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RISK AVERSE ASYMPTOTICS AND THE OPTIONAL DECOMPOSITION¹⁾

Рассматривается задача максимизации ожидаемой полезности для общей функции полезности на $\mathbf{R}$, когда степень неприятия риска агентом возрастает. Показывается, что предельная стратегия является особой, единственной суперхеджирующей стратегией — так называемой балансовой стратегией. Исследуется связь с опциональным разложением и классом минимальных хеджирующих стратегий, описанными в [5].

Ключевые слова и фразы: хеджирование, экспоненциальная функция полезности, неприятие риска, опциональное разложение.

1. Introduction. In this paper we consider a financial market with an agent who has to satisfy a contingent claim, e.g., an option, at final time T . The agent, who has the possibility of investing in an asset of the underlying financial market, has to decide on a trading strategy and there are, at least, two natural, but different possibilities: utility maximization and superhedging. We will investigate the relationship between these two concepts and show how superhedging can be interpreted as the limit of utility maximization. This interpretation will give some new insight into the notion of superhedging.

For example the notion of superhedging in general does not yield a unique outcome, i.e., there are different superhedging strategies that lead to different terminal wealth at time T . But by looking at superhedging as a limit of utility maximization one is naturally led to a special, unique outcome.

Let us assume we are given a finite probability space $\Omega = \{\omega_1, \dots, \omega_N\}$, a finite time horizon $T \in \mathbf{N}$, a filtration $(\mathcal{F}_t)_{t=0,1,\dots,T}$, and a probability measure $\mathbf{P}$. For finite Ω we always assume that $\mathbf{P}[\omega_i] > 0$ for all $i = 1, \dots, N$. Furthermore we assume, as is typically done in arbitrage pricing theory, that $\mathcal{F}_0$ is trivial. The market consists of a stock $S = (S_t)_{t=0,1,\dots,T}$, i.e., an $\mathbf{R}$ -valued adapted process, and a riskless bond B . We assume without loss of generality that the stock S represents the discounted price process, i.e., that it is denoted in terms of units of the riskless bond, and that the bond is constant equal to 1.

The stock process S can be represented by an event tree. Each node in the tree corresponds to an atom $F \in \mathcal{F}_t$, $t = 0, 1, \dots, T$. By an atom $F \in \mathcal{F}_t$ we mean an element $F \in \mathcal{F}_t$, $F \neq \emptyset$ such that $G \in \mathcal{F}_t$ with $G \subseteq F$ implies $G = \emptyset$. By a slight abuse of notation we say node F at time t for an atom $F \in \mathcal{F}_t$. G is a successor of $F \in \mathcal{F}_t$, if $G \in \mathcal{F}_{t+1}$ and $G \subset F$, $G \neq \emptyset$.

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In this finite setting a trading strategy ϑ , i.e., an $\mathcal{F}_{t-1}$ -measurable process, can be viewed as a vector $\vec{\vartheta} \in \mathbf{R}^M$, where M is the number of nodes in the tree from time 0 to time $T - 1$. These are the nodes at which one can make a decision on how to trade. By a slight abuse of notation we will write ϑ also for the vector, not just for the process. We denote by $\vartheta_F \in \mathbf{R}$ the value of the strategy ϑ at the node $F \in \mathcal{F}_t$, $t = 0, 1, \dots, T$. This means that ϑ_F denotes the number of shares the agent holds in the risky asset from time t up to time $t + 1$, if the state of the world ω satisfies $\omega \in F$, which is, since $F \in \mathcal{F}_t$, known at time t . The final gain of a trading strategy ϑ , in general the stochastic integral $G_T(\vartheta) = \int_0^T \vartheta dS$, is in the finite setting just a finite sum. Note that $G_T(\vartheta)(\omega_i)$ is linear in ϑ .

We assume that our market is arbitrage free (see, e.g., [4]), this is a standing assumption which we always make. In the current finite setting this simply means that at each time step the stock is not allowed to move only up or only down. In mathematical terms this means that for each atom $F \in \mathcal{F}_t$, $t = 0, 1, \dots, T - 1$, either $\mathbf{P}[\Delta S_{t+1} = 0; F] = \mathbf{P}[S_{t+1} = S_t; F] = 1$ (this corresponds to the case when the stock does not move at all), or $\mathbf{P}[\Delta S_{t+1} > 0; F] > 0$ and $\mathbf{P}[\Delta S_{t+1} < 0; F] > 0$.

Let us now introduce an economic agent who has a short position in an $\mathcal{F}_T$ -measurable contingent claim C , $C_i := C(\omega_i)$, and starts with an initial endowment $c \in \mathbf{R}$. So by employing strategy ϑ the agent achieves an output

$$Z_i(\vartheta) := Z(\vartheta)(\omega_i) := c + G_T(\vartheta)(\omega_i) - C_i.$$

We define the set

$$\mathcal{Z} := \{Z(\vartheta) \mid \vartheta \in \mathbf{R}^M\}$$

of all possible outcomes.

2. Balanced strategies. We will now define a special outcome $\bar{Z} = Z(\bar{\vartheta}) \in \mathcal{Z}$, introducing a *balanced strategy* and the corresponding *balanced outcome*. To our knowledge this is a new concept. One nice feature of the definition will be that it is very easy and straightforward to determine, given the stock price process and the contingent claim, the balanced strategy and the balanced outcome. But the definition will also turn out to be useful from a theoretical point of view.

Before giving the formal definition, let us describe the idea behind a balanced strategy. At each time when the agent has to make a decision about his strategy, he considers two cases: the stock going up or down. Within each of these two cases, the agent looks at the worst possible outcome for him, taking into account his short position in the contingent claim C . He then chooses a strategy to balance those two worst case scenarios, i.e., he takes the strategy that makes the worst outcome in the case of a rising stock equal the worst outcome in the case of a falling stock.

