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Corollary 2. Suppose $\{X_i: i \geq 1\}$ is a sequence of identically distributed random variables such that $E|X_1| < +\infty$, $n \in \mathbf{N}$. If there exists δ , $0 < \delta < 1$, such that for sufficiently large k

$$\int_{\mathbf{R}^j} \left| \mathbf{P} \left(\bigcap_{i=1}^j \{X_{v_i} \leq x_{v_i}\} \right) - \prod_{i=1}^j \mathbf{P} \{Y_{v_i} \leq x_{v_i}\} \right| dx_{v_1} \cdots dx_{v_j} \leq (1 - k^{-\delta/2})^{k - k^{\delta/2} - j}, \quad (38)$$

where $v_1, \dots, v_j$ is any choice of indices such that $k^{\delta/2} < v_1 < \dots < v_j \leq k$, then

$$\frac{1}{\sigma\sqrt{k}} \sum_{i=1}^k (X_i - EX_i) \xrightarrow{D} N(0, 1) \quad \text{as } k \rightarrow +\infty.$$

The proof follows directly from Theorem 4.

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SOME REMARKS ON AN INTERPOLATION PROBLEM OF A. M. YAGLOM

В статье рассматривается проблема наилучшей интерполяции в смысле среднеквадратической ошибки для непрерывного q -мерного слабо стационарного процесса x на $\mathbf{R}$, если известны x_k , $k \in \mathbf{Z}$. Такая проблема впервые изучалась А. М. Ягломом [14] в случае $q = 1$. Пусть $\mathcal{J}_{\mathbf{Z}}$ обозначает семейство всех подмножеств множества $\mathbf{R}$, которые получаются сдвигами множества $\mathbf{Z}$. С помощью спектральной меры процесса x выводятся критерии $\mathcal{J}_{\mathbf{Z}}$ -сингулярности и $\mathcal{J}_{\mathbf{Z}}$ -регулярности. Аналогичные результаты сформулируются для стационарных случайных последовательностей на $\mathbf{Z}$.

Ключевые слова и фразы: q -мерный слабо стационарный процесс, лучшая интерполяция в смысле среднеквадратической ошибки, $\mathcal{J}_{\mathbf{Z}}$ -регулярность и $\mathcal{J}$ -сингулярность, линейная фильтрация.

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1. Introduction. Let q be an element of the set $\mathbf{N}$ of positive integers and $\mathbf{x} := \{x_t\}_{t \in \mathbf{R}}$ a continuous q -variate weakly stationary random process over the group $\mathbf{R}$ of real numbers. In the univariate case, i.e., if $q = 1$, Yaglom [14] studied the following linear interpolation problem. For $s \in \mathbf{R}$, find the best mean-square approximation of x_s by the elements of the space spanned by x_k , $k \in \mathbf{Z}$. Here $\mathbf{Z}$ denotes the group of integers. Yaglom [14, Theorem 1] gave a formula for the approximation error. Salehi [12, Theorem 3.2] completed Yaglom's result by computing the element of best approximation and at the same time extended all assertions to arbitrary q . Generalizations to Hilbert-Schmidt operator-valued weakly stationary processes were given in [6, Theorems 4.8 and 4.10].

All these results are obtained by transforming the interpolation problem from the time domain of the process to its spectral domain via Kolmogorov's isomorphism. Roughly speaking, the spectral domain of $\mathbf{x}$ is a left Hilbert module $L^2(F)$ of $(q \times q)$ -matrix valued functions which are square-integrable with respect to a nonnegative hermitian $(q \times q)$ -matrix valued measure F , the so-called nonstochastic spectral measure of $\mathbf{x}$. The image of x_t under Kolmogorov's isomorphism is the function $e^{it\cdot}E$, $t \in \mathbf{R}$, where E denotes the identity matrix of order q . Thus, the problem above can be given the following form. Find the orthogonal projection in $L^2(F)$ of the function $e^{is\cdot}E$ onto the closed left submodule H_0 generated by the functions $e^{ik\cdot}E$, $k \in \mathbf{Z}$.

Using the method of [6] one can derive a description of H_0 . This description, which is given in Lemmas 3.1 and 3.2 of the present paper is the corner stone of our investigations and enables us to solve the mentioned interpolation problem as well as some related problems. For example, we can compute not only the orthogonal projection of $e^{is\cdot}E$ onto H_0 , but also the orthogonal projection of an arbitrary function of $L^2(F)$. The corresponding assertion as well as an application to a linear filtration problem of Yaglom type are contained in Section 4.

As is well known, the notions of regular and singular processes play an important role in the theory of weakly stationary random processes. Salehi and Scheidt [13, Definition 2.10] introduced the following and even more general extensions of these concepts. For a nonempty subset A of $\mathbf{R}$, denote by $\mathbf{x}(A)$ the subspace of the time domain of $\mathbf{x}$ spanned by x_t , $t \in A$. Let $\mathcal{J}$ be a family of subsets of $\mathbf{R}$ invariant under translation, i.e., if $A \in \mathcal{J}$ and $s \in \mathbf{R}$, then $s + A \in \mathcal{J}$. The process $\mathbf{x}$ is called $\mathcal{J}$ -regular if $\bigcap_{A \in \mathcal{J}} \mathbf{x}(A) = \{0\}$. It is called $\mathcal{J}$ -singular if $\mathbf{x}(A)$ coincides with the time domain of $\mathbf{x}$ for all $A \in \mathcal{J}$.

