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REFERENCES

1. *Barndorff-Nielsen O. E.* Information and Exponential Families in Statistical Theory. Chichester: Wiley, 1978.
2. *Barndorff-Nielsen O. E.* Parametric statistical models and likelihood. Lect. Notes in Statist., v. 50, 1988.
3. *Barndorff-Nielsen O. E., Blæsild P.* Exponential models with affine dual foliations. — Ann. Statist., 1983, v. 11, p. 753–769.
4. *Barndorff-Nielsen O. E., Cox D. R.* Inference and Asymptotics. London: Chapman and Hall, 1994.
5. *Barndorff-Nielsen O. E., Jupp P. E.* Differential geometry, profile likelihood, L -sufficiency and composite transformation models. — Ann. Statist., 1988, v. 16, p. 1009–1043.
6. *Rémon M.* On a concept of partial sufficiency: L -sufficiency. — Internat. Statist. Rev., 1984, v. 52, p. 127–136.

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NECESSARY CONDITIONS FOR STABLE
CONVERGENCE OF SEMIMARTINGALES

В статье доказан результат обратной теореме об устойчивой сходимости семимартингалов, полученный в [1]. Как следствие утверждается (при условии некоторого контроля за скачками), что последовательность мартингалов X^n устойчиво сходится к непрерывному мартингалу X с условно независимыми приращениями тогда и только тогда, когда квадратичные отклонения X^n сходятся по вероятности к квадратичному отклонению X для каждого $t \in \mathbf{R}^+$.

Ключевые слова и фразы: семимартингал, устойчивая сходимость, независимые приращения.

1. Introduction and results. The subject of the present paper is the characterization of the behaviour of a sequence of semimartingales when they converge stably to a continuous semimartingale with conditionally independent increments. In view of the results due to [3], and more recently to [1], it remains only to prove some inverse theorems, and this is done, based on the extension of existing results for weak convergence, and a convenient characterization of stable convergence. The results presented here were announced in [4].

For general concepts about semimartingales and their characteristics we refer to ([2, Chapters I, II]).

Let us now define the sequence of semimartingales and the limit.

Let h denote a truncation function, and all the characteristics will be considered with respect to h .

(S) Consider for each $n \in \mathbf{N}$ a d -dimensional semimartingale X^n defined on a stochastic basis $\mathcal{B}^n = (\Omega, \mathcal{F}, \mathbb{F}^n, \mathbf{P})$, with $\mathbb{F}^n = (\mathcal{F}_t^n: t \geq 0)$. Denote by $(B^n, \tilde{C}^n, \nu^n)$ the triplet of modified characteristics of X^n under $\mathbf{P}$, and by $[X^n, X^n]$ the $\mathbf{R}^d \otimes \mathbf{R}^d$ -valued processes whose components are the (co-)quadratic variation $[X^{n,j}, X^{n,k}]$. Set for simplicity $X_0^n = 0$.

As for the limit consider a transition probability $\mathbf{Q}(\omega, d\gamma)$ from $(\Omega, \mathcal{F})$ into the Skorohod space

$$(\mathbf{D}(\mathbf{R}^d), \mathcal{D}(\mathbf{R}^d), (\mathcal{D}_t: t \geq 0)),$$

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such that for each $\omega \in \Omega$ the canonical process on $\mathbf{D}(\mathbf{R}^d)$ defined by $X_t(\gamma) = \gamma_t$ is a semimartingale with independent increments, and characteristics $(B(\omega), C(\omega), 0)$.

Now we set

$$\bar{\Omega} = \Omega \times \mathbf{D}(\mathbf{R}^d), \quad \bar{\mathcal{F}} = \mathcal{F} \otimes \mathcal{D}(\mathbf{R}^d),$$

$$\bar{\mathbf{P}}(d\omega, d\gamma) = \mathbf{P}(d\omega)\mathbf{Q}(\omega, d\gamma), \quad \bar{\mathbb{F}} = \left(\cap_{s>t} (\mathcal{F} \otimes \mathcal{D}_s): t \geq 0 \right),$$

and every process, variable or σ -field defined on $(\Omega, \mathcal{F})$ or $(\mathbf{D}(\mathbf{R}^d), \mathcal{D}(\mathbf{R}^d))$ is naturally extended to $(\bar{\Omega}, \bar{\mathcal{F}})$ with the same symbol, e.g., $X(\omega, \gamma) = X(\gamma)$, $X^n(\omega, \gamma) = X^n(\omega)$, etc.

