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ON THE CLT FOR MEANS UNDER THE ROTATION ACTION. II¹⁾

Предлагается метод, позволяющий построить для различных типичных средних, порожденных действием любого иррационального вращения круга, примеры L^2 -функций, удовлетворяющих центральной предельной теореме. В качестве примера рассматриваются нелинейные средние и средние по последовательности квадратов. Во втором случае

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используется метод круга Харди–Литтлвуда. Мы также даем пример непрерывного случайного гауссовского ряда Фурье, траектории которого удовлетворяют как центральной предельной теореме, так и центральной предельной теореме в смысле сходимости почти наверное.

Ключевые слова и фразы: центральная предельная теорема с вероятностью единица, иррациональные вращения, нелинейные средние, квадратичные средние, взвешенные средние, гауссовская рандомизация, случайные ряды Фурье, метод круга.

6. Other ergodic sums. It is worth observing here that the only characteristics of the ergodic averages used throughout the paper are the sole estimates (2.5) concerning their spectral kernels. Consequently, for any other type of averages satisfying the same type of estimates, all the statements proved in the previous sections apply as well. Here, we show how to adapt the construction operated in Section 2 to treat other examples.

6.1. Nonlinear ergodic sums. Consider the nonlinear ergodic sums

$$S_N^\tau(f, g)(t) = \sum_{k=0}^{N-1} f(\tau^k t) g(\tau^{2k} t) \quad (N \geq 1, t \in X), \quad (6.1)$$

for $f(t) = \sum_{k=1}^{\infty} a_k \cos 2\pi l_k t$, $g(t) = \sum_{k=1}^{\infty} b_k \cos 2\pi l_k t$ ($f, g \in L^2(m)$). These sums are classical objects in multiple recurrence theory. Sequences $e = (\varepsilon_j)_{j \geq 1}$, $\mathcal{L} = (l_n)_{n \geq 1}$, and $\mathcal{N} = (N_j)_{j \geq 1}$ used here are defined in (2.1) and (2.2).

Theorem 6.1. Assume that the following conditions are fulfilled:

$$\begin{aligned} \text{(i)} \quad & \lim_{j \rightarrow \infty} \frac{\varepsilon_j}{(\sum_{k, i > j} a_k^2 b_i^2)^{1/2}} = 0, \\ \text{(ii)} \quad & \lim_{j \rightarrow \infty} \frac{|a_j|}{(\sum_{k > j} a_k^2)^{1/2}} = \lim_{j \rightarrow \infty} \frac{|b_j|}{(\sum_{i > j} b_i^2)^{1/2}} = 0. \end{aligned}$$

Then $S_N^\tau(f, g) / \|S_N^\tau(f, g)\|_2 \xrightarrow{\mathcal{D}} N(0, 1)$.

P r o o f. Plainly

$$S_N^\tau(f, g)(t) = \frac{N}{2} \sum_{k, i=1}^{\infty} a_k b_i \Delta_{k, i}(t, \vartheta),$$

where we put

$$\begin{aligned} \Delta_{k, i}(t, \vartheta) = & \cos 2\pi t(l_k + l_i) \operatorname{Re}[V_N((l_k - 2l_i)\vartheta)] + \cos 2\pi t(l_k - l_i) \operatorname{Re}[V_N((l_k + 2l_i)\vartheta)] \\ & - \sin 2\pi t(l_k + l_i) \operatorname{Im}[V_N((l_k + 2l_i)\vartheta)] - \sin 2\pi t(l_k - l_i) \operatorname{Im}[V_N((l_k - 2l_i)\vartheta)]. \end{aligned} \quad (6.2)$$

Since $\mathcal{L}$ is assumed to be ρ -lacunary with $\rho \geq 3$, the decomposition $n = l_k + l_i$ is unique; hence

$$\|S_N^\tau(f, g)\|_2^2 = \frac{N^2}{4} \sum_{k, i=1}^{\infty} a_k^2 b_i^2 \left(|V_N((l_k + 2l_i)\vartheta)|^2 + |V_N((l_k - 2l_i)\vartheta)|^2 \right).$$

Next, we observe that the partial sums follow the same rule of decomposition as operated in Section 2. Put, for any $j \geq 1$ and $N \in I_j$

$$\begin{aligned} \check{S}_N^\tau(f, g) &= \frac{N}{2} \sum_{k, i > j} a_k b_i \Delta_{k, i}(t, \vartheta), \quad S'^\tau_N(f, g) = \frac{N}{2} \sum_{k \text{ or } i < j} a_k b_i \Delta_{k, i}(t, \vartheta), \\ s_N^\tau(f, g) &= \frac{N}{2} a_j \sum_{i \geq j} b_i \Delta_{j, i}(t, \vartheta) + \frac{N}{2} b_j \sum_{k \geq j} a_k \Delta_{k, j}(t, \vartheta), \\ R_j^\tau(f, g) &= \frac{N}{2} \sum_{k, i > j} a_k b_i \left\{ \cos 2\pi t(l_k + l_i) + \cos 2\pi t(l_k - l_i) \right. \\ &\quad \left. - \sin 2\pi t(l_k + l_i) - \sin 2\pi t(l_k - l_i) \right\}. \end{aligned} \quad (6.3)$$

Then

$$\left\| \tilde{S}_N^\tau(f, g) - R_j^\tau(f, g) \right\|_2^2 = \frac{N^2}{4} \sum_{k, i > j} a_k^2 b_i^2 \left(|V_N((l_k + 2l_i)\vartheta) - 1|^2 + |V_N((l_k - 2l_i)\vartheta) - 1|^2 \right).$$