Definition 2.1. Given a finite probability space Ω , $|\Omega| = N$, a finite time horizon $T \in \mathbf{N}$, a filtration $(\mathcal{F}_t)_{t=0, \dots, T}$, an adapted $\mathbf{R}$ -valued stock process $(S_t)_{t=0, \dots, T}$, and a probability measure $\mathbf{P}$, we define for an atom $F \in \mathcal{F}_t$, $t = 0, 1, \dots, T - 1$,

$$F_+ := \{\omega \in F : \Delta S_{t+1}(\omega) > 0\} \in \mathcal{F}_{t+1}, \quad F_- := \{\omega \in F : \Delta S_{t+1}(\omega) < 0\} \in \mathcal{F}_{t+1}.$$

Assume we are further given an initial endowment $c \in \mathbf{R}$ and a claim $C \in \mathcal{F}_T$. We then call a strategy $\bar{\vartheta}$ (respectively, an outcome $\bar{Z} := Z(\bar{\vartheta}) = c + G_T(\bar{\vartheta}) - C$) *balanced*, if

$$z_{F_+}(\bar{\vartheta}) := \min_{\omega \in F_+} Z(\bar{\vartheta})(\omega) = \min_{\omega \in F_-} Z(\bar{\vartheta})(\omega) =: z_{F_-}(\bar{\vartheta})$$

for all atoms $F \in \mathcal{F}_t$, $t = 0, 1, \dots, T - 1$.

Remark 2.1. The probability measure $\mathbf{P}$ does not enter into the definition. The only important fact about the probability measure is that $\mathbf{P}[\omega] > 0$ for all $\omega \in \Omega$, a condition we always impose on a probability measure on a finite probability space Ω .

Also the initial endowment $c \in \mathbf{R}$ has no influence on the definition of a balanced strategy.

The first thing we have to check is whether a balanced strategy $\bar{\vartheta}$ really exists. For that we go backwards through the tree described by S . At an atom $F \in \mathcal{F}_{T-1}$, i.e., a node at time $T-1$, one has to find a $\bar{\vartheta}_F \in \mathbf{R}$ such that

$$\min_{\omega \in F_+} (\bar{\vartheta}_F \Delta S_T(\omega) - C(\omega)) = \min_{\omega \in F_-} (\bar{\vartheta}_F \Delta S_T(\omega) - C(\omega)). \quad (1)$$

If $F_+ = \emptyset$, i.e., the stock does not move up, then the no-arbitrage condition implies that $F_- = \emptyset$, and the above equation simplifies to $0 = 0$, holding true for all $\bar{\vartheta}_F \in \mathbf{R}$. This is a special case in which the strategy ϑ_F is irrelevant.

We want to exclude this trivial case and impose thus the standing assumption that the tree generated by the process S is nondegenerate, see, e.g., [8]. This means intuitively that at each node the stock moves to at least two values with positive probability, mathematically we impose that for all atoms $F \in \mathcal{F}_t$, $t = 0, 1, \dots, T-1$,

$$\mathbf{E}[(\Delta S_{t+1})^2; F] - \mathbf{E}[\Delta S_{t+1}]^2 \neq 0.$$

Assumption 2.1. *From now on we will always impose the standing assumption that the tree generated by the stock process S is nondegenerate. Thus we assume, together with the standing assumption of no-arbitrage, $\mathbf{P}[\Delta S_{t+1} > 0; F] > 0$ and $\mathbf{P}[\Delta S_{t+1} < 0; F] > 0$ for all atoms F in $\mathcal{F}_t$, $t = 0, 1, \dots, T-1$.*

The moment's reflection reveals that in our setting the assumption of nondegeneracy is equivalent to a one-to-one correspondence between strategies $\vartheta \in \mathbf{R}^M$ and outcomes $Z \in \mathcal{Z}$.

Since we assume that the tree is nondegenerate,

$$\vartheta_F \mapsto \min_{\omega \in F_+} \vartheta_F \Delta S_T(\omega)$$

is the minimum over a strictly positive, finite number of affine, strictly increasing functions and therefore continuous, strictly increasing and the limits as ϑ_F tends to ∞ (respectively, $-\infty$) are

$$\lim_{\vartheta_F \rightarrow \infty} \min_{\omega \in F_+} \vartheta_F \Delta S_T(\omega) = \infty \quad \left(\lim_{\vartheta_F \rightarrow -\infty} \min_{\omega \in F_+} \vartheta_F \Delta S_T(\omega) = -\infty \right).$$

Analogously, by no-arbitrage also $F_- \neq \emptyset$ a.s., $\vartheta_F \mapsto \min_{\omega \in F_-} \vartheta_F \Delta S_T(\omega)$ is continuous, strictly decreasing in ϑ , with limits

$$\lim_{\vartheta_F \rightarrow \infty} \min_{\omega \in F_-} \vartheta_F \Delta S_T(\omega) = -\infty \quad \text{and} \quad \lim_{\vartheta_F \rightarrow -\infty} \min_{\omega \in F_-} \vartheta_F \Delta S_T(\omega) = \infty.$$

Therefore, there exists a unique $\bar{\vartheta}_F \in \mathbf{R}$ that gives us equality in (1).

So we get a unique $\bar{\vartheta}_F$ for each atom $F \in \mathcal{F}_{T-1}$. Having fixed the strategy at time $T-1$, we can, following the same argument as above, determine $\bar{\vartheta}_F$ for each atom $F \in \mathcal{F}_{T-2}$. Continuing this way, using backward induction we get the existence of a unique balanced strategy $\bar{\vartheta}$. The corresponding balanced outcome $Z(\bar{\vartheta})$ is also unique.

Summing up the above considerations, we get the following proposition.