(L) Let then X be a process defined on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbf{P}})$ with $(B, C, 0)$ as characteristics, that is, a process with $\mathcal{F}$ -conditionally independent increments. Denote by $[X, X]$ the quadratic variation of X (defined as above), and observe that $\tilde{C} = C$

(N) (Nesting condition.) Assume there exists a sequence of real numbers (α_n) decreasing to 0, such that

- (1) $\mathcal{F}_{\alpha_n}^n \subset \mathcal{F}_{\alpha_{n+1}}^{n+1}$,
- (2) $\mathcal{F} = \bigvee_n \mathcal{F}_{\alpha_n}^n$.

Then consider the conditions:

$$[\gamma - \mathbf{R}^+]: \tilde{C}_t^n \xrightarrow{\mathbf{P}} C_t \quad \forall t \in \mathbf{R},$$

$$[\delta - \mathbf{R}^+]: \mathbb{I}_{\{|x|>\varepsilon\}} * \nu_t^n \xrightarrow{\mathbf{P}} 0 \quad \forall t \in \mathbf{R} \quad \forall \varepsilon > 0.$$

Denote $\text{Var}(B^n)_t = \sum_{i=1}^d \text{Var}(B^{n,i})_t$, where $\text{Var}(A)_t$ stands for the total variation of a function A on $[0, t]$.

We will say that the sequence X^n converges $\mathcal{F}$ -stably to X in $\mathbf{D}(\mathbf{R}^d)$ (denoted by $X^n \xrightarrow{d} X$ ($\mathcal{F}$ -stably)), if $\mathbf{E}(Yf(X^n)) \rightarrow \mathbf{E}(Yf(X))$ as $n \rightarrow +\infty$ for all bounded and $\mathcal{F}$ -measurable random variables Y and continuous and bounded functions $f: \mathbf{D}(\mathbf{R}^d) \rightarrow \mathbf{R}$.

Theorem 1. Assume conditions (S) and (L) are fulfilled. If

- (i) $X^n \xrightarrow{d} X$ ($\mathcal{F}$ -stably),
- (ii) $(\text{Var}(B^n)_t \mid \bar{\mathbf{P}})$ is tight for all $t \geq 0$,

then

$$([X^n, X^n], X^n) \xrightarrow{d} ([X, X], X) \quad (\mathcal{F}\text{-stably})$$

and, in particular,

$$[X^n, X^n] \xrightarrow{d} [X, X] \quad (\mathcal{F}\text{-stably}).$$

Corollary 1. Under the assumptions of Theorem 1 $[X^n, X^n]_t \xrightarrow{\mathbf{P}} C_t \quad \forall t \geq 0$.

Theorem 2. Assume conditions (S) and (L) are fulfilled. If

- (i) $X^n \xrightarrow{d} X$ ($\mathcal{F}$ -stably),
- (ii) $(\text{Var}(B^n)_t \mid \bar{\mathbf{P}})$ is tight;

then

- (1) $[\delta - \mathbf{R}^+]: \mathbb{I}_{\{|x|>\varepsilon\}} * \nu_t^n \xrightarrow{\mathbf{P}} 0 \quad \forall t \in \mathbf{R}^+ \quad \forall \varepsilon > 0$,
- (2) $[\gamma - \mathbf{R}^+]: \tilde{C}_t^n \xrightarrow{\mathbf{P}} C_t \quad \forall t \in \mathbf{R}^+$.

Corollary 2. Assume conditions (S), (L), and (N) are fulfilled. Suppose that X^n 's are local martingales satisfying $|\Delta X_t^n| \leq c$ for some constant c , and X is a continuous $\mathcal{F}$ -conditionally Gaussian martingale (for each fixed ω , X is a Gaussian martingale)

Then the following statements are equivalent:

- (1) $X^n \xrightarrow{d} X$ ($\mathcal{F}$ -stably),
- (2) $[X^n, X^n]_t \xrightarrow{\mathbf{P}} [X, X]_t \quad \forall t \in \mathbf{R}^+$,
- (3) $[\gamma - \mathbf{R}^+] + [\delta - \mathbf{R}^+]$.

2. Proofs. Let us first characterize stable convergence in a convenient way. Let $\mathcal{G}$ be a sub- σ -field of $\mathcal{F}$. Denote by $\mathcal{P}(\mathcal{G})$ the set of probability measures $\bar{\mathbf{P}}$ on $(\Omega, \mathcal{F})$ such that $d\bar{\mathbf{P}} = Yd\mathbf{P}$ for any $\mathcal{G}$ -measurable, positive and bounded random variable Y . Given X a random variable, let $\sigma(X)$ denote the σ -field generated by X .