Define for $u \in \mathbf{R}$, $|u| = \text{dist}(u, \mathbf{Z})$ ($|u| \in [0, \frac{1}{2}]$ and $|u + v| \leq |u| + |v|$). Then $V_N(\psi) = V_N(|\psi|)$. We assume that N_1 is chosen so that $\varepsilon_1/(16N_1) \leq \frac{1}{8}$. Thus $|l_i\vartheta| = \{l_i\vartheta\}$ and $|2l_i\vartheta| = \{2l_i\vartheta\} = 2\{l_i\vartheta\}$ ($i \geq 1$). Combining these elementary observations with (2.5) gives

$$\begin{aligned} |V_N((l_k \pm 2l_i)\vartheta) - 1| &= |V_N(|(l_k \pm 2l_i)\vartheta|) - 1| \\ &\leq 32N|(l_k \pm 2l_i)\vartheta| \leq 32N\{|l_k\vartheta| + 2|l_i\vartheta|\} \leq 64N_{j+1} \left\{ \frac{\varepsilon_1\varepsilon_2^2 \cdots \varepsilon_k^2}{16^k N_1} + \frac{\varepsilon_1\varepsilon_2^2 \cdots \varepsilon_i^2}{16^i N_1} \right\} \\ &\leq \frac{6416^j N_1}{\varepsilon_1\varepsilon_2^2 \cdots \varepsilon_j^2 \varepsilon_{j+1}} \left\{ \frac{\varepsilon_1\varepsilon_2^2 \cdots \varepsilon_k^2}{16^k N_1} + \frac{\varepsilon_1\varepsilon_2^2 \cdots \varepsilon_i^2}{16^i N_1} \right\} \leq 8\varepsilon_{j+1}. \end{aligned} \quad (6.4)$$

Therefore, $\|\tilde{S}_N^\tau(f, g) - R_j^\tau(f, g)\|_2^2 \leq 4N^2 \sum_{k, i > j} a_k^2 b_i^2 \varepsilon_{j+1}^2$ and so

$$\left\| \tilde{S}_N^\tau(f, g) - R_j^\tau(f, g) \right\|_2 \leq 2N\|f\|_2 \|g\|_2 \varepsilon_{j+1}. \quad (6.5)$$

Concerning $S'^\tau_N(f, g)$, we have

$$\|S'^\tau_N(f, g)\|_2^2 = \frac{N^2}{4} \sum_{k \text{ or } i < j} a_k^2 b_i^2 \left(|V_N((l_k + 2l_i)\vartheta)|^2 + |V_N((l_k - 2l_i)\vartheta)|^2 \right).$$

Observe that $\{l_k\vartheta\} = |l_k\vartheta| \leq \frac{1}{8}$, $\{2l_i\vartheta\} = 2\{l_i\vartheta\} = |2l_i\vartheta| \leq \frac{1}{4}$. Hence

$$|(l_k \pm 2l_i)\vartheta| = |\{l_k\vartheta\} \pm 2\{l_i\vartheta\}|.$$

Thus, by (2.5) $|V_N((l_k \pm 2l_i)\vartheta)| \leq \frac{1}{2}N_j|\{l_k\vartheta\} \pm 2\{l_i\vartheta\}|$. By distinguishing three cases: $k = i$, $k > i$, and $k < i$, it is easy to show that $|\{l_k\vartheta\} \pm 2\{l_i\vartheta\}| \geq \frac{3}{4}\{l_h\vartheta\}$, $h = k \wedge i$. Since $h < j$, we have

$$\{l_h\vartheta\} \geq \frac{1}{2} \frac{16^{j-1-h}}{16^{j-1}N_1} \varepsilon_1\varepsilon_2^2 \cdots \varepsilon_h^2 \geq \frac{\varepsilon_1\varepsilon_2^2 \cdots \varepsilon_{j-1}^2}{2 \cdot 16^{j-1}N_1} \implies |\{l_k\vartheta\} \pm 2\{l_i\vartheta\}| \geq \frac{3}{8} \frac{\varepsilon_1\varepsilon_2^2 \cdots \varepsilon_{j-1}^2}{16^{j-1}N_1}.$$

Therefore,

$$|V_N((l_k \pm 2l_i)\vartheta)| \leq \frac{8}{3} \frac{\varepsilon_1\varepsilon_2^2 \cdots \varepsilon_{j-1}^2 \varepsilon_j}{16^{j-1}N_1} \frac{16^{j-1}N_1}{\varepsilon_1\varepsilon_2^2 \cdots \varepsilon_{j-1}^2} \leq 3\varepsilon_j. \quad (6.6)$$

Thus,

$$\|S'^\tau_N(f, g)\|_2^2 \leq \frac{9N^2\varepsilon_j^2}{4} \left(\sum_{k \text{ or } i < j} a_k^2 b_i^2 \right) \implies \|S'^\tau_N(f, g)\|_2 \leq \frac{3N\varepsilon_j}{2} \|f\|_2 \|g\|_2. \quad (6.7)$$

Finally,

$$\|s_N^\tau(f, g)\|_2 \leq N \left(|a_j| \left(\sum_{i > j} b_i^2 \right)^{1/2} + |b_j| \left(\sum_{k > j} a_k^2 \right)^{1/2} \right). \quad (6.8)$$