Proposition 2.1. *Given a market as in Definition 2.1 that satisfies in addition Assumption 2.1, there exist a unique balanced strategy $\bar{\vartheta}$ and a unique balanced outcome $\bar{Z} = Z(\bar{\vartheta})$.*

Example 2.1. Let us consider a one time-step model with 4 states, i.e., $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. The stock starts at $S_0 = 3$ and $\Delta S_1 = (3, 0, -1, -2)$. The contingent claim is given by $C = (3, 3, 2, 0)$ and the initial endowment is assumed to be $c = 3$. So we get

$$Z(\vartheta)(\omega_1) = 3\vartheta, \quad Z(\vartheta)(\omega_2) = 0, \quad Z(\vartheta)(\omega_3) = -\vartheta + 1, \quad Z(\vartheta)(\omega_4) = -2\vartheta + 3.$$

The only atom $F \in \mathcal{F}_0$ is $F = \Omega$ and we have $F_+ = \{\omega_1\}$, $F_- = \{\omega_3, \omega_4\}$. So

$$z_{F_+}(\vartheta) = 3\vartheta \quad \text{and} \quad z_{F_-}(\vartheta) = \begin{cases} -\vartheta + 1, & \vartheta \leq 2, \\ -2\vartheta + 3, & 2 < \vartheta. \end{cases}$$

Therefore the unique balanced strategy is $\bar{\vartheta} = \frac{1}{4}$. Note however that any strategy $\vartheta \in [0, 1]$ is a superhedging strategy.

3. General utility functions. As indicated in the introduction, we are interested in utility maximization, thus we are going to look at the underlying utility function. We will also introduce the concept of increasing risk aversion.

Definition 3.1. Let $U(x)$ denote a general utility function on $\mathbf{R}$, which is twice differentiable. By this we mean a function $U: \mathbf{R} \rightarrow \mathbf{R}$ which is strictly increasing and strictly concave, i.e., $U'(x) > 0$ and $U''(x) < 0$, and which satisfies the Inada condition²⁾ $U'(-\infty) = \infty$. Then we denote by

$$r(x) := -\frac{U''(x)}{U'(x)} > 0$$

the (absolute) *risk aversion*. This notion goes back to Arrow [1] and Pratt [6].

We consider a family of utility functions $(U_{\alpha_n})_{n \geq 1}$, whose risk aversion tends to infinity uniformly on $\mathbf{R}$ as $n \rightarrow \infty$. By that we mean that for all $K > 0$ there exists n_0 such that

$$r_{\alpha_n}(x) = -\frac{U''_{\alpha_n}(x)}{U'_{\alpha_n}(x)} > K$$

for all x and for all $n \geq n_0$. As an example one can think of the exponential utility function, since for $U_{\alpha_n}(x) = -e^{-\alpha_n x}$ the risk aversion equals $r_{\alpha_n}(x) = \alpha_n = n$.

Using the setting described in Section 1 and being interested in utility maximization, we look at

$$\mathbf{E}U_{\alpha}(Z(\vartheta)) = \mathbf{E}U_{\alpha}(c + G_T(\vartheta) - C) \mapsto \max_{\vartheta \in \mathbf{R}^M}.$$

Let us denote by ϑ^{α} the optimal strategy, i.e., $\mathbf{E}U_{\alpha}(Z^{\alpha}) = \mathbf{E}U_{\alpha}(Z(\vartheta^{\alpha})) = \sup_{\vartheta \in \mathbf{R}^M} \mathbf{E}U_{\alpha}(Z(\vartheta))$, where $Z^{\alpha} := Z(\vartheta^{\alpha})$ is the optimal outcome.

Remark 3.1. How do we know that the optimal strategy ϑ^{α} really exists? One could deduce this fact from general results, e.g., [3, Theorem 2.1] for the exponential utility function. But for the finite setting we can use some tools from convex analysis to easily deduce the existence of a unique maximizing strategy ϑ^* , since the function $\vartheta \mapsto \mathbf{E}U(Z(\vartheta))$ is a continuous, concave function on $\mathbf{R}^M$.

Proposition 3.1. *In the setting described in Section 1 there exists a unique utility maximizing strategy $\vartheta^* \in \mathbf{R}^M$, i.e., $\mathbf{E}U(Z(\vartheta^*)) = \sup_{\vartheta \in \mathbf{R}^M} \mathbf{E}U(Z(\vartheta))$ for a utility function $U(x)$.*

Proof. $-\mathbf{E}U(Z(\vartheta))$ is a continuous, thus closed, proper convex function. We can therefore use [7, Theorem VI.27.1(d)], which guarantees the existence of a minimizer provided $-\mathbf{E}U(Z(\vartheta))$ has no direction of recession. Remember that a vector ρ is called a direction of recession, if for every ϑ the function $t \mapsto -\mathbf{E}U(Z(\vartheta + t\rho))$ is nonincreasing in $t \in \mathbf{R}$.

To show that no direction of recession exists, we use [7, Proposition II.8.6 together with the definition following Corollary 8.6.2] and show that

$$\limsup_{\lambda \rightarrow \infty} \mathbf{E}U(Z(\lambda\vartheta)) = -\infty$$

for any $\vartheta \neq 0$. Since $\vartheta \neq 0$ we know by the no-arbitrage condition together with the nondegeneracy condition, see Assumption 2.1, that there has to exist an $\omega_1 \in \Omega$ such that $G_T(\vartheta)(\omega_1) < 0$. We also get that $\max_{\omega \in \Omega} G_T(\vartheta)(\omega) > 0$. So

$$\begin{aligned} \mathbf{E}U(Z(\lambda\vartheta)) &\leq \mathbf{P}[\omega_1] U(c + \lambda G_T(\vartheta)(\omega_1) - C(\omega_1)) \\ &\quad + (1 - \mathbf{P}[\omega_1]) U\left(c + \lambda \max_{\omega \in \Omega} G_T(\vartheta)(\omega) - \min_{\omega \in \Omega} C(\omega)\right), \end{aligned}$$

and therefore there exist constants c_1, c_2, d_1, d_2 , where $d_1, d_2 > 0$ such that

$$\begin{aligned} \mathbf{E}U(Z(\lambda\vartheta)) &\leq \mathbf{P}[\omega_1] U(c_1 - \lambda d_1) + (1 - \mathbf{P}[\omega_1]) U(c_2 + \lambda d_2) \\ &= \mathbf{P}[\omega_1] U(c_1 - \lambda d_1) \left[1 + \frac{1 - \mathbf{P}[\omega_1]}{\mathbf{P}[\omega_1]} \frac{U(c_2 + \lambda d_2)}{U(c_1 - \lambda d_1)} \right]. \end{aligned} \quad (2)$$

²⁾ Normally the Inada conditions also imply the assumption $U'(\infty) = 0$, but we do not need this additional assumption in the following.