In connection with the linear interpolation problem above it is natural to discuss $\mathcal{J}_{\mathbf{Z}}$ -regularity and $\mathcal{J}_{\mathbf{Z}}$ -singularity of $\mathbf{x}$, where $\mathcal{J}_{\mathbf{Z}}$ stands for the set of shifts of $\mathbf{Z}$. We obtain criteria of $\mathcal{J}_{\mathbf{Z}}$ -singularity and $\mathcal{J}_{\mathbf{Z}}$ -regularity, respectively, of a process $\mathbf{x}$ in terms of its nonstochastic spectral measure, see Theorems 3.1 and 3.2, respectively.

The results presented for processes over $\mathbf{R}$ in Sections 3 and 4 have analogues for q -variate weakly stationary random sequences over $\mathbf{Z}$. Section 5 deals with the corresponding assertions. We mention that similar problems for (univariate) harmonizable symmetric p -stable random sequences, $p \in (1, 2]$, are discussed in [8].

2. Preliminaries. If X is a matrix with entries from the field of complex numbers $\mathbf{C}$, the symbols $\text{Ker}X$, $\mathcal{R}(X)$, X^* , and X^+ stand for the kernel, range, the adjoint, and the Moore-Penrose inverse of X , respectively. If X is quadratic, $\text{tr}X$ denotes its trace. For $q \in \mathbf{N}$, let $\mathcal{M}_q$ be the algebra of $(q \times q)$ -matrices and $\mathcal{M}_q^{\geq}$ the subset of nonnegative hermitian matrices. The set $\mathcal{M}_q^{\geq}$ will be equipped with Loewner's partial ordering, i.e., $X \leq Y$ if and only if $Y - X \in \mathcal{M}_q^{\geq}$, $X, Y \in \mathcal{M}_q^{\geq}$.

The following well-known result is basic in the theory of nonnegative hermitian matrices.

Lemma 2.1 ([1, Theorem 1], see also [3, § 2, Section 4]). *Let $q, r \in \mathbf{N}$. A block matrix $\begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix} \in \mathcal{M}_{q+r}$ belongs to $\mathcal{M}_{q+r}^{\geq}$ if and only if the following conditions are satisfied:*

- (i) $\mathcal{R}(Y^*) \subseteq \mathcal{R}(Z)$,
- (ii) $Z \in \mathcal{M}_r^{\geq}$,
- (iii) $X - YZ^+Y^* \in \mathcal{M}_q^{\geq}$.

For an $\mathcal{M}_q$ -valued function on a Borel subset of $\mathbf{R}$, measurability means its Borel measurability. Relations between measurable functions are to be understood as relations which are true a.e. with respect to the underlying measure although this will not be

emphasized each time. The following result is a consequence of the fact that «measurability is invariant under many operations of linear algebra», cf. [2, Corollary 2].

Lemma 2.2. *Let Φ be a measurable $\mathcal{M}_q$ -valued function on a Borel subset of $\mathbf{R}$. Then the function $P_{\mathfrak{A}(\Phi)}$, whose values are orthogonal projections in $\mathbf{C}^q$ onto the ranges of Φ , as well as the function Φ^+ are measurable.*

3. $\mathcal{I}_Z$ -regularity and $\mathcal{I}_Z$ -singularity. Let $q \in \mathbf{N}$, let F be an $\mathcal{M}_q^\geq$ -valued Borel measure on $\mathbf{R}$ and ν a nonnegative σ -finite Borel measure on $\mathbf{R}$ such that F is absolutely continuous with respect to ν . Denote by $F' := \frac{dF}{d\nu}$ the Radon–Nikodým derivative of F with respect to ν . Then the set $L^2(F)$ of all (equivalence classes of) measurable $\mathcal{M}_q$ -valued functions Φ such that $\int \Phi F' \Phi^* d\nu$ exists forms a left Hilbert- $\mathcal{M}_q$ -module with $\mathcal{M}_q$ -valued inner product

$$\langle \Phi, \Psi \rangle_F := \int \Phi F' \Psi^* d\nu, \quad \Phi, \Psi \in L^2(F).$$

(Throughout Sections 3 and 4, integration is over $\mathbf{R}$ if not indicated otherwise.) The definition of $L^2(F)$ does not depend on the choice of ν .

In the present paper we will obtain some approximation results concerning $L^2(F)$, which can be given a probabilistic sense if one considers $L^2(F)$ as the spectral domain of a continuous q -variate weakly stationary random process $\mathbf{x}$ over $\mathbf{R}$ and takes into account Kolmogorov's isomorphism between the time and the spectral domains of $\mathbf{x}$.

Remark 3.1. We find it convenient to work with a space $L^2(F)$ according to the definition above, which was given by Rosenberg [10]. However, it should be mentioned that perhaps the first definition of $L^2(F)$ -spaces was given by Kac [5] in a slightly different form. Rozanov used it in his book [11], which we refer to for more information on q -variate stationary random processes.