Summarizing, we find by combining estimates (6.5), (6.7), and (6.8)

$$\begin{aligned} \|S_N^\tau(f, g) - R_j^\tau(f, g)\|_2 &\leq \|\tilde{S}_N^\tau(f, g) - R_j^\tau(f, g)\|_2 + \|S'^\tau_N(f, g)\|_2 + \|s_N^\tau(f, g)\|_2 \\ &\leq N\|f\|_2 \|g\|_2 \varepsilon_j + N \left[|a_j| \left(\sum_{i > j} b_i^2 \right)^{1/2} + |b_j| \left(\sum_{k > j} a_k^2 \right)^{1/2} \right]. \end{aligned} \quad (6.9)$$

Therefore,

$$\left| \frac{\|S_N^T(f, g)\|_2}{\|R_j^T(f, g)\|_2} - 1 \right| \leq \left\| \frac{S_N^T(f, g)}{\|R_j^T(f, g)\|_2} - \frac{R_j^T(f, g)}{\|R_j^T(f, g)\|_2} \right\|_2$$

$$\leq 4\|f\|_2 \|g\|_2 \frac{\varepsilon_j}{(\sum_{k, i > j} a_k^2 b_i^2)^{1/2}} + \left[\frac{|a_j|}{(\sum_{k > j} a_k^2)^{1/2}} + \frac{|b_j|}{(\sum_{i > j} b_i^2)^{1/2}} \right].$$

Under the assumptions made, we deduce

$$\lim_{j \rightarrow \infty} \sup_{N \in I_j} \left\| \frac{S_N^T(f, g)}{\|R_j^T(f, g)\|_2} - \frac{R_j^T(f, g)}{\|R_j^T(f, g)\|_2} \right\|_2 = 0,$$

$$\lim_{j \rightarrow \infty} \sup_{N \in I_j} \left| \frac{\|S_N^T(f, g)\|_2}{N(\sum_{k, i > j} a_k^2 b_i^2)^{1/2}} - 1 \right| = 0. \quad (6.10)$$

Thus the normalized partial sums are close (in the $L^2(m)$ norm) to the sequence of normalized rests $(R_j^T(f, g)/\|R_j^T(f, g)\|_2)_j$. We are led to the same situation as the one treated in details in Section 5, and we deduce from the Salem-Zygmund result that

$$\frac{S_N^T(f, g)}{\|S_N^T(f, g)\|_2} \xrightarrow{\mathcal{D}} N(0, 1) \quad \text{as } N \rightarrow \infty. \quad (6.11)$$

We pass, now, to the study of another example.

6.2. Weighted ergodic sums. Let $\sigma = \{\sigma_k, k \geq 0\}$ be a sequence of nonnegative reals and put $S_N = \sum_{k=0}^{N-1} \sigma_k$. Consider for $f \in L^2(m)$ the weighted sums

$$U_N^T f = \sum_{k=0}^{N-1} \sigma_k f \circ \tau^k \quad (N \geq 1). \quad (6.12)$$

The kernels corresponding to their averages are defined by $W_N(t) = (1/S_N) \sum_{k=0}^{N-1} \sigma_k e_k(t)$. We have the easy estimates valid for any $M \geq N \geq 0$ and $t \in X$,

$$(i) \quad |W_M(t) - W_N(t)| \leq 4\pi M t \frac{S_M - S_N}{S_M}, \quad (ii) \quad |W_M(t) - 1| \leq 4\pi M t,$$

$$(iii) \quad |W_N(t)| \leq \frac{2\sigma_\infty}{S_N |\sin \pi t|} \quad \left(\sigma_\infty = \sigma_0 + \sum_{k=0}^{\infty} |\sigma_k - \sigma_{k+1}| \right). \quad (6.13)$$

The first estimate comes from the writing $W_M(t) - W_N(t) = \sum_{k=0}^{M-1} (\sigma_k/S_M) \times (e_k(t) - 1) - \sum_{k=0}^{N-1} (\sigma_k/S_N)(e_k(t) - 1)$. The second is obvious and the last is obtained by Abel's summation. Let $\mathcal{N} = \{N_j, j \geq 1\}$ be some rapidly increasing sequence of integers greater than 2 and write again $I_j = [N_j, N_{j+1}[$. We assume that $\eta_j := N_j/N_{j+1}$ tends to 0 as j tends to infinity. We choose a sequence $\mathcal{L} = \{l_j, j \geq 1\}$ so that

$$\frac{1}{N_k} - \frac{1}{N_k^2} \leq \{l_k \vartheta\} < \frac{1}{N_k}.$$

Let $\beta^1 = (\beta_k^1)_{k \geq 1}$, $\beta^2 = (\beta_k^2)_{k \geq 1}$, $\beta^1, \beta^2 \in \ell^2$, and put $\rho = (\rho_k)_{k \geq 1}$ with $\rho_k = \beta_k^1 + i\beta_k^2$ and consider

$$F(t) = \sum_{k=1}^{\infty} \operatorname{Re}(\rho_k \overline{e_{l_k}(t)}) \quad (\forall t \in X).$$

Theorem 6.2. Assume that $\sum_{k=0}^{\infty} |\sigma_k - \sigma_{k+1}| < \infty$ and $S_N \geq N^\alpha$ for some positive α . If

$$(i) \quad \lim_{j \rightarrow \infty} \frac{\eta_j}{(\sum_{k > j} \rho_k^2)^{1/2}} = 0, \quad (ii) \quad \lim_{j \rightarrow \infty} \frac{\sum_{a_j \leq k \leq j} \rho_k^2}{(\sum_{k > j} \rho_k^2)^{1/2}} = 0 \quad (\forall 0 < a \leq 1),$$

then $U_N^T F / \|U_N^T F\|_2 \xrightarrow{\mathcal{D}} N(0, 1)$.