Lemma 1. Suppose on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ a sequence of random elements X^n and X with values in a metric space (E, δ) is given. Let $\mathcal{G} \subset \mathcal{F}$. Then (1) and (2) are equivalent:

- (1) $X^n \xrightarrow{\mathbf{d}} X$ ($\mathcal{G}$ -stably),
 (2) $X^n \xrightarrow{\mathcal{L}(\tilde{\mathbf{P}})} X \quad \forall \tilde{\mathbf{P}} \in \mathcal{P}(\mathcal{G})$.

If $E = \mathbf{R}^d$, δ is the Euclidean metric in $\mathbf{R}^d$, and $\sigma(X) \subset \mathcal{G}$, then (1) or (2) are equivalent to (3):

- (3) $X^n \xrightarrow{\mathbf{P}} X$.

P r o o f. Equivalence (1) $\Leftrightarrow$ (2) follows from linearity and normalization.

(1) $\Rightarrow$ (3). We know that $\mathbf{E}(\exp\{i(\lambda X + \mu X^n)\}) \rightarrow \mathbf{E}(\exp\{i(\lambda X + \mu X)\})$ for the real and continuous functions $\cos(\lambda x)$, $\cos(\mu x)$, $\sin(\lambda x)$, $\sin(\mu x)$ and the bounded and $\mathcal{G}$ -measurable random variables $\cos(\lambda X)$, $\cos(\mu X)$, $\sin(\lambda X)$, $\sin(\mu X)$.

From this fact it follows that

$$(X, X^n) \xrightarrow{\mathcal{L}(\mathbf{P})} (X, X) \quad \text{and} \quad X^n - X \xrightarrow{\mathbf{P}} 0.$$

- (3) $\Rightarrow$ (2). If $\tilde{\mathbf{P}} \in \mathcal{P}$ then $\mathbf{P} \sim \tilde{\mathbf{P}}$. So $X^n \xrightarrow{\tilde{\mathbf{P}}} X$ and $X^n \xrightarrow{\mathcal{L}(\tilde{\mathbf{P}})} X$.

P r o o f o f T h e o r e m 1. According to Lemma 1, it suffices to verify

$$([X^n, X^n], X^n) \xrightarrow{\mathcal{L}(\tilde{\mathbf{P}})} ([X, X], X),$$

where $\tilde{\mathbf{P}} \in \mathcal{P}$.

We know that the quadratic variation is invariant under an absolutely continuous change of measure, so it is natural to apply Theorem VI.6.1 of [2] under $\tilde{\mathbf{P}}$. Let us verify its hypothesis. We know $X^n \xrightarrow{\mathcal{L}(\tilde{\mathbf{P}})} X$ ((1) $\Rightarrow$ (2) in Lemma 1). In order to prove $(\text{Var}(B^n)_t | \tilde{\mathbf{P}})$ is tight, let us fix an $\varepsilon > 0$. As $\tilde{\mathbf{P}} \sim \mathbf{P}$, there exists a $\delta > 0$ such that $\tilde{\mathbf{P}}(A) < \delta \Rightarrow \mathbf{P}(A) < \varepsilon$, $A \in \mathcal{F}$. Let K_ε be such that $\tilde{\mathbf{P}}(\text{Var}(B^n)_t > K_\varepsilon) < \delta \quad \forall n \in \mathbf{N}$, then

$$\tilde{\mathbf{P}}(\text{Var}(B^n)_t > K_\varepsilon) < \varepsilon \quad \forall n \in \mathbf{N}.$$

Now the application of Theorem VI.6.1 of [2] completes the proof

$$([X^n, X^n], X^n) \xrightarrow{\mathcal{L}(\tilde{\mathbf{P}})} ([X, X], X).$$

P r o o f o f C o r o l l a r y 1. In view of Theorem 1 and implication (1) $\Rightarrow$ (3) in Lemma 1 the result follows from the fact that C_t is $\mathcal{F}$ -measurable.

P r o o f o f T h e o r e m 2. (1) As X is continuous,

$$\sup_{s \leq t} |\Delta X_s^n| \xrightarrow{\mathbf{P}} 0, \tag{2.1}$$

which is equivalent to $[\delta - \mathbf{R}^+]$ in virtue of the Lenglart–Rebolledo inequalities (see [2, Section VI.4.22]).

(2) Let $X^n = M^n + B^n + \tilde{X}^n(h)$ be the semimartingale decomposition of X^n :

$$\tilde{X}^n(h) = \sum_{s \leq t} \Delta X_s^n - h(\Delta X_s^n),$$

M^n is a local martingale with bounded jumps and $M_0^n = 0$,

B^n is a predictable process with finite variation in intervals.