Let us introduce some further notation. For $k \in \mathbf{Z}$, denote by I_k the interval $[2k\pi, 2(k+1)\pi)$. Let τ_k be the shift by $2k\pi$, i.e., if D is a subset of $\mathbf{R}$, set $\tau_k D := 2k\pi + D$, and if f is a function on D , denote by $\tau_k f$ the function $\tau_k f(t) := f(t - 2k\pi)$, $t \in \tau_k D$. If a function f is defined on $\mathbf{R}$, let f_k be its restriction to the interval I_k , $k \in \mathbf{Z}$. The symbol $[g_k]_{k \in \mathbf{Z}} =: [g_k]$ stands for a two-sided sequence of functions such that the domain of definition of the function g_k at the k -th place is I_k , $k \in \mathbf{Z}$. Note that a function Φ of $L^2(F)$ can be identified with the sequence $[\Phi_k]$ of its restrictions to I_k , $k \in \mathbf{Z}$. On the other hand, if $[\Phi_k]$ is a sequence of measurable $\mathcal{M}_q$ -valued functions Φ_k on I_k such that $\sum \int_{I_k} \Phi_k F'_k \Phi_k^* d\nu$ exists, we can consider Φ_k as the restriction of a function $\Phi \in L^2(F)$ to I_k , $k \in \mathbf{Z}$, and we have $\langle \Phi, \Phi \rangle_F = \sum \int_{I_k} \Phi_k F'_k \Phi_k^* d\nu$. (Throughout the paper, the summation index is denoted by k and it runs through $\mathbf{Z}$ if not indicated otherwise.)

Given F , for a Borel subset B of I_0 , define

$$\widehat{F}(B) := \sum F(\tau_k B) \quad \text{and} \quad \widehat{\nu}(B) := \text{tr } \widehat{F}(B).$$

For a Borel subset B of $\mathbf{R}$, set

$$\nu(B) := \sum \widehat{\nu}(\tau_{-k}(B \cap I_k)).$$

Note that $\widehat{\nu}$ is a nonnegative finite Borel measure on I_0 such that the $\mathcal{M}_q^\geq$ -valued Borel measure $\widehat{F}$ is absolutely continuous with respect to $\widehat{\nu}$. Similarly, ν is a nonnegative σ -finite Borel measure on $\mathbf{R}$ and F is absolutely continuous with respect to ν . Note further that

$$\widehat{F}' := \frac{d\widehat{F}}{d\widehat{\nu}} = \sum \tau_{-k} F'_k.$$

Let $I := [0, 1)$ and $\xi_s := e^{2\pi s i}$ if $s \in I$.

For $\widehat{\Phi} \in L^2(\widehat{F})$, consider the sequence $[\xi_s^k \tau_k \widehat{\Phi}]$, $s \in I$.

Lemma 3.1. *Let $s \in I$. The map $\widehat{\Phi} \mapsto [\xi_s^k \tau_k \widehat{\Phi}]$ is $\mathcal{M}_q$ -linear and isometric from $L^2(\widehat{F})$ into $L^2(F)$.*

P r o o f. Obviously, the map is $\mathcal{M}_q$ -linear. Lemma 4.2 of [6] yields that it is isometric.

From Lemma 3.1 it follows that the set

$$H_s := \left\{ [\xi_s^k \tau_k \hat{\Phi}] : \hat{\Phi} \in L^2(\hat{F}) \right\}, \quad s \in I,$$

is a closed submodule of $L^2(F)$.

For $t \in \mathbf{R}$, denote by e_t the function $e_t(u) := e^{itu}$, $u \in \mathbf{R}$. Let $\mathcal{T}$ be the set of all $\mathcal{M}_q$ -valued trigonometric polynomials, i.e., the set of all $\mathcal{M}_q$ -linear combinations of e_k , $k \in \mathbf{Z}$. For $s \in I$, set $\mathcal{T}_s := \{e_s T : T \in \mathcal{T}\}$. Clearly, $\mathcal{T}_s \subseteq H_s$. Moreover, using Lemma 3.1 and [6, Lemma 4.1] we get the following result.

Lemma 3.2. *For $s \in I$, the set $\mathcal{T}_s$ is dense in H_s with respect to the topology of $L^2(F)$.*

Let $\mathcal{J}$ be a family of subsets of $\mathbf{R}$ invariant under translation. Salehi and Scheidt [13] studied $\mathcal{J}$ -regularity and $\mathcal{J}$ -singularity of q -variate weakly stationary random processes. In particular, they established the existence and uniqueness of the Wold decomposition into a $\mathcal{J}$ -regular and a $\mathcal{J}$ -singular part, cf. [13, Theorem 2.13]. The present paper deals with the family $\mathcal{J} =: \mathcal{J}_{\mathbf{Z}}$, which consists of all shifts of $\mathbf{Z}$, i.e.,

$$\mathcal{J}_{\mathbf{Z}} := \{s + \mathbf{Z} : s \in I\}.$$

If we take into account Kolmogorov's isomorphism and Lemma 3.2, we see that the following definitions of $\mathcal{J}_{\mathbf{Z}}$ -regularity and $\mathcal{J}_{\mathbf{Z}}$ -singularity correspond to Salehi's and Scheidt's definitions, see [13, Definition 2.10].

Definition 3.1. The space $L^2(F)$ is called $\mathcal{J}_{\mathbf{Z}}$ -regular if

$$\bigcap_{s \in I} H_s = \{0\}$$

and $\mathcal{J}_{\mathbf{Z}}$ -singular if

$$H_s = L^2(F)$$

for some and, hence, for all $s \in I$.

We wish to describe these notions in terms of F . To do this let us introduce the following conditions:

(S) For all $j \in \mathbf{Z}$, $\mathcal{R}(\tau_{-j} F'_j) \cap \mathcal{R}(\sum_{k \neq j} \tau_{-k} F'_k) = \{0\}$ $\hat{\nu}$ -a.e.,

(R) for all $j \in \mathbf{Z}$, $\mathcal{R}(\tau_{-j} F'_j) \subseteq \mathcal{R}(\sum_{k \neq j} \tau_{-k} F'_k)$ $\hat{\nu}$ -a.e.