We know from concavity that $U(-\infty) = -\infty$. If $U(\infty) < \infty$ it follows directly, otherwise we deduce from de L'Hospital and $U'(-\infty) = \infty$, i.e., the Inada condition, that the expression in the square bracket tends to 1 as $\lambda \rightarrow \infty$. Therefore the right-hand side in (2) goes to $-\infty$. This proves that $-EU(Z(\vartheta))$ has no direction of recession.

To show uniqueness, we just assume that there are two different maximizing strategies and consider a linear combination of those with coefficients $\frac{1}{2}$. Due to nondegeneracy this provides us with an outcome that differs from the two maximizing ones on a set with probability greater than zero (cf. Remark 2.1). The utility function is by assumption strictly concave and thus the expected utility of the new strategy would be greater than the one of the maximizing strategies — the desired contradiction. Proposition 3.1 is proved.

Our goal is to show that Z^α converges to the balanced outcome Z and determine the speed of convergence. First we will show that the optimal outcome Z^α becomes «almost balanced» as the risk aversion $r_\alpha(x)$ tends to infinity.

Proposition 3.2. *Let $Z^{\alpha n} = c + G_T(\vartheta^{\alpha n}) - C$ be the optimal outcome for the utility maximization problem*

$$EU_{\alpha n}(c + G_T(\vartheta) - C) \mapsto \max_{\vartheta}.$$

Let further $r_{\alpha n}(x)$ denote the risk aversion for the utility function $U_{\alpha n}(x)$, $n \geq 1$. Then for any atom $F \in \mathcal{F}_t$, $t = 0, 1, \dots, T-1$, there exists a constant $K(F)$ such that

$$\left| \min_{\omega \in F_+} Z^{\alpha n}(\omega) - \min_{\omega \in F_-} Z^{\alpha n}(\omega) \right| \leq \frac{K(F)}{\inf_{x \in I} r_{\alpha n}(x)}$$

holds true for all $n \geq 1$, where $I := (-\infty, \min_{\omega \in F_-} Z^{\alpha n}(\omega)) \cup (-\infty, \min_{\omega \in F_+} Z^{\alpha n}(\omega))$.

P r o o f. Fix an atom $F \in \mathcal{F}_t$, $t = 0, 1, \dots, T-1$, and denote by $\vartheta_F^{\alpha n}$ the entry of the optimal strategy $\vartheta^{\alpha n}$ that corresponds to the node given by F . Since $\vartheta^{\alpha n}$ is optimal, we get, if we differentiate $EU_{\alpha n}(Z(\vartheta))$ with respect to the entry ϑ_F ,

$$0 = \mathbf{E}[U'_{\alpha n}(Z^{\alpha n}(\omega)) \Delta S_{t+1}(\omega); F],$$

where $\mathbf{E}[\cdot; F]$ denotes the expected value with respect to the atom F . We can split this sum into a sum over $\omega \in F_+$ and a sum over $\omega \in F_-$, since ω which belong to $F \setminus \{F_+ \cup F_-\}$ (i.e., ω for which $\Delta S_{t+1}(\omega) = 0$) do not contribute to the sum. Noting that $U'_{\alpha n}(x) > 0$ is strictly decreasing we get

$$\begin{aligned} 0 &\geq \mathbf{E}[U'_{\alpha n}(Z^{\alpha n}(\omega)) \Delta S_{t+1}(\omega); F_+] + \mathbf{E}\left[\left(\max_{\tilde{\omega} \in F_-} U'_{\alpha n}(Z^{\alpha n}(\tilde{\omega}))\right) \Delta S_{t+1}(\omega); F_-\right] \\ &= U'_{\alpha n}\left(\min_{\tilde{\omega} \in F_-} Z^{\alpha n}(\tilde{\omega})\right) \\ &\quad \times \left\{ \mathbf{E}\left[\frac{U'_{\alpha n}(Z^{\alpha n}(\omega))}{U'_{\alpha n}(\min_{\tilde{\omega} \in F_-} Z^{\alpha n}(\tilde{\omega}))} \Delta S_{t+1}(\omega); F_+\right] + \mathbf{E}[\Delta S_{t+1}; F_-] \right\}. \end{aligned} \quad (3)$$

This inequality must hold true for all $n \geq 1$, thus

$$\frac{U'_{\alpha n}(Z^{\alpha n}(\omega))}{U'_{\alpha n}(\min_{\tilde{\omega} \in F_-} Z^{\alpha n}(\tilde{\omega}))} \Delta S_{t+1}(\omega) > 0$$

must, for all $n \geq 1$, be bounded from above for $\omega \in F_+$. It follows from the definition of risk aversion, i.e.,

$$r(x) = -\frac{U''(x)}{U'(x)} = -\frac{d}{dx} \ln U'(x),$$

that $U'(a)/U'(b) = \exp\{\int_a^b r(x) dx\}$. Therefore we get, if $\min_{\tilde{\omega} \in F_-} Z^{\alpha n}(\tilde{\omega}) \geq Z^{\alpha n}(\omega)$,

$$\begin{aligned} \frac{U'_{\alpha n}(Z^{\alpha n}(\omega))}{U'_{\alpha n}(\min_{\tilde{\omega} \in F_-} Z^{\alpha n}(\tilde{\omega}))} &= \exp\left\{\int_{Z^{\alpha n}(\omega)}^{\min_{\tilde{\omega} \in F_-} Z^{\alpha n}(\tilde{\omega})} r_{\alpha n}(x) dx\right\} \\ &\geq \exp\left\{\left(\min_{\tilde{\omega} \in F_-} Z^{\alpha n}(\tilde{\omega}) - Z^{\alpha n}(\omega)\right) \inf_{x \in I} r_{\alpha n}(x)\right\}. \end{aligned}$$