We give an example of application. If $\rho_k^2 = k^{-1}(\ln k)^{-2}$, then condition (ii) is satisfied, and condition (i) means $\lim_{j \rightarrow \infty} \eta_j (\ln j)^{-1/2} = 0$. Theorem 6.2 applies for this example provided that $\mathcal{N}$ grows sufficiently fast.

P r o o f. By estimates (6.13) for $N \in I_j$

$$\begin{aligned} |W_N(l_k \vartheta) - 1| &= |W_N(\{l_k \vartheta\}) - 1| \leq 4\pi\eta_{j+1} \quad (k > j+1), \\ |W_N(l_k \vartheta)| &\leq \frac{8\sigma_\infty}{\pi} \eta_{[ja]-1} \quad (k < [ja]). \end{aligned}$$

Put for any $j \geq 1$, $N \in I_j$, and $t \in X$:

$$\begin{aligned} \check{U}_N(F) &= S_N \sum_{k > j+1} \operatorname{Re} [\rho_k \overline{W_N(l_k \vartheta)} e_{l_k}], \quad U'_N(F) = S_N \sum_{k \leq [ja]} \operatorname{Re} [\rho_k \overline{W_N(l_k \vartheta)} e_{l_k}], \\ u_N(F) &= S_N \sum_{[ja] < k \leq j+1} \operatorname{Re} [\rho_k \overline{V_N(l_k \vartheta)} e_{l_k}], \quad F_j(t) = \sum_{k > j} \operatorname{Re} [\rho_k \overline{e_{l_k}(t)}]. \end{aligned} \quad (6.14)$$

Then

$$\begin{aligned} \|\check{U}_N f - S_N F_j\| &\leq 4\pi\eta_{j+1} S_N \left(\sum_{k > j+1} \rho_k^2 \right)^{1/2}, \\ \|U'_N\| &\leq S_N \|F\| \leq \frac{8\sigma}{\pi} N \eta_{[ja]-1}, \quad \|u_N(F)\| \leq S_N \left(\sum_{[ja] \leq k \leq j+1} \rho_k^2 \right)^{1/2}. \end{aligned}$$

Consequently,

$$\lim_{j \rightarrow \infty} \sup_{N \in I_j} \left| \frac{\|U_N(F)\|_2}{S_N (\sum_{k > j+1} \rho_k^2)^{1/2}} - 1 \right| = \lim_{j \rightarrow \infty} \sup_{N \in I_j} \left\| \frac{U_N(F)}{S_N (\sum_{k > j+1} \rho_k^2)^{1/2}} - \frac{F_j}{\|F_j\|_2} \right\| = 0.$$

Here again we are led to the situation treated in details in Section 5 and we deduce from the Salem-Zygmund result that $U_N(F)/\|U_N(F)\|_2 \xrightarrow{\mathcal{D}} N(0, 1)$.

6.3. Ergodic sums along the squares. Consider for $f \in L^2(m)$ the sums along the sequence of squares

$$\Gamma_N^r f = \sum_{k=0}^{N-1} f \circ \tau^{k^2} \quad (N \geq 1) \quad (6.15)$$

and the spectral kernels $Q_N(t) = N^{-1} \sum_{k=0}^{N-1} e_{k^2}(t)$. It is well known that the ergodic averages $N^{-1} \Gamma_N^r f$ converge almost everywhere for any $f \in L^r(m)$, with $r > 1$. We refer to the remarkable and concise paper [2] for the result and references concerning the estimates of exponential sums $\sum_{k=0}^{N-1} e_{k^2}(t)$ we use in what follows.

Theorem 6.3. *There exists an $F \in L^2(m)$ such that*

$$\frac{\Gamma_N^r F}{\|\Gamma_N^r F\|_2} \Rightarrow N(0, 1).$$

The proof of that result easily extends to the more general sums $\sum_{k=0}^{N-1} f \circ \tau^{P(k)}$, where $P(x) = b_1 x + \dots + b_d x^d$ with $b_j \in \mathbb{Z}$ and $b_d > 0$.

P r o o f. We use the circle method. Let $\mathcal{N} = \{N_k, k \geq 1\}$ be increasing to infinity and put $\Delta_k = [1/N_k - 1/N_k^2, 1/N_k]$. Let $0 < h < \frac{1}{3}$ be fixed. Put for $k \geq 1$ and $N \geq N_k^{1/h}$

$$\Delta_k(N) = \left\{ \alpha \in \Delta_k : \exists q \leq N^h \text{ and } a \text{ with } (a, q) = 1 : \left| \alpha - \frac{a}{q} \right| < \left(\frac{2}{N} \right)^{2-h} \right\}.$$

Then

$$m(\Delta_k(N)) \leq \sum_{q \leq N^h} \sum_{\substack{(a, q)=1 \\ \operatorname{dist}(a/q, \Delta_k) < (2/N)^{2-h}}} m\left(\Delta_k \cap \left] \frac{a}{q} - \left(\frac{2}{N} \right)^{2-h}, \frac{a}{q} + \left(\frac{2}{N} \right)^{2-h} \right[\right).$$

And since $\text{dist}(a/q, \Delta_k) < (2/N)^{2-h}$, we have

$$m(\Delta_k(N)) \leq C_h \left(\sum_{q \leq N^h} q \right) \frac{1}{N_k} \frac{1}{N^{2-h}} \leq C_h N_k^{-1} N^{-1+2h}.$$