Remark 3.2. Note that if $X, Y \in \mathcal{M}_q^{\geq}$, the algebraic sum $\mathcal{R}(X) + \mathcal{R}(Y)$ is equal to $\mathcal{R}(X + Y)$. Thus, condition (S) is equivalent to the following: For $\hat{\nu}$ -a.a. $t \in I_0$, there exists a unique set of at most q spaces $\mathcal{R}(\tau_{-k} F'_k(t))$, $k \in \mathbf{Z}$, which are different from $\{0\}$ and the direct sum of which equals $\mathcal{R}(\hat{F}'(t))$. Similarly, condition (R) implies that for $j \in \mathbf{Z}$ and $\hat{\nu}$ -a.a. $t \in I_0$, there exist q spaces $\mathcal{R}(\tau_{-k} F'_k(t))$, $k \in \mathbf{Z}$, $k \neq j$, such that their algebraic sum includes $\mathcal{R}(\tau_{-j} F'_j(t))$.

Theorem 3.1. *Condition (S) is necessary and sufficient for the $\mathcal{J}_{\mathbf{Z}}$ -singularity of $L^2(F)$.*

P r o o f. **N e c e s s i t y.** For $j \in \mathbf{Z}$, set $P_j = P_{\mathcal{R}(\tau_{-j} F'_j)}$. According to Lemma 2.2 P_j is measurable. The sequence $[\delta_{kj} \tau_k P_j]_{k \in \mathbf{Z}}$, where δ_{kj} denotes the Kronecker symbol, belongs to $L^2(F)$. Suppose $H_0 = L^2(F)$. Then there exists $\hat{\Phi} \in L^2(\hat{F})$ such that $[\tau_k \hat{\Phi}] = [\delta_{kj} \tau_k P_j]_{k \in \mathbf{Z}}$. This yields $\text{Ker } \hat{\Phi} \cap \mathcal{R}(\tau_{-j} F'_j) = \{0\}$ and $\mathcal{R}(\tau_{-k} F'_k) \subseteq \text{Ker } \hat{\Phi}$, $k \in \mathbf{Z}$, $k \neq j$, hence, condition (S).

S u f f i c i e n c y. Let $[\Phi_k] \in L^2(F)$. Define an $\mathcal{M}_q$ -valued measurable function $\hat{\Phi}$ on I_0 in such a way that for $\hat{\nu}$ -a.a. $t \in I_0$ and $x \in \mathcal{R}(\tau_{-k} F'_k(t))$ one has $\hat{\Phi}(t)x = \tau_{-k} \Phi_k(t)x$, $k \in \mathbf{Z}$. From Remark 3.2 it follows that this is possible if condition (S) is satisfied. Moreover, it is easy to see that $\hat{\Phi} \in L^2(\hat{F})$ and $[\tau_k \hat{\Phi}] = [\Phi_k]$, which means that $[\Phi_k]$ belongs to H_0 .

Theorem 3.1 is proved.

Theorem 3.2. *Condition (R) is necessary and sufficient for the $\mathcal{J}_{\mathbf{Z}}$ -regularity of $L^2(F)$.*

P r o o f. N e c e s s i t y. For $j \in \mathbf{Z}$ and $t \in I_0$, let $Q(t)$ be the orthogonal projector in $\mathcal{C}^q$ onto the space $\mathcal{R}(\widehat{F}'(t)) \ominus \mathcal{R}(\sum_{k \neq j} \tau_{-k} F'_k(t))$. The measurability of the function Q_j follows from Lemma 2.2 and from von Neumann's alternating projections theorem, cf. [4, Problem 96]. Assume that (R) is not true. Then there exist $j \in \mathbf{Z}$ and a Borel subset B of I_0 such that $\widehat{\nu}(B) > 0$ and $Q_j(t) \neq 0$ for $\widehat{\nu}$ -a.a. $t \in B$. For $s \in I$, the sequence $[\xi_s^{k-j} \tau^k Q_j]_{k \in \mathbf{Z}}$ is a nonzero element of H_s . It is easy to see that for $s, r \in I$, $[\xi_s^{k-j} \tau_k Q_j]_{k \in \mathbf{Z}} = [\xi_r^{k-j} \tau_k Q_j]_{k \in \mathbf{Z}}$ as elements of $L^2(F)$. Thus, $L^2(F)$ is not $\mathcal{J}_{\mathbf{Z}}$ -regular.

S u f f i c i e n c y. Suppose $[\Phi_k] \in \bigcap_{s \in I} H_s$. Then for $s \in I$, there exists $\widehat{\Phi}^{(s)} \in L^2(\widehat{F})$ such that $[\Phi_k] = [\xi_s^k \tau_k \widehat{\Phi}^{(s)}]$ with respect to the topology of $L^2(F)$. This yields

$$\tau_{-k} \Phi_k x = \xi_s^k \widehat{\Phi}^{(s)} x \quad \text{for } x \in \mathcal{R}(\tau_{-k} F'_k), \quad s \in I, \quad k \in \mathbf{Z}. \quad (3.1)$$