So we get from (3) the necessary condition that for each $\omega \in F_+$ (note that for $\min_{\tilde{\omega} \in F_-} Z^{\alpha_n}(\tilde{\omega}) < Z^{\alpha_n}(\omega)$ the following holds trivially) there exists a constant $K_1(\omega)$ such that for any $n \geq 1$

$$\left(\min_{\tilde{\omega} \in F_-} Z^{\alpha_n}(\tilde{\omega}) - Z^{\alpha_n}(\omega) \right) \inf_{x \in I} r_{\alpha_n}(x) \leq K_1(\omega).$$

So in particular, since $|F_+| < \infty$, there exists a constant $K_1(F_+)$ such that for any $\alpha_n \geq 1$

$$\left(\min_{\tilde{\omega} \in F_-} Z^{\alpha_n}(\tilde{\omega}) - \min_{\tilde{\omega} \in F_+} Z^{\alpha_n}(\tilde{\omega}) \right) \inf_{x \in I} r_{\alpha_n}(x) \leq K_1(F_+),$$

i.e., $\min_{\tilde{\omega} \in F_-} Z^{\alpha_n}(\tilde{\omega}) - \min_{\tilde{\omega} \in F_+} Z^{\alpha_n}(\tilde{\omega}) \leq K_1(F_+)/\inf_{x \in I} r_{\alpha_n}(x)$, for all $\alpha_n \geq 1$.

If one uses

$$0 \leq \mathbf{E} \left[\left(\max_{\tilde{\omega} \in F_+} U'_{\alpha_n}(Z^{\alpha_n}(\tilde{\omega})) \right) \Delta S_{t+1}(\omega); F_+ \right] + \mathbf{E} [U'_{\alpha_n}(Z^{\alpha_n}(\omega)) \Delta S_{t+1}; F_-]$$

instead of (3), one analogously gets

$$\min_{\tilde{\omega} \in F_+} Z^{\alpha_n}(\tilde{\omega}) - \min_{\tilde{\omega} \in F_-} Z^{\alpha_n}(\tilde{\omega}) \leq \frac{K_1(F_-)}{\inf_{x \in I} r_{\alpha_n}(x)} \quad \forall \alpha_n \geq 1,$$

thus completing the proof for the constant $K(F) = \max\{K_1(F_+), K_1(F_-)\}$. Proposition 3.2 is proved.

The previous proposition showed that the optimal outcome Z^{α_n} gets, for large risk aversion, «almost balanced». Now we have to investigate whether this property of the outcome also implies that the corresponding optimal trading strategy ϑ^{α_n} becomes «almost balanced». Actually we do not need the fact the strategy ϑ^{α_n} is optimal. The following proposition can be formulated for any strategy ϑ^{β_n} and outcome $Z^{\beta_n} = Z(\vartheta^{\beta_n})$.

Proposition 3.3. *Let β_n be a sequence in $\mathbf{R}$ and $r: \mathbf{R} \mapsto \mathbf{R}$ a function such that $r(\beta_n)$ tends to infinity as $n \geq 1$ tends to ∞ . If for all atoms $F \in \mathcal{F}_t$, $t = 0, 1, \dots, T-1$,*

$$\left| \min_{\tilde{\omega} \in F_+} Z^{\beta_n}(\tilde{\omega}) - \min_{\tilde{\omega} \in F_-} Z^{\beta_n}(\tilde{\omega}) \right| = O\left(\frac{1}{r(\beta_n)}\right)$$

as $n \rightarrow \infty$, then

$$\|\vartheta^{\beta_n} - \bar{\vartheta}\|_{\infty} = O\left(\frac{1}{r(\beta_n)}\right),$$

where $\|\cdot\|_{\infty}$ denotes the supremum norm on $\mathbf{R}^M$.

P r o o f. Suppose the proposition would not hold true. Then for $n = 1, 2, \dots$ there exist nodes F_n and a function $g: \mathbf{N} \mapsto \mathbf{N}$ tending to infinity as $n \rightarrow \infty$ such that $|\vartheta_{F_n}^{\beta_n} - \bar{\vartheta}_{F_n}| \geq g(n)/r(\beta_n)$. Since there are only finitely many nodes F_n , there have to exist one fixed node F and a subsequence n_k tending to ∞ , such that

$$|\vartheta_F^{\beta_{n_k}} - \bar{\vartheta}_F| \geq \frac{g(n_k)}{r(\beta_{n_k})}. \quad (4)$$

If there are more such atoms F_i , which correspond to times t_i , we choose a node $F = F_i$ such that $t = t_i$ is maximal. This way we get for nodes $G \subsetneq F$, i.e., for nodes G following F , $|\vartheta_G^{\beta_{n_k}} - \bar{\vartheta}_G| = O(1/r(\beta_{n_k}))$.

We will consider the subtree starting at this specially chosen node F at time t , and we will deduce a contradiction by showing that for any given $K_1 > 0$ there exists an $\beta_0 = \beta_0(K_1)$ such that

$$\left| \min_{\tilde{\omega} \in F_+} Z^{\beta_0}(\tilde{\omega}) - \min_{\tilde{\omega} \in F_-} Z^{\beta_0}(\tilde{\omega}) \right| > \frac{K_1}{r(\beta_0)}.$$

To do this, we define the outcome $\tilde{Z}^{\beta} = Z(\tilde{\vartheta}^{\beta})$, which we get if we follow the strategy $\tilde{\vartheta}$ everywhere except on the subtree starting at F . There we use the balanced strategy $\bar{\vartheta}$

just for the first time step $(t, t + 1]$ and after that we follow the strategy ϑ^β , i.e., $\tilde{\vartheta}^\beta = \tilde{\vartheta}$ everywhere except on nodes $G \subsetneq F$, where $\tilde{\vartheta}_G^\beta = \vartheta_G^\beta$.