By (2.1) we know $\mathbf{P}(\sup_{s \leq t} |\tilde{X}^n(h)_s| > 0) \rightarrow 0$. Furthermore, on the set $\{\sup_{s \leq t} |\Delta \tilde{X}_s^n| = 0\}$ we have

$$[X^n, X^n] = [M^n, M^n] + [B^n, B^n] + 2[M^n, B^n]$$

and

$$\sup_{s \leq t} |[X^n, X^n] - [M^n, M^n]| \leq \text{Var}(B^n)_t \sup_{s \leq t} (|\Delta B_s^n| + 2|\Delta M_s^n|).$$

We know that $\text{Var}(B^n)_t$ is tight. Let us first check that

$$\sup_{s \leq t} |\Delta B_s^n| \xrightarrow{\mathbf{P}} 0. \tag{2.2}$$

Observe that

$$|\Delta B_s^n| = h(x) * \nu^n(\{s\}, dx) \leq \varepsilon + \Delta \mathbb{I}_{\{|x| > \varepsilon\}} * \nu_s^n \leq \varepsilon + \mathbb{I}_{\{|x| > \varepsilon\}} * \nu_t^n$$

(assuming $h(x) = x$ on the set $\{|x| < \varepsilon\}$ and $|h(x)| \leq 1$). So

$$\sup_{s \leq t} |\Delta B_s^n| \leq \varepsilon + \mathbb{I}_{\{|x| > \varepsilon\}} * \nu_t^n$$

which yields (2.2) in virtue of $[\delta - \mathbf{R}^+]$.

Let us then prove

$$\sup_{s \leq t} |\Delta M_s^n| \xrightarrow{\mathbf{P}} 0. \quad (2.3)$$

We have

$$\Delta M_s^n = \Delta X_s^n - \Delta B_s^n - \Delta \tilde{X}(h)_s = h(\Delta X_s^n) - \Delta B_s^n,$$

so, as $|h(x)| \leq |x|$,

$$|\Delta M_s^n| \leq |\Delta X_s^n| + |\Delta B_s^n|$$

and (2.3) follows.

Thus, we have then proved that:

$$\sup_{s \leq t} |[X^n, X^n] - [M^n, M^n]| \xrightarrow{\mathbf{P}} 0$$

and, as Corollary 1 gives $[X^n, X^n]_t \xrightarrow{\mathbf{P}} C_t$, we obtain $[M^n, M^n]_t \xrightarrow{\mathbf{P}} C_t$. Now $[\gamma - \mathbf{R}^+]$ follows from a standard argument given in [5] that establishes, that under (2.1)

$$\sup_{s \leq t} |(M^n, M^n) - [M^n, M^n]| \xrightarrow{\mathbf{P}} 0. \quad (2.4)$$

Proof of Corollary 2. Let us first take a truncation function h that satisfies $h(x) = x$ for $|x| \leq c$: we then have $B^n(h) = 0$.

(1) $\Rightarrow$ (2) is a particular case of Corollary 1.

(3) $\Rightarrow$ (1) is a particular case of Feigin's Theorem (1985), see also [2, Section VIII.5.42].

(2) $\Rightarrow$ (3). Since (2.1) $\forall t \in \mathbf{R}^+$ is equivalent to $[\delta - \mathbf{R}^+]$ and under (2.1) convergence (2.4) takes place, it is enough to prove that (2) $\Rightarrow$ (2.1).

Now it follows from (2) that

$$\sup_{s \leq t} |\Delta[X^n, X^n]_s| \xrightarrow{\mathbf{P}} 0.$$

Since $|\Delta X^n|^2 = \sum_{i=1}^d \Delta[X^{n,i}, X^{n,i}]$, (2.1) holds.

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REFERENCES

1. Feigin P. D. Stable convergence of semimartingales. — Stoch. Process. Appl., 1985, v. 19, p. 125–134.
2. Jacod J., Shiryaev A. N. Limit Theorems for Stochastic Processes. Berlin, Heidelberg: Springer-Verlag, 1987.
3. Луцкер Р. Ш., Ширяев А. Н. Слабая сходимость последовательности семимартингалов к процессу диффузионного типа. — Матем. сб., 1983, т. 121 (163), № 2, с. 176–200.
4. Мордеки Э. Необходимые условия устойчивой сходимости семимартингалов. — Успехи матем. наук., 1993, т. 48, № 2, с. 193–194.
5. Rebolledo R. Central limit theorems for local martingales. — Z. Wahrscheinlichkeitstheor. verw. Geb., 1980, v. 51, p. 269–286.