Let $p_0 = \inf\{p: 2^p \geq N_k^{1/h}\}$. Then

$$\begin{aligned} m\left(\bigcup_{N=2^p, N \geq N_k^{1/h}} \Delta_k(N)\right) &\leq C_h \sum_{p=p_0}^{\infty} N_k^{-1} \frac{1}{2^{p(1-2h)}} \leq C_h N_k^{-1} N_k^{-(1-2h)/h} \\ &= C_h N_k^{-(1/h-1)} \leq \frac{N_k^{-2}}{2}, \end{aligned}$$

assuming N_1 sufficiently large (depending on h only). Therefore,

$$m\left(\bigcup_{N=2^p, N \geq N_k^{1/h}} \Delta_k(N)\right) \leq \frac{1}{2} m(\Delta_k).$$

Let $\Delta_k^* = \{\alpha \in \Delta_k: \forall N \geq N_k^{1/h}, \forall q \leq N^h \text{ and } a \text{ with } (a, q) = 1: |\alpha - a/q| > 1/N^{2-h}\}$. Let $\alpha \in \bigcap_{N=2^p, N \geq N_k^{1/h}} \Delta_k(N)^c$; then for all $p \geq p_0$, $q \leq (2^p)^h$, and $(a, q) = 1$ we have $|\alpha - a/q| < 2^{2-h}/(2^p)^{2-h}$. Let $2^p \leq N < 2^{p+1}$. For $q \leq N^h$, we have $q \leq (2^p)^h$ and for a such that $(a, q) = 1$

$$\left| \alpha - \frac{a}{q} \right| \geq \frac{2^{2-h}}{(2^p)^{2-h}} \geq \frac{1}{N^{2-h}}.$$

Therefore, for any $p \geq p_0$ and $2^p \leq N < 2^{p+1}$ the inequality $|\alpha - a/q| \geq 1/N^{2-h}$ is fulfilled for any $q \leq N^h$ and a with $(a, q) = 1$. This implies that $m(\Delta_k^*) \geq \frac{1}{2} m(\Delta_k) > 0$. By using Birkhoff's theorem we obtain: for almost all $x \in X$ and integers $k \geq 1$,

$$\lim_{J \rightarrow \infty} \frac{\#\{0 \leq j < J: x + j\vartheta \in \Delta_k^* \pmod{1}\}}{J} = m(\Delta_k^*).$$

We can thus find an $x \in X$ and an increasing sequence of positive integers $\mathcal{L} = \{l_k, j \geq 1\}$ such that $x + l_k\vartheta \in \Delta_k^* \pmod{1}$ for all $k \geq 1$. Define for $\rho = \{\rho_k, k \geq 1\} \in \ell^2$: $F = \sum_{k=1}^{\infty} \text{Re}(\rho_k e_{l_k+x/\vartheta})$. The system of functions $(e_{l_k+x/\vartheta}, k \geq 1)$ is orthonormal, and

$$\Gamma_N^T F = N \sum_{k=1}^{\infty} \text{Re}\left(\rho_k \overline{e_{l_k+x/\vartheta} Q_N(l_k\vartheta+x)}\right).$$

a) We fix N . We first consider the summation block corresponding to $N^h \leq N_k < N^{2-h}$. We have the following estimate: if $|\alpha - a/q| < 1/q^2$ and $(a, q) = 1$, then

$$\left| \sum_{n=1}^N e^{2i\pi\alpha n^2} \right| < C_\varepsilon N^{1+\varepsilon} \left\{ \frac{1}{q} + \frac{1}{N} + \frac{q}{N^2} \right\}^{1/2}.$$

We choose $\varepsilon = h/4$ and assume $N_1 > 2$. Since $x + l_k\vartheta \in \Delta_k^* \subset \Delta_k \pmod{1}$, we have

$$\frac{1}{N_k} - \frac{1}{N_k^2} \leq |x + l_k\vartheta| = \{x + l_k\vartheta\} < \frac{1}{N_k},$$

and so

$$\left| \sum_{n=1}^N e^{2i\pi(x+l_k\vartheta)n^2} \right| < C_h N^{1+h/4} \left\{ \frac{1}{N_k} + \frac{1}{N} + \frac{N_k}{N^2} \right\}^{1/2} \leq C_h N^{1-h/4}.$$

Therefore, $|Q_N(x + l_k \vartheta)| \leq C_h N^{-h/4}$, $N^h \leq N_k < N^{2-h}$ and

$$\begin{aligned}
 & \int_X \left| \sum_{k: N^h \leq N_k < N^{2-h}} \operatorname{Re} \left(\overline{\rho_k e_{l_k + x/\vartheta} N Q_N(l_k \vartheta + x)} \right) \right|^2 dt \\
 & \leq \int_X \left| \sum_{k: N^h \leq N_k < N^{2-h}} \rho_k \overline{e_{l_k + x/\vartheta} N Q_N(l_k \vartheta + x)} \right|^2 dt \\
 & = \sum_{k, j: N^h \leq N_k, N_j < N^{2-h}} N^2 \rho_k \overline{\rho_j} \overline{Q_N(l_k \vartheta + x)} Q_N(l_j \vartheta + x) \langle \overline{e_{l_k + x/\vartheta}}, e_{l_j + x/\vartheta} \rangle \\
 & = \sum_{k: N^h \leq N_k < N^{2-h}} N^2 |\rho_k|^2 |Q_N(l_k \vartheta + x)|^2 \|e_{l_k + x/\vartheta}\|^2 \leq C_h N^{2-h/2} \sum_{k: N^h \leq N_k < N^{2-h}} |\rho_k|^2.
 \end{aligned}$$