We know that there exists $K_2 > 0$ such that

$$\left| \min_{\tilde{\omega} \in F_+} \tilde{Z}^{\beta_{n_k}}(\tilde{\omega}) - \min_{\tilde{\omega} \in F_-} \tilde{Z}^{\beta_{n_k}}(\tilde{\omega}) \right| < \frac{K_2}{r(\beta_{n_k})}$$

holds for all β_{n_k} . This follows since the node F was chosen such that t is maximal and since the outcome $\tilde{Z}^{\beta_{n_k}} = c + G_T(\tilde{\vartheta}^{\beta_{n_k}}) - C$ depends in the affine way on the strategy.

If we set

$$m = \min\{|\Delta S_{t+1}(\omega)| : |\Delta S_{t+1}(\omega)| > 0 \text{ and } \omega \in F\},$$

we know from the nondegeneracy condition that $m > 0$ and we can take $g(n_k) \geq (K_1 + K_2)/(2m)$.

Let us now suppose $\vartheta_F^{\beta_{n_k}} > \bar{\vartheta}_F$, the other case work analogously, just the roles of F_+ and F_- have to be interchanged. By (4) this implies $\vartheta_F^{\beta_{n_k}} \geq \bar{\vartheta}_F + g(n_k)/r(\beta_{n_k})$, and we get $\min_{\tilde{\omega} \in F_+} \tilde{Z}^{\beta_{n_k}}(\tilde{\omega}) > \min_{\tilde{\omega} \in F_+} \tilde{Z}^{\beta_{n_k}}(\tilde{\omega}) + mg(n_k)/r(\beta_{n_k})$, $\min_{\tilde{\omega} \in F_-} \tilde{Z}^{\beta_{n_k}}(\tilde{\omega}) < \min_{\tilde{\omega} \in F_-} \tilde{Z}^{\beta_{n_k}}(\tilde{\omega}) - mg(n_k)/r(\beta_{n_k})$. Therefore

$$\begin{aligned} \min_{\tilde{\omega} \in F_+} \tilde{Z}^{\beta_{n_k}}(\tilde{\omega}) - \min_{\tilde{\omega} \in F_-} \tilde{Z}^{\beta_{n_k}}(\tilde{\omega}) &> \min_{\tilde{\omega} \in F_+} \tilde{Z}^{\beta_{n_k}}(\tilde{\omega}) - \min_{\tilde{\omega} \in F_-} \tilde{Z}^{\beta_{n_k}}(\tilde{\omega}) + \frac{2mg(n_k)}{r(\beta_{n_k})} \\ &> -\frac{K_2}{r(\beta_{n_k})} + \frac{K_1 + K_2}{r(\beta_{n_k})} = \frac{K_1}{r(\beta_{n_k})}, \end{aligned}$$

the desired contradiction with $\beta_0 = \beta_{n_k}$. Proposition 3.3 is proved.

To get our main theorem, we now just have to apply Proposition 3.3 for the optimal outcome Z^{α_n} , where Proposition 3.2 guarantees that the necessary assumptions are fulfilled. Since $\vartheta \mapsto Z(\vartheta)(\omega)$ is affine in ϑ , convergence of the strategy ϑ^{α_n} implies convergence of the outcome Z^{α_n} at the same rate of convergence.

Theorem 3.1. Assume we are given a finite probability space $(\Omega, (\mathcal{F})_{t=0,\dots,T}, \mathbf{P})$ with a finite time horizon $T \in \mathbf{N}$, an $\mathbf{R}$ -valued adapted stochastic process $(S_t)_{t=0,\dots,T}$ representing the discounted stock price process, a riskless bond $B = 1$, the contingent claim $C \in \mathcal{F}_T$, and the initial endowment c for an agent holding a short position in the contingent claim C . Assume further that Assumption 2.1 is satisfied, i.e., S is nondegenerate.

Given a family of utility functions $(U_{\alpha_n}(x))_{n \geq 1}$ whose risk aversion tends to infinity uniformly as n tends to ∞ , let ϑ^{α_n} (respectively, Z^{α_n}) denote the optimal strategy (respectively, outcome) for the optimization problem

$$\mathbf{E}U_{\alpha_n}(c + G_T(\vartheta) - C) \mapsto \max_{\vartheta}.$$

Then

$$\|\vartheta^{\alpha_n} - \bar{\vartheta}\|_\infty = O\left(\frac{1}{\inf_{x \in \mathbf{R}} r_{\alpha_n}(x)}\right) \quad \text{and} \quad \sup_{\omega \in \Omega} |Z^{\alpha_n}(\omega) - \bar{Z}(\omega)| = O\left(\frac{1}{\inf_{x \in \mathbf{R}} r_{\alpha_n}(x)}\right).$$

4. Different notions of superhedging. Now we are going to use the notion of a balanced strategy to show the connections to the concept of superhedging. As indicated in the introduction there are some intuitive connections between increasing risk aversion and superhedging. We saw however in the quite easy Example 2.1 that there might be more than one superhedging strategy, whereas the notation of a balanced strategy is, in the setting presented so far, unique. Therefore we look at a more restrictive notion of superhedging that is already known in the literature.

