+n} X_l - na \right| > \lambda \sqrt{n} \right\} \leq \frac{\varepsilon}{\lambda^2} \quad (2.7)$$

for all k and n . Moreover, similarly to (2.4)

$$\begin{aligned} & \left| \mathbf{P}_p \left\{ \left| \sum_{l=k+1}^{k+n} X_l - na \right| > \lambda \sqrt{n} \right\} - \mathbf{P} \left\{ \left| \sum_{l=k+1}^{k+n} X_l - na \right| > \lambda \sqrt{n} \right\} \right| \\ & = \left| \sum^* (p_{r_{k+1}} - \pi_{r_1} p_{r_1 r_{k+1}}(k)) p_{r_{k+1} r_{k+2}} \cdots p_{r_{k+n-1} r_{k+n}} \right| \leq p_0^{-1} C \rho^k, \end{aligned} \quad (2.8)$$

where $\sum^*$ denotes the summation over $r_1, r_{k+1}, \dots, r_{k+n} \in S$, $|\sum_{l=k+1}^{k+n} r_l - na| > \lambda \sqrt{n}$. Let K be large enough so that $p_0^{-1} C \rho^K < \varepsilon / \lambda^2$. Then for all $k \geq K$ and $n \geq 1$, from (2.7) and (2.8) we have

$$\mathbf{P} \left\{ \left| \sum_{l=k+1}^{k+n} X_l - na \right| > \lambda \sqrt{n} \right\} \leq \frac{2\varepsilon}{\lambda^2}. \quad (2.9)$$

For $k < K$, there exists an integer n_1 such that for $n \geq n_1$

$$\begin{aligned} & \mathbf{P} \left\{ \left| \sum_{l=k+1}^{k+n} X_l - na \right| > \lambda \sqrt{n} \right\} \leq \mathbf{P} \left\{ \left| \sum_{l=k+1}^K X_l - (K-k)a \right| > \frac{\lambda}{2} \sqrt{n} \right\} \\ & + \mathbf{P} \left\{ \left| \sum_{l=K+1}^{k+n} X_l - (n+k-K)a \right| > \frac{\lambda}{2} \sqrt{n} \right\} \leq \frac{9\varepsilon}{\lambda^2}. \end{aligned} \quad (2.10)$$

Fix k . Let

$$C_i = \left\{ \max_{j < i} \left| \sum_{l=k+1}^{k+j} X_l - ja \right| < 3\lambda \sqrt{n} \leq \left| \sum_{l=k+1}^{k+i} X_l - ia \right| \right\}.$$

Then for $n \geq n_1$

$$\mathbf{P} \left\{ \max_{j \leq n} \left| \sum_{l=k+1}^{k+j} X_l - ja \right| \geq 3\lambda \sqrt{n} \right\} \leq \mathbf{P} \left\{ \left| \sum_{l=k+1}^{k+n} X_l - na \right| \geq \lambda \sqrt{n} \right\}$$

$$\begin{aligned}
& + \sum_{i=1}^{n-1} \mathbf{P} \left(C_i \cap \left\{ \left| \sum_{l=k+i+1}^{k+n} X_l - (n-i)a \right| \geq 2\lambda\sqrt{n} \right\} \right) \\
& \leq \frac{9\varepsilon}{\lambda^2} + \sum_{i=1}^{n-n'-1} \left(\mathbf{P} \left\{ \left| \sum_{l=k+i+1}^{k+i+n'} X_l - n'a \right| \geq \lambda\sqrt{n} \right\} \right. \\
& \quad \left. + \mathbf{P} \left(C_i \cap \left\{ \left| \sum_{l=k+i+n'+1}^{k+n} X_l - (n-i-n')a \right| \geq \lambda\sqrt{n} \right\} \right) \right) \\
& + \sum_{i=n-n'}^{n-1} \mathbf{P} \left\{ \left| \sum_{l=k+i+1}^{k+n} X_l - (n-i)a \right| \geq \lambda\sqrt{n} \right\} \\
& \leq \frac{9\varepsilon}{\lambda^2} + n \sup_k \mathbf{P} \left\{ \sum_{l=k+1}^{k+n'} |X_l| + n'|a| \geq \lambda\sqrt{n} \right\} \\
& + \sum_{i=1}^{n-n'-1} \left(\mathbf{P}(C_i) \mathbf{P} \left\{ \left| \sum_{l=k+i+n'+1}^{k+n} X_l - (n-i-n')a \right| \geq \lambda\sqrt{n} \right\} \right. \\
& \quad \left. + \sum_{j \in S} \sum' \pi_j p_{j r_{k+1}}(k) p_{r_{k+1} r_{k+2}} \cdots p_{r_{k+i-1} r_{k+i}} \right. \\
& \quad \left. \times |p_{r_{k+i}, r_{k+i+n'+1}}(n'+1) - p_{r_{k+i+n'+1}}| \right. \\
& \quad \left. \times p_{r_{k+i+n'+1} r_{k+n}}(n-i-n'-1) \right), \quad (2.11)
\end{aligned}$$

where $\sum'$ denotes the sum over $(r_{k+1}, \dots, r_{k+i})$ in $\{\max_{j < i} |\sum_{l=k+1}^{k+j} r_l - ja| < 3\lambda\sqrt{n} \leq |\sum_{l=k+1}^{k+i} r_l - ia|\}$ and over $(r_{k+i+n'+1}, \dots, r_{k+n})$ in $\{|\sum_{l=k+i+n'+1}^{k+n} r_l - (n-i-n')a| \geq \lambda\sqrt{n}\}$. By the Markov inequality, (2.10) and (2.2), there exists $n_2 \geq n_1$ such that for $n \geq n_2$, the right-hand side of (2.11) does not exceed

$$\frac{19\varepsilon}{\lambda^2} + p_0^{-1} C \rho^{n+1} \leq \frac{20\varepsilon}{\lambda^2}.$$

Thus (2.5) is proven except for the irrelevant factors 9 and 20. The proof of the theorem is completed.

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