Hence

$$\left\| \sum_{k: N^h \leq N_k < N^{2-h}} \operatorname{Re} \left(\overline{\rho_k e_{l_k + x/\vartheta} N Q_N(l_k \vartheta + x)} \right) \right\|_2 \leq C_h N^{1-h/4} \left(\sum_{k: N^h \leq N_k < N^{2-h}} |\rho_k|^2 \right)^{1/2}.$$

b) Next we consider the summation rest corresponding to $N_k \geq N^{2+h}$. We use the elementary estimate $|Q_N(t) - 1| \leq 2\pi N^2 t$. This is easily deduced from the fact $|Q_N(t) - 1| = |N^{-1} \sum_{n=0}^{N-1} (e_{n^2}(t) - 1)| \leq (2/N) \sum_{n=0}^{N-1} |\sin \pi t n^2|$. Then

$$\begin{aligned}
 & \int_X \left| \sum_{k: N_k \geq N^{2+h}} \operatorname{Re} \left(\overline{\rho_k e_{l_k + x/\vartheta} N [Q_N(l_k \vartheta + x) - 1]} \right) \right|^2 dt \\
 & \leq \int_X \left| \sum_{k: N_k \geq N^{2+h}} \rho_k \overline{e_{l_k + x/\vartheta} N [Q_N(l_k \vartheta + x) - 1]} \right|^2 dt \\
 & = \sum_{k: N_k \geq N^{2+h}} N^2 |\rho_k|^2 |Q_N(l_k \vartheta + x) - 1|^2 \|e_{l_k + x/\vartheta}\|^2 \\
 & \leq N^2 \sum_{k: N_k \geq N^{2+h}} |\rho_k|^2 \left(2\pi N^2 \{x + l_k \vartheta\} \right)^2 \\
 & \leq 4\pi^2 N^6 \sum_{k: N_k \geq N^{2+h}} |\rho_k|^2 N_k^{-2} \leq 4\pi^2 N^{2-2h} \sum_{k: N_k \geq N^{2+h}} |\rho_k|^2.
 \end{aligned}$$

Hence

$$\left\| \sum_{k: N_k \geq N^{2+h}} \operatorname{Re} \left(\overline{\rho_k e_{l_k + x/\vartheta} N [Q_N(l_k \vartheta + x) - 1]} \right) \right\|_2 \leq 2\pi N^{1-h} \left(\sum_{k: N_k \geq N^{2+h}} |\rho_k|^2 \right)^{1/2}.$$

c) Next we continue with the trivial bound

$$\left\| \sum_{k: N^{2-h} \leq N_k < N^{2+h}} \operatorname{Re} \left(\overline{\rho_k e_{l_k + x/\vartheta} N Q_N(l_k \vartheta + x)} \right) \right\|_2 \leq N \left(\sum_{k: N^{2-h} \leq N_k < N^{2+h}} |\rho_k|^2 \right)^{1/2}.$$

d) Finally, we treat the last summation block corresponding to the values of k for which $N_k \leq N^h$ by using the construction of the sequence $\mathcal{L}$ performed before. Plainly

$$\left\| \sum_{k: N_k \leq N^h} \operatorname{Re} \left(\overline{\rho_k e_{l_k + x/\vartheta} N Q_N(l_k \vartheta + x)} \right) \right\|_2^2 \leq N^2 \sum_{k: N_k \leq N^h} |\rho_k|^2 |Q_N(x + l_k \vartheta)|^2.$$

Since $x + l_k \vartheta \in \Delta_k^* \pmod{1}$ and $N_k \leq N^h$, it follows that $x + l_k \vartheta$ does not belong to a major arc; and so by using Lemma 5.6 of [2],

$$|Q_N(x + l_k \vartheta)| \leq CN^{-h'} \quad (h' > 0).$$

Reporting that estimate into the concerned sum gives

$$\left\| \sum_{k: N_k \leq N^h} \operatorname{Re} \left(\overline{\rho_k e_{l_k + x/\vartheta} N Q_N(l_k \vartheta + x)} \right) \right\|_2 \leq CN^{1-h'} \left(\sum_{k \geq 1} |\rho_k|^2 \right)^{1/2}.$$

Summarizing, if $F_N = \sum_{k: N_k \geq N^{2+h}} \operatorname{Re}(\rho_k \overline{e_{l_k + x/\vartheta}})$, then

$$\begin{aligned} \left\| \Gamma_N^T F - N F_N \right\|_2 &\leq C_h N^{1-h/4} \left(\sum_{k: N^h \leq N_k < N^{2-h}} |\rho_k|^2 \right)^{1/2} \\ &\quad + 2\pi N^{1-h} \left(\sum_{k: N_k \geq N^{2+h}} |\rho_k|^2 \right)^{1/2} \\ &\quad + N \left(\sum_{k: N^{2-h} \leq N_k < N^{2+h}} |\rho_k|^2 \right)^{1/2} + CN^{1-h'} \left(\sum_{k \geq 1} |\rho_k|^2 \right)^{1/2}. \end{aligned}$$

We choose a sequence $\rho = \{\rho_k, k \geq 1\}$ such that

$$\begin{aligned} \text{(i)} \quad &\lim_{N \rightarrow \infty} \frac{\sum_{N^{2-h} \leq N_k < N^{2+h}} |\rho_k|^2}{\sum_{N_k \geq N^{2+h}} |\rho_k|^2} = 0, \\ \text{(ii)} \quad &\lim_{N \rightarrow \infty} N^{-\varepsilon} \sum_{N_k \geq N^{2+h}} |\rho_k|^2 = 0 \quad (\forall \varepsilon > 0). \end{aligned}$$