In his paper [5] Kramkov considers two notions of superhedging (in fact he uses the word «hedging», not «superhedging»), superhedging and minimal superhedging. The setting of the paper is a very general, continuous time model. But we are going to apply the results just to our finite Ω model, described by $(\Omega, (\mathcal{F}_t)_{t=0,1,\dots,T}, \mathbf{P})$, the stock process $(S_t)_{t=0,1,\dots,T}$, the initial endowment c (which will, for consistency reasons, in this section be denoted by V_0), the contingent claim $C \in \mathcal{F}_T$, and the riskless bond $B = 1$. Let us consider a trading strategy ϑ , which in our finite setting can be interpreted as a

vector, together with the value process $V_t = V_0 + G_t(\vartheta) - K_t$, where $K_t \geq 0$ is (in our finite setting) an adapted nondecreasing process, which can be interpreted as cumulative consumption. Note that until now we only looked at the final gain of a trading strategy, i.e., $G_T(\vartheta) = \int_0^T \vartheta dS$. Now we also have to consider intermediate wealth, thus we use $G_t(\vartheta) = \int_0^t \vartheta dS$ as the intermediate gain of the trading strategy ϑ .

We call ϑ_t and K_t , together with the initial wealth V_0 , a *superhedging portfolio* for the claim C , if

$$C \leq V_T,$$

i.e., if the strategy gives us, provided we start with initial endowment V_0 , enough to satisfy the claim at final time T . We call a strategy $\hat{\vartheta}_t$, a nondecreasing consumption process $\hat{K}_t \geq 0$ and an initial wealth $\hat{V}_0$ with value process $\hat{V}_t = \hat{V}_0 + G_t(\hat{\vartheta}) - \hat{K}_t$ *minimal superhedging portfolio*, if it satisfies

$$\hat{V}_t \leq V_t, \quad t \in [0, T], \quad (5)$$

for all superhedging portfolios (V_0, ϑ_t, K_t) with value processes V_t .

Kramkov shows that a minimal superhedging portfolio exists such that its value process $\hat{V}_t$ is given by $\hat{V}_t = \sup_{Q \in \mathcal{M}^e} E_Q[C | \mathcal{F}_t]$, and that it gives rise to the so-called optional decomposition $\hat{V}_0 + G_t(\hat{\vartheta}) - \hat{K}_t = \hat{V}_t$. $\mathcal{M}^e$ denotes, see [5] for details, the set of martingale measures equivalent to $\mathbf{P}$. At final time T this gives

$$\hat{V}_0 + G_T(\hat{\vartheta}) - \hat{K}_T = \hat{V}_T.$$

Note that $C = \sup_{Q \in \mathcal{M}^e} E_Q[C | \mathcal{F}_T] = \hat{V}_T$ holds and therefore the cumulative consumption at time T

$$\hat{K}_T = \hat{V}_0 + G_T(\hat{\vartheta}) - C$$

can be interpreted as the outcome of an agent starting with an initial endowment of $\hat{V}_0$ and a short position in the contingent claim C , using strategy $\hat{\vartheta}$. Since by assumption $\hat{K}_T \geq 0$, this is a superhedging strategy.

What are the relations between the minimal superhedging portfolio introduced by Kramkov and the concept of balanced strategies? Note that the balanced strategy is independent of the initial endowment. It becomes however a superhedging strategy provided the initial endowment is big enough to be able to superhedge, i.e., if the endowment equals $\sup_{Q \in \mathcal{M}^e} E_Q C$. We will actually show that it is a *minimal* superhedging strategy, i.e., the strategy, belonging to a minimal superhedging portfolio.

The minimal superhedging portfolio seems to be more restrictive, since it imposes the condition (5), which is a condition concerning the *whole time-period* $[0, T]$, whereas the balanced strategy is only concerned with the outcome at the *final time* T . We will however show that every balanced outcome $\bar{Z}$ is given by the outcome $\bar{Z} = Z(\bar{\vartheta})$ for some minimal superhedging strategy $\bar{\vartheta}$. Before proving this, the following remark shows that an implication in the other direction, namely that every minimal superhedging strategy is a balanced strategy, does for blatant reasons not hold true.

R e m a r k 4.1. Note that the minimal superhedging strategy does not have to be unique. This can be seen by considering the following easy example: $S_0 = 2$, $\Delta S_1 = (1, 0, -1)$, $c = 1 = V_0$, and $C = (0, 1, 0)$. In this setting any $\vartheta \in [-1, 1]$ is a minimal superhedging strategy with $V_0 = 1$, $K_0 = 0$, $V_1 = (0, 1, 0) = C$, and $K_1 = (1 + \vartheta, 0, 1 - \vartheta)$. So in this example the restriction to minimal superhedging strategies does not provide a superhedging agent with any help for deciding which strategy to choose, whereas the balanced strategy $\bar{\vartheta} = 0$ is unique.

Let us now show that the balanced outcome $\bar{Z}$ corresponds, as claimed above, indeed to a minimal superhedging strategy.

Theorem 4.1. *Given a market as in Definition 2.1 that satisfies in addition Assumption 2.1. Then the balanced strategy $\bar{\vartheta}$, defined in Section 2, is a minimal superhedging strategy and the balanced outcome $\bar{Z} = Z(\bar{\vartheta})$ equals the consumption $\hat{K}_T$ for a minimal superhedging portfolio.*

P r o o f. A strategy $\hat{\vartheta}$, a consumption process $\hat{K}$, and an initial wealth $\hat{V}_0$ are a minimal superhedging portfolio if and only if they satisfy

$$\hat{V}_t = \sup_{Q \in \mathcal{H}} \mathbf{E}_Q[C \mid \mathcal{F}_t]$$

and $\hat{K}_t \geq 0$, determined by

$$\hat{V}_0 + G_t(\hat{\vartheta}) - \hat{K}_t = \hat{V}_t,$$

is nondecreasing.

To prove the theorem we will look at the balanced strategy $\bar{\vartheta}$ and choose the corresponding value process $\bar{V}_t$ (respectively, the corresponding consumption process $\bar{K}_t = \bar{V}_0 - \bar{V}_t + G_t(\bar{\vartheta})$) in such a way that the superhedging portfolio $(\bar{V}_0, \bar{\vartheta}, \bar{K})$ is a minimal superhedging portfolio.