Assume that condition (R) is true. Then Remark 3.2 asserts that for $j_0 \in \mathbf{Z}$ and $\widehat{\nu}$ -a.a. $t \in I_0$, there exist q integers $j_k \neq j_0$, $k = 1, \dots, q$, such that the algebraic sum $\sum_{k=1}^q \mathcal{R}(\tau_{-j_k} F'_{j_k}(t))$ includes $\mathcal{R}(\tau_{-j_0} F'_{j_0}(t))$. Hence, for $x_{j_0} \in \mathcal{R}(\tau_{-j_0} F'_{j_0}(t))$, there exist $x_{j_k} \in \mathcal{R}(\tau_{-j_k} F'_{j_k}(t))$, $k = 1, \dots, q$, such that

$$\sum_{k=0}^q x_{j_k} = 0. \quad (3.2)$$

Choose $q+1$ numbers $s_l \in I$, $l = 0, 1, \dots, q$, such that the set $\mathcal{S} := \{\xi_{s_k} : k = 0, 1, \dots, q\}$ is algebraically independent over the field of rational numbers. We have $\sum_{k=0}^q \widehat{\Phi}^{(s_l)}(t) x_{j_k} = 0$ and, taking into account (3.1),

$$\sum_{k=0}^q \xi_{s_l}^{-j_k} \tau_{-j_k} \Phi_{j_k}(t) x_{j_k} = 0, \quad l = 0, 1, \dots, q. \quad (3.3)$$

Since the set $\mathcal{S}$ is algebraically independent over the field of rational numbers, the coefficient matrix of system (3.3) is invertible. It follows $\tau_{-j_0} \Phi_{j_0}(t) x_{j_0} = 0$. Since $x_{j_0} \in \mathcal{R}(\tau_{-j_0} F'_{j_0}(t))$ was arbitrary, we get $\mathcal{R}(\tau_{-j_0} F'_{j_0}(t)) \subseteq \text{Ker}(\tau_{-j_0} \Phi_{j_0}(t))$ for $\widehat{\nu}$ -a.a. $t \in I_0$. Since $j_0 \in \mathbf{Z}$ was arbitrary, it follows $[\Phi_k] = 0$. This means that $L^2(F)$ is $\mathcal{J}_{\mathbf{Z}}$ -regular.

Theorem 3.2 is proved.

R e m a r k 3.3. Examining the proof of the sufficiency part of Theorem 3.2, we can conclude that if $L^2(F)$ is $\mathcal{J}_{\mathbf{Z}}$ -regular, there exist $q+1$ numbers $s_l \in I$, $l = 0, 1, \dots, q$, such that $\bigcap_{l=0}^q H_{s_l} = \{0\}$.

Let us mention some special cases of Theorems 3.1 and 3.2.

Corollary 3.1. *If the support of F is included in an interval of length 2π and the intersection of the ranges of the point masses at the boundary points is $\{0\}$, then $L^2(F)$ is $\mathcal{J}_{\mathbf{Z}}$ -singular.*

Corollary 3.2. *If there exist integers j, k , $j \neq k$, such that $\mathcal{R}(\tau_{-l} F'_l) \subseteq \mathcal{R}(\tau_{-j} F'_j) = \mathcal{R}(\tau_{-k} F'_k)$ $\widehat{\nu}$ -a.e. for $l \in \mathbf{Z}$, then $L^2(F)$ is $\mathcal{J}_{\mathbf{Z}}$ -regular. In particular, if F' is invertible ν -a.e., $L^2(F)$ is $\mathcal{J}_{\mathbf{Z}}$ -regular.*

Given F , it is of interest to find the spectral measures of the $\mathcal{J}_{\mathbf{Z}}$ -regular and $\mathcal{J}_{\mathbf{Z}}$ -singular parts of the Wold decomposition of the corresponding q -variate weakly stationary process. If $q = 1$, this is easy since conditions (S) and (R) simplify to the following condition (S_1) and (R_1), respectively, for the scalar function F' :

(S_1) For $\widehat{\nu}$ -a.a. $t \in I_0$, there is a unique $k \in \mathbf{Z}$ such that $\tau_{-k} F'_k(t) \neq 0$;

(R_1) for $\widehat{\nu}$ -a.a. $t \in I_0$, there exist at least two integers j, k , $j \neq k$, such that $\tau_{-j} F'_j(t) \neq 0$ and $\tau_{-k} F'_k(t) \neq 0$.

Thus, if we set

$$\begin{aligned} A &:= \{t \in I_0: \tau_{-j}F'_j(t) \neq 0 \text{ and } \tau_{-k}F'_k(t) \neq 0 \\ &\quad \text{for at least two different integers } j, k\}, \\ B &:= \{t \in I_0: \tau_{-k}F'_k(t) \neq 0 \text{ for at most one } k \in \mathbf{Z}\}, \\ C &:= \bigcup_{k \in \mathbf{Z}} \tau_k A, \quad D := \bigcup_{k \in \mathbf{Z}} \tau_k B, \end{aligned}$$

the restrictions of F to C and D are the spectral measures of the $\mathcal{J}_{\mathbf{Z}}$ -regular and $\mathcal{J}_{\mathbf{Z}}$ -singular parts of the Wold decomposition, respectively. It seems to be difficult to obtain a similar procedure for $q \geq 2$.