This is achieved with the choices $\rho_k^2 = k^{-1}(\ln k)^{-2}$ and $N_k = 2^k$, $k \geq 1$. Since

$$\begin{aligned} \left\| \frac{\Gamma_N^T F}{N \|F_N\|_2} - \frac{F_N}{\|F_N\|_2} \right\|_2 &\leq C_h N^{-h/4} \left(\frac{\sum_{N^h \leq N_k < N^{2-h}} |\rho_k|^2}{\sum_{N_k \geq N^{2+h}} |\rho_k|^2} \right)^{1/2} + 2\pi N^{-h} \\ &\quad + \left(\frac{\sum_{k: N^{2-h} \leq N_k < N^{2+h}} |\rho_k|^2}{\sum_{N_k \geq N^{2+h}} |\rho_k|^2} \right)^{1/2} + CN^{-h'} \left(\frac{\sum_{k \geq 1} |\rho_k|^2}{\sum_{N_k \geq N^{2+h}} |\rho_k|^2} \right)^{1/2}, \end{aligned}$$

we deduce

$$\lim_{N \rightarrow \infty} \left| \frac{\|\Gamma_N^T F\|}{N \|F_N\|_2} - 1 \right| = \lim_{N \rightarrow \infty} \left\| \frac{\Gamma_N^T F}{N \|F_N\|_2} - \frac{F_N}{\|F_N\|_2} \right\|_2 = 0.$$

So, the study of the CLT for the averages $\Gamma_N^T F / \|\Gamma_N^T F\|_2$ reduces to the one of the sequence of rests $F_N / \|F_N\|_2$, $N \geq 1$. This is done by Salem-Zygmund's result.

7. ASCLT. Let us consider Example 3 from Section 5. We already know that almost all sample paths of $\mathcal{X}$ satisfy the CLT. We prove here that they satisfy the ASCLT as well. A first idea would have been to directly prove the ASCLT by arranging partial sums $S_N(\mathcal{X})$ in blocks forming a quasi-orthogonal system like in [20].

Indeed, since $|\langle S_N(f), S_M(f) \rangle| \leq \sum_k \beta_k^2 |V_N(l_k \vartheta)| |V_M(l_k(t))| NM$, it is easy to show for $N \in I_n$ and $M \in I_m$ that (by decomposing the above sum according to $\sum_{k < n} + \sum_{n \leq k < m} + \sum_{k \geq m}$)

$$|\langle S_N(f), S_M(f) \rangle| \leq 4n^{-2}m^{-2} + b_m b_n^{-1} + b_n m^{-2}.$$

But this is not enough to produce a quasi-orthogonal system with suitable blocks. We thus will follow a different approach. Define on the product space $\Omega \times X$ with notation $x = (\omega, t) \in \Omega \times X$, $N, j \geq 1$:

$$\tilde{\Theta}_j(x) = \frac{\sum_{k: k \geq j+1} \beta_k \operatorname{Re}(\gamma_k(\omega) \overline{e_{l_k}(t)})}{b_j}, \quad \tilde{S}_N(x) = S_N(\mathcal{X}_f)(\omega, t).$$

Let also $\mathcal{F}$ be some countable set of continuous functions $\varphi: \mathbf{R} \rightarrow \mathbf{R}$ such that $|\varphi(x)| \leq e^{a_\varphi x^2}$, $x \in \mathbf{R}$, with $a_\varphi < \frac{1}{4}$ depending on φ only. Put $\Phi(A) = \mathbf{P}\{N(0, 1) \in A\}$, $A \in \mathcal{B}(\mathbf{R})$, and $\Phi(dx) = e^{-x^2/2}(\sqrt{2\pi})^{-1} dx$.

Theorem 7.1. *The following properties are realized:*

$$m \times \mathbf{P} \left\{ x \left| \forall \varphi \in \mathcal{F} \lim_{J \rightarrow \infty} \frac{1}{\ln N_J} \sum_{j=1}^J d_j \varphi(\tilde{\Theta}_j(x)) = \int \varphi(y) \Phi(dy) \right. \right\} = 1,$$

$$\mathbf{P} \left\{ \omega \left| \lim_{N \rightarrow \infty} \frac{1}{\ln N} \sum_{n=1}^N \frac{1}{n} \delta_{\tilde{S}_N(\omega, \cdot)/\tau_N} \stackrel{\mathcal{D}}{=} N(0, 1) \right. \right\} = 1.$$

A straightforward consequence of Theorems 5.2 and 7.1 is the following assertion.

Theorem 7.2 (CLT & ASCLT for sample paths). *Almost all sample paths of $\mathcal{X}$ are continuous and satisfy both CLT and ASCLT.*

Proof of Theorem 7.1. Let $\nu = \mathbf{P} \times m$ be the product measure on $\Omega \times X$ and denote by $\mathbf{E}_\nu$ the corresponding expectation symbol. We shall operate in two steps: first we prove a kind of strong law of large numbers for the process $(\tilde{\Theta}_j)_j$, next we show that this in turn implies the desired property. It is plain that $(\tilde{\Theta}_j)_j$ is a Gaussian sequence with correlation given by $\mathbf{E} \tilde{\Theta}_j \tilde{\Theta}_k = b_k b_j^{-1}$ ($j \leq k$). Put for any integer $l \geq 1$, $S_l = \sum_{j=1}^l [\varphi(\tilde{\Theta}_j) - \mathbf{E}_\nu \varphi(\tilde{\Theta}_j)] d_j$. By virtue of correlation properties of Gaussian variables, we have

$$\begin{aligned} \mathbf{E}_\nu [S_{n+m} - S_m]^2 &\leq C \sum_{j,u=m+1}^{n+m} d_j d_u |\mathbf{E}_\nu \tilde{\Theta}_j \tilde{\Theta}_u| \\ &\leq C \left(\sum_{j=m+1}^{n+m} d_j^2 + 2 \sum_{j=m+1}^{n+m} d_j b_j^{-1} \left(\sum_{u=j+1}^{m+n} d_u b_u \right) \right). \end{aligned}$$