We start building up a strategy to get the balanced outcome $\bar{Z}$ by working backwards through the tree induced by the stock process S_t . We set $\bar{K}_T = \bar{Z}_T$ and therefore $\bar{V}_T = C = \bar{V}_T$. For each node F at time $T-1$ we have to determine the corresponding strategy $\bar{\vartheta}_F$ to get the balanced outcome $\bar{Z} = \bar{K}_T \geq 0$. But this simply means that we have to take the unique $\bar{\vartheta}_F$ to get $\bar{\vartheta}_F \Delta S_T - \bar{V}_T$ balanced, i.e.,

$$\min_{\omega \in F_+} (\bar{\vartheta}_F \Delta S_T(\omega) - \bar{V}_T(\omega)) = \min_{\omega \in F_-} (\bar{\vartheta}_F \Delta S_T(\omega) - \bar{V}_T(\omega)).$$

For a node $F \in \mathcal{F}_{T-1}$ we denote by $\bar{V}_{T-1}(F) \in \mathbf{R}$ the value of $\bar{V}_{T-1}(\omega)$ for $\omega \in F$, which is, since $F \in \mathcal{F}_{T-1}$ is an atom, equal for all $\omega \in F$. Now we set for each such node $F \in \mathcal{F}_{T-1}$ the value $\bar{V}_{T-1}(F) \in \mathbf{R}$ minimal, such that

$$\bar{V}_{T-1}(F) \geq \bar{V}_T(\omega) - \bar{\vartheta}_F \Delta S_T(\omega) \quad (6)$$

for all $\omega \in F_+ \cup F_-$. If we repeat this procedure for all the previous time steps $T-2, T-3, \dots, 0$, we get a superhedging strategy $\bar{\vartheta}$ and a nondecreasing consumption process $\bar{K}_t$, since

$$\bar{K}_t(\omega) - \bar{K}_{t-1}(F) = \bar{V}_{t-1}(F) - (\bar{V}_t(\omega) - \bar{\vartheta}_F \Delta S_t(\omega)) \geq 0,$$

because of (6).

Now it just remains to show that this superhedging strategy $\bar{\vartheta}$ is indeed a minimal superhedging strategy. We will do this by showing that, for some minimal superhedging portfolio $\hat{V}$, $\bar{V}_{T-1}(F) = \hat{V}_{T-1}(F)$ for each node F at time $T-1$ and then one just has to work again back in time. This gives $\bar{V}_t = \hat{V}_t$ for all $t \in [0, T]$, thus proving the theorem.

To show that $\bar{V}_{T-1}(F) = \hat{V}_{T-1}(F)$, first note that by the definition of a *minimal* superhedging strategy $\hat{V}_{T-1}(F) \leq \bar{V}_{T-1}(F)$.

If we assume that $\bar{V}_{T-1}(F) > \hat{V}_{T-1}(F)$, there exists by the choice of $\bar{V}_{T-1}(F)$ an $\omega \in F_+ \cup F_-$ such that $\bar{V}_T(\omega) - \bar{\vartheta}_F \Delta S_T(\omega) > \hat{V}_{T-1}(F)$, i.e.,

$$-\hat{V}_{T-1}(F) > \bar{\vartheta}_F \Delta S_T(\omega) - \bar{V}_T(\omega) \geq \min_{\tilde{\omega} \in F_+ \cup F_-} (\bar{\vartheta}_F \Delta S_T(\tilde{\omega}) - \bar{V}_T(\tilde{\omega})). \quad (7)$$

Since $\bar{\vartheta}$ was chosen to give a balanced outcome, this implies that there exist $\omega_+ \in F_+$ and $\omega_- \in F_-$ for which the minimum on the right-hand side of the last inequality is attained.

If we now assume that $\hat{\vartheta}_F \leq \bar{\vartheta}_F$ it is enough to take $\omega = \omega_+$, otherwise just take $\omega = \omega_-$, to get from (7), using $\bar{V}_T = \hat{V}_T$,

$$-\hat{V}_{T-1}(F) > \bar{\vartheta}_F \Delta S_T(\omega) - \hat{V}_T(\omega) \geq \hat{\vartheta}_F \Delta S_T(\omega) - \hat{V}_T(\omega). \quad (8)$$

Now we just have to use the fact that the consumption process $\hat{K}_t$ is nondecreasing, to get finally

$$\hat{\vartheta}_F \Delta S_T(\omega) - \hat{V}_T(\omega) \geq -\hat{V}_{T-1}(F).$$

Together with (8) this now leads to $\hat{V}_{T-1}(F) < \hat{V}_{T-1}(F)$, the desired contradiction.

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CENTRAL LIMIT THEOREM FOR CERTAIN CLASSES OF DEPENDENT RANDOM VARIABLES¹⁾

Наш главный интерес в этой статье — центральная предельная теорема для зависимых классов случайных величин. Мы вычисляем точную разность между характеристической функцией суммы зависимых случайных величин и характеристической функцией суммы независимых случайных величин с теми же самыми распределениями. Полученная разность служит для определения класса зависимых случайных величин, для которых имеет место обычная сходимости по распределению к нормально распределенной случайной величине. Структура зависимости в том виде, как она определена в нашем классе случайных величин, отвечает определенным физическим явлениям.

Ключевые слова и фразы: центральная предельная теорема, зависимые случайные величины.

1. Introduction. During the past century the central limit theorem, CLT, for dependent random variables was one of the most active areas of research. Since Bernstein's paper (1926), many authors have occupied themselves with the problem of extending the central limit theorem for classes of dependent random variables, see for example, [1], [3], or [4]. Various conditions of dependence were derived from the estimation of the difference between characteristic function of averages of dependent and independent random variables. These conditions had some appealing physical interpretations, most often concerned with the passing of time. Dependence structures were defined with the idea that events far separated in time should be stochastically less dependent than close events.

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