4. Orthogonal projection onto H_0 . Let $\mathbf{x} := \{x_t\}_{t \in \mathbf{R}}$ be a continuous q -variate weakly stationary random process over $\mathbf{R}$, $L^2(F)$ its spectral domain, and $s \in I$. The problem of the best mean-square approximation of x_s by x_k , $k \in \mathbf{Z}$, leads to the problem of finding the orthogonal projection of $e_s E \in L^2(F)$ onto H_0 . Here we shall establish a more general result by calculating the orthogonal projection $\Phi_{\mathbf{Z}}$ of an arbitrary function $\Phi = [\Phi_k] \in L^2(F)$ onto H_0 as well as the interpolation error matrix $\langle \Phi - \Phi_{\mathbf{Z}}, \Phi - \Phi_{\mathbf{Z}} \rangle_F$. Using the notation of Section 3 we set

$$\hat{\Phi} := \sum \tau_{-k}(\Phi_k F'_k) \hat{F}'^+. \quad (4.1)$$

Lemma 4.1. *The function $\hat{\Phi}$ belongs to $L^2(\hat{F})$.*

P r o o f. We have

$$\begin{aligned} &\begin{pmatrix} \hat{F}' & \sum \tau_{-k}(F'_k \Phi_k^*) \\ \sum \tau_{-k}(\Phi_k F'_k) & \sum \tau_{-k}(\Phi_k F'_k \Phi_k^*) \end{pmatrix} \\ &= \sum \begin{pmatrix} E & 0 \\ 0 & \tau_{-k} \Phi_k \end{pmatrix} \begin{pmatrix} \tau_{-k} F'_k & \tau_{-k} F'_k \\ \tau_{-k} F'_k & \tau_{-k} F'_k \end{pmatrix} \begin{pmatrix} E & 0 \\ 0 & \tau_{-k} \Phi_k^* \end{pmatrix} \in \mathcal{M}_{2q}^{\geq}. \end{aligned}$$

Thus, Lemma 2.1 yields

$$\sum \tau_{-k}(\Phi_k F'_k) \hat{F}'^+ \sum \tau_{-k}(F'_k \Phi_k^*) \leq \sum \tau_{-k}(\Phi_k F'_k \Phi_k^*). \quad (4.2)$$

Since the left-hand side of (4.2) coincides with $\hat{\Phi} \hat{F}'^+ \hat{\Phi}^*$ and since the trace of the right-hand side of (4.2) is integrable with respect to $\hat{\nu}$, the lemma follows.

Theorem 4.1. *The orthogonal projection of a function $\Phi = [\Phi_k] \in L^2(F)$ onto H_0 is equal to*

$$\Phi_{\mathbf{Z}} = [\tau_k \hat{\Phi}], \quad (4.3)$$

where $\hat{\Phi}$ is the function (4.1). The interpolation error matrix equals

$$\langle \Phi, \Phi \rangle_F - \langle \hat{\Phi}, \hat{\Phi} \rangle_{\hat{F}}.$$

P r o o f. If $\Phi_{\mathbf{Z}}$ denotes the function (4.3), Lemma 4.1 implies that

$$\Phi_{\mathbf{Z}} \in H_0. \quad (4.4)$$

Let $\Psi = [\tau_k \hat{\Psi}] \in H_0$, $\hat{\Psi} \in L^2(\hat{F})$. By straightforward computations we get

$$\begin{aligned} &\langle \Phi - \Psi, \Phi - \Psi \rangle_F \\ &= \int_{I_0} \sum \left(\tau_{-k}(\Phi_k F'_k \Phi_k^*) - \tau_{-k}(\Phi_k F'_k) \hat{\Psi}^* - \hat{\Psi} \tau_{-k}(F'_k \Phi_k^*) + \hat{\Psi} \tau_{-k} F'_k \hat{\Psi}^* \right) d\hat{\nu}. \end{aligned} \quad (4.5)$$

Taking into account (4.1) and the fact that $\mathcal{R}(\tau_{-k} F'_k) \subseteq \mathcal{R}(\hat{F}')$, $k \in \mathbf{Z}$, we can write the right-hand side of (4.5) in the form

$$\begin{aligned} &\int_{I_0} \left(\sum \tau_{-k}(\Phi_k F'_k \Phi_k^*) - \hat{\Phi} \hat{F}'^+ \hat{\Psi}^* - \hat{\Psi} \hat{F}'^+ \hat{\Phi}^* + \hat{\Psi} \hat{F}'^+ \hat{\Psi}^* \right) d\hat{\nu} \\ &= \langle \Phi, \Phi \rangle_F - \langle \hat{\Phi}, \hat{\Phi} \rangle_{\hat{F}} + \langle \hat{\Phi} - \hat{\Psi}, \hat{\Phi} - \hat{\Psi} \rangle_{\hat{F}}. \end{aligned}$$

Using this and (4.4), we obtain the assertions of the theorem.

Theorem 4.1 can be applied to the following linear filtration problem. Let $\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \left\{ \begin{pmatrix} \mathbf{x}_t \\ \mathbf{y}_t \end{pmatrix} \right\}_{t \in \mathbf{R}}$ be a continuous $(q+r)$ -variate weakly stationary random process over $\mathbf{R}$ with spectral measure $\mathbf{F} := \begin{pmatrix} F & F_{\mathbf{x}\mathbf{y}} \\ F_{\mathbf{y}\mathbf{x}} & F_{\mathbf{y}\mathbf{y}} \end{pmatrix}$, and let $L^2_{r,q}(F)$ be the Hilbert space of $(r \times q)$ -matrix valued functions which are square-integrable with respect to F , $r, q \in \mathbf{N}$, and ν a nonnegative σ -finite Borel measure on $\mathbf{R}$ such that $\mathbf{F}$ is absolutely continuous with respect to ν . For $s \in I$, find the best mean square approximation of y_s by x_k , $k \in \mathbf{Z}$. The problem can be solved in two steps.