But

$$\sum_{u=j+1}^{m+n} d_u b_u \leq C \sum_{u=j+1}^{m+n} u L(u)^{-4} \exp[-u L(u)^{-4}] \leq C j \exp[-j L(j)^{-4}],$$

and thus

$$2 \sum_{j=m+1}^{n+m} d_j b_j^{-1} \left(\sum_{u=j+1}^{m+n} d_u b_u \right) \leq C \sum_{u=j+1}^{m+n} j^2 L(j)^{-4} \leq C((m+n)^3 - m^3).$$

Estimating similarly $\sum_{j=m+1}^{n+m} d_j^2$ finally gives

$$\mathbf{E}_\nu [S_{n+m} - S_m]^2 \leq C((m+n)^3 - m^3). \quad (7.1)$$

Observing now that $\ln N_J \sim J^2 L(J)^{-4}$ and applying Gál-Koksma inequality (see [17, Theorem A1, p. 134]) gives for any $0 \leq r < \frac{1}{2}$:

$$\frac{1}{\ln N_J} \sum_{j=1}^J d_j [\varphi(\tilde{\Theta}_j(x)) - \mathbf{E}_\nu \varphi(\tilde{\Theta}_j)] = O(J^{-r}) \quad x\text{-a.s.} \quad (7.2)$$

This proves the first part of the theorem. Now let A be some open set of $\mathbf{R}$ and put for any $\varepsilon > 0$,

$$A^\varepsilon = \{y = x + b: x \in A, |b| < \varepsilon\}, \quad A_\varepsilon = \{a \in A: a + x \in A, |x| < \varepsilon\}$$

and

$$L_J(A) = \frac{1}{\ln N_{J+1}} \sum_{j=1}^J B_j(A), \quad B_j(A) = \sum_{N \in I_j} \frac{1}{N} \mathbf{1}[\tilde{S}_N \in A]. \quad (7.3)$$

By means of Proposition 3.1, we may find $J_\varepsilon < \infty$ such that for all $j \geq J_\varepsilon$, $\sup_{N \in I_j} |\tilde{S}_N/\tau_N - \tilde{\Theta}_j| < \varepsilon$. Thus we have the implication: $j \geq J_\varepsilon$ and $\tilde{\Theta}_j \in A \Rightarrow \tilde{S}_N/\tau_N \in A^\varepsilon$ for all $N \in I_j$. And so $B_j(A^\varepsilon) = d_j$. Hence,

$$\liminf_{J \rightarrow \infty} \frac{1}{\ln N_{J+1}} \sum_{j=1}^J B_j(A^\varepsilon) \geq \liminf_{J \rightarrow \infty} \frac{1}{\ln N_{J+1}} \sum_{j=1}^J \mathbf{1}[\tilde{\Theta}_j \in A].$$

Conversely, if $j \geq J_\varepsilon$ and if there exists $N \in I_j$ such that $\tilde{S}_N/\tau_N \in A$, then $\tilde{\Theta}_j \in A^\varepsilon$ and for all $M \in I_j$, we have $\tilde{S}_M/\tau_M \in A^\varepsilon$. Thus,

$$\begin{aligned} \frac{1}{\ln N_{J+1}} \sum_{j=J_\varepsilon+1}^J B_j(A) &= \frac{1}{\ln N_{J+1}} \sum_{j=J_\varepsilon+1}^J B_j(A) \mathbf{1}[B_j(A) > 0] \\ &\leq \frac{1}{\ln N_{J+1}} \sum_{j=J_\varepsilon+1}^J B_j(A^\varepsilon) \mathbf{1}[B_j(A) > 0] \\ &= \frac{1}{\ln N_{J+1}} \sum_{j=J_\varepsilon+1}^J d_j \mathbf{1}[B_j(A) > 0] \leq \frac{1}{\ln N_{J+1}} \sum_{j=J_\varepsilon+1}^J d_j \mathbf{1}[\tilde{\Theta}_j \in A^\varepsilon]. \end{aligned}$$

And consequently

$$\limsup_{J \rightarrow \infty} (\ln N_{J+1})^{-1} \sum_{j=1}^J B_j(A) \leq \limsup_{J \rightarrow \infty} (\ln N_{J+1})^{-1} \sum_{j=1}^J \mathbf{1}[\tilde{\Theta}_j \in A^\varepsilon].$$

Noticing that $m \times \mathbf{P}\{\tilde{\Theta}_j \in A\} = \Phi(A)$, we deduce

$$\Phi(A_\varepsilon) \leq \liminf_{J \rightarrow \infty} \frac{1}{\ln N_{J+1}} \sum_{j=1}^J B_j(A) \leq \limsup_{J \rightarrow \infty} \frac{1}{\ln N_{J+1}} \sum_{j=1}^J B_j(A) \leq \Phi(A^\varepsilon). \quad (7.4)$$

And when A is Φ -continuous

$$\lim_{J \rightarrow \infty} \frac{1}{\ln N_{J+1}} \sum_{j=1}^J B_j(A)(x) = \Phi(A) \quad x\text{-a.s.}$$

The proof is now complete.

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