First find the orthogonal projection $\tilde{y}_s$ of y_s onto the time domain of $\mathbf{x}$, then project $\tilde{y}_s$ onto the space spanned by x_k , $k \in \mathbf{Z}$. Taking into account [10, Theorem (1.8) and formula (1.9)] we see that this leads to the problem of projecting the function $\Phi := e_s \frac{dF_{\mathbf{y}\mathbf{x}}}{d\nu} \left(\frac{dF}{d\nu} \right)^+$ in the space $L^2_{r,q}(F)$ onto the closed submodule generated by $e_k C$, where $k \in \mathbf{Z}$ and C is an arbitrary $(r \times q)$ -matrix. This can be done similarly to Theorem 4.1.

5. Stationary sequences over $\mathbf{Z}$. Let M be an $\mathcal{M}_q^\geq$ -valued Borel measure on I_0 . In this section we state some trigonometric approximation results on the space $L^2(M)$ similar to those of Sections 3 and 4. They can be given a probabilistic sense by interpreting $L^2(M)$ as the spectral domain of a q -variate weakly stationary random sequence over $\mathbf{Z}$. In what follows we consider I_0 as an additive group, i.e., addition is mod 2π .

Let $n \in \mathbf{N}$, $n \geq 2$. Denote by $\mathbf{N}_n$ the set of the first n nonnegative integers and by $\mathcal{J}_{(n)}$ the family of subsets of $\mathbf{Z}$ which are shifts of $n\mathbf{Z}$, i.e., $\mathcal{J}_{(n)} := \{m + n\mathbf{Z} : m \in \mathbf{N}_n\}$. For $m \in \mathbf{N}_n$, let $H_m^{(n)}$ be the closed submodule of $L^2(M)$ spanned by $e_{kn+m}E$, $k \in \mathbf{Z}$. The space $L^2(M)$ is called $\mathcal{J}_{(n)}$ -regular if $\bigcap_{m=0}^{n-1} H_m^{(n)} = \{0\}$, and $\mathcal{J}_{(n)}$ -singular if $H_m^{(n)} = L^2(M)$ for some and, hence, for all $m \in \mathbf{N}_n$.

If $k \in \mathbf{Z}$, D is a subset of I_0 , and f is a function defined on D , set $\tau_k^{(n)} D := D + 2k\pi/n$, $\tau_k^{(n)} f(t) := f(t - 2k\pi/n)$, $t \in \tau_k^{(n)} D$. If f is defined on I_0 , let $f_{(k)}$ be its restriction to $[2k\pi/n, 2(k+1)\pi/n]$, $k \in \mathbf{N}_n$.

For a Borel subset B of $[0, 2\pi/n]$, set $\widehat{M}(B) := \sum_{k=0}^{n-1} M(\tau_k^{(n)} B)$ and $\widehat{\mu}(B) := \text{tr } \widehat{M}(B)$.

For a Borel subset B of I_0 , set $\mu(B) := \sum_{k=0}^{n-1} \widehat{\mu}(\tau_k^{(n)}(B \cap [2k\pi/n, 2(k+1)\pi/n]))$.

Let $M' := \frac{dM}{d\mu}$ be the Radon-Nikodým derivative of M with respect to μ and $\widehat{M}' := \frac{d\widehat{M}}{d\widehat{\mu}}$.

Similarly to Theorems 3.1 and 3.2 the following results can be established.

Theorem 5.1. *Let $n \in \mathbf{N}$, $n \geq 2$. The space $L^2(M)$ is $\mathcal{J}_{(n)}$ -singular if and only if for $j \in \mathbf{N}_n$,*

$$\mathcal{R}(\tau_{-j}^{(n)} M'_{(j)}) \cap \mathcal{R}\left(\sum_{k=0, k \neq j}^{n-1} \tau_{-k}^{(n)} M'_{(k)}\right) = \{0\} \quad \widehat{\mu}\text{-a.e.}$$

The space $L^2(M)$ is $\mathcal{J}_{(n)}$ -regular if and only if for $j \in \mathbf{N}_n$,

$$\mathcal{R}(\tau_{-j}^{(n)} M'_{(j)}) \subseteq \mathcal{R}\left(\sum_{k=0, k \neq j}^{n-1} \tau_{-k}^{(n)} M'_{(k)}\right) \quad \widehat{\mu}\text{-a.e.}$$

Let $\mathbf{x}$ be a q -variate weakly stationary random sequence over $\mathbf{Z}$ whose spectral measure is M . Using Theorem 5.1 we can compute the spectral measures of the $\mathcal{J}_{(2)}$ -regular and $\mathcal{J}_{(2)}$ -singular parts, respectively, of the Wold decomposition of $\mathbf{x}$.

To simplify the notation, set $L := M'_{(0)}$ and $N := \tau_{-1}^{(2)} M'_{(1)}$. Let $L := \begin{pmatrix} L_{11} & L_{12} \\ L_{12}^* & L_{22} \end{pmatrix}$ be the block representation of L with respect to the orthogonal decomposition

$$\mathbf{C}^q = (\mathcal{R}(L) \cap \mathcal{R}(N)) \oplus (\text{Ker } L + \text{Ker } N).$$

Note that this decomposition depends on t , $t \in I_0$.

Let

$$L_R := \begin{pmatrix} L_{11} - L_{12}L_{22}^+L_{12}^* & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad L_S := \begin{pmatrix} L_{12}L_{22}^+L_{12}^* & L_{12} \\ L_{12}^* & L_{22} \end{pmatrix}.$$

Define N_R and N_S analogously and set

$$M'_R(t) := \begin{cases} L_R(t), & t \in [0, \pi), \\ \tau_1^{(2)} N_R(t), & t \in [\pi, 2\pi), \end{cases} \quad M'_S(t) := \begin{cases} L_S(t), & t \in [0, \pi), \\ \tau_1^{(2)} N_S(t), & t \in [\pi, 2\pi). \end{cases}$$

Note that Lemma 2.1 implies that M'_R and M'_S are $\mathcal{M}_q^\geq$ -valued functions on I_0 . The measurability of these functions follows from Lemma 2.2 and from von Neumann's alternating projections theorem.

Theorem 5.2. *The measures $M'_R d\mu$ and $M'_S d\mu$ are the spectral measures of the $\mathcal{J}_{(2)}$ -regular and $\mathcal{J}_{(2)}$ -singular parts, respectively, of the Wold decomposition.*

P r o o f. Theorem 4.4 of [7] implies that it is enough to show that $L^2(M'_R d\mu)$ has the following two properties:

- (i) $L^2(M'_R d\mu)$ is $\mathcal{J}_{(2)}$ -regular;
- (ii) if W is a measurable $\mathcal{M}_q^\geq$ -valued function on I_0 such that $W \leq M'$ μ -a.e. and $L^2(W d\mu)$ is $\mathcal{J}_{(2)}$ -regular, then $W \leq M'_R$ μ -a.e.

Proof of (i): Clearly $\begin{pmatrix} 0 \\ y \end{pmatrix} \in \text{Ker } L_R$ for $y \in (\text{Ker } L + \text{Ker } N)$. Suppose $\begin{pmatrix} x \\ 0 \end{pmatrix} \in \text{Ker } L_R$. Then $L_{11}x = L_{12}L_{22}^+L_{12}^*x$. Setting $z := -L_{22}^+L_{12}^*x$, by Lemma 2.1 we get $\begin{pmatrix} x \\ z \end{pmatrix} \in \text{Ker } L_S$, hence, $\begin{pmatrix} x \\ z \end{pmatrix} \in \text{Ker } L$. But $\begin{pmatrix} x \\ 0 \end{pmatrix}$ is orthogonal to $\text{Ker } L$ which yields $x = 0$. Thus, $\text{Ker } L_R = \left\{ \begin{pmatrix} 0 \\ y \end{pmatrix} : y \in (\text{Ker } L + \text{Ker } N) \right\}$. Since $\text{Ker } N_R$ has the same description, the $\mathcal{J}_{(2)}$ -regularity of $L^2(M'_R d\mu)$ follows from Theorem 5.1.

Proof of (ii): Let $B := \{t \in [0, \pi) : W(t) \leq M'_R(t) \text{ is not true}\}$ and let $t \in B$. Lemma 2.1 implies that there exists $y \in (\text{Ker } L(t) + \text{Ker } N(t))$ such that $W(t)y \neq 0$. Since by assumption $W(t) \leq M'(t) = L(t)$ and $\tau_{-1}^{(2)}W(t) \leq N(t)$, we have $L(t) \neq 0$ and $\text{Ker } N(t) \subseteq \text{Ker}(\tau_{-1}^{(2)}W(t))$, respectively. This yields $y \in \text{Ker}(\tau_{-1}^{(2)}W(t))$. Thus,

$$\text{Ker } W(t) \neq \text{Ker}(\tau_{-1}^{(2)}W(t)) \quad \text{for } t \in B.$$

Since $L^2(W d\mu)$ is supposed to be $\mathcal{J}_{(2)}$ -regular, Theorem 5.1 gives $\widehat{\mu}(B) = \mu(B) = 0$. Analogously one can show that $\mu(\{t \in [\pi, 2\pi) : W(t) \leq M'_R(t) \text{ is not true}\}) = 0$.

Theorem 5.2 is proved.

Let us state an analogue of Theorem 4.1. Let $n \in \mathbb{N}$, $n \geq 2$.

For $\Phi \in L^2(M)$, set

$$\widehat{\Phi}_{\{n\}} := \sum_{k=0}^{n-1} \tau_{-k}^{(n)}(\Phi_{(k)} M'_{(k)}) \widehat{M}'^+ \quad (5.1)$$

and define

$$\Phi_{nZ}(t) := \tau_k^{(n)} \widehat{\Phi}_{\{n\}}(t), \quad t \in \left[\frac{2k\pi}{n}, \frac{2(k+1)\pi}{n} \right), \quad k \in \mathbb{N}_n. \quad (5.2)$$

Theorem 5.3. *Let $n \in \mathbb{N}$, $n \geq 2$, and $\Phi \in L^2(M)$. The orthogonal projection of Φ onto $H_0^{(n)}$ is equal to the function Φ_{nZ} as defined in (5.2). The interpolation error matrix equals*

$$\langle \Phi, \Phi \rangle_M - \langle \widehat{\Phi}_{\{n\}}, \widehat{\Phi}_{\{n\}} \rangle_{\widehat{M}},$$

where $\widehat{\Phi}_{\{n\}}$ is defined as in (5.1).

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ON THE CLT FOR MEANS UNDER THE ROTATION ACTION. II¹⁾

Предлагается метод, позволяющий построить для различных типичных средних, порожденных действием любого иррационального вращения круга, примеры L^2 -функций, удовлетворяющих центральной предельной теореме. В качестве примера рассматриваются нелинейные средние и средние по последовательности квадратов. Во втором случае

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