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NOQUET C.*

INEQUALITIES FOR THE TOTAL VARIATION BETWEEN THE DISTRIBUTIONS OF A SEQUENCE AND ITS TRANSLATE AND APPLICATIONS

Пусть $\xi = (\xi_k)_{k \in \mathbb{N}^*}$ — однородная стационарная цепь Маркова и $\xi + a = (\xi_k + a_k)_{k \in \mathbb{N}^*}$ — ее сдвиг на вещественную последовательность. В статье доказывается неравенство для полной вариации между распределениями ξ и $\xi + a$. Этот результат позволяет дать достаточные условия абсолютной непрерывности этих распределений. Далее, мы рассматриваем последовательность $\xi = (\xi_k)_{k \in \mathbb{N}^*}$ независимых и одинаково распределенных случайных величин и другую последовательность $\eta = (\eta_k)_{k \in \mathbb{N}^*}$ независимых величин, которая не зависит от ξ . Приводится оценка полной вариации между распределениями ξ и $\xi + \eta$. Результаты применяются к задаче абсолютной непрерывности.

Ключевые слова и фразы: полная вариация, цепь Маркова, случайный сдвиг, абсолютная непрерывность.

1. Introduction

The origin of this paper is a work made by Yu. A. Davydov [2], who proposed an inequality for the total variation between the distributions of a sequence of independent and identically distributed random variables (i.i.d.r.v.'s) and its translate.

Indeed, let $\xi = (\xi_k)_{k \in \mathbb{N}^*}$ be an i.i.d.r.v. sequence, defined on some probability space $(\Omega, \mathcal{A}, \mathbf{P})$ and taking values in $(\mathbf{R}, \mathcal{B}(\mathbf{R}), \lambda)$, where $\mathcal{B}(\mathbf{R})$ is the σ -field of Borel subsets of $\mathbf{R}$ and λ is the Lebesgue measure. For all $k \in \mathbb{N}^*$, ξ_k has the density p , assumed to be absolutely continuous (a.c.). Let $I(p)$ denote the Fisher quantity associated with p , defined by

$$I(p) = \int_{\mathbf{R}} \left[\frac{p'}{p}(x) \right]^2 p(x) dx.$$

Let $a = (a_k)_{k \in \mathbb{N}^*}$ be a real sequence. We denote by $\mathcal{P}$ and $\mathcal{P}^a$ respectively the distributions of the sequences ξ and $\xi + a$. In this case, if $I(p) < \infty$, Yu. A. Davydov obtained

$$\|\mathcal{P}^a - \mathcal{P}\| \leq \sqrt{I(p)} \sqrt{\sum_{k \geq 1} a_k^2}. \quad (1.1)$$

In the same way, Yu. A. Davydov proved a similar inequality for the case of a sequence of non-identically distributed r.v.'s.

The aim of this article is at first to generalize those inequalities to the case of stationary homogeneous Markov chains and to the case of a sequence of i.i.d.r.v.'s with a random translation.

Next, we consider the different applications of those inequalities. In fact, there are two main applications. On the one hand, we can use our results in the research in local limit theorems for stochastic functionals in invariance principle, but it will be the subject of another paper.

On the other hand, those inequalities can be used to study absolute continuity between the distributions of the sequence and its translate. It will constitute the second subject of this note.

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2. Inequalities

2.1. Markov chains. Let $\xi = (\xi_k)_{k \in \mathbf{N}^*}$ be a stationary homogeneous Markov chain, defined on $(\Omega, \mathcal{A}, \mathbf{P})$ and taking values in $(\mathbf{R}, \mathcal{B}(\mathbf{R}), \lambda)$. Let Π denote its stationary probability distribution with density π assumed to be a.c. and P denote the probability transition kernel with the family of transition densities $\{p(x, \cdot); x \in \mathbf{R}\}$. For all $x \in \mathbf{R}$, we suppose that $p(x, \cdot)$ is a.c. Moreover, π' is the derivative of π and p'_x, p'_y are the partial derivatives of p .

Let us denote by q the transition density for time-reversed (with respect to Π) chain. This transition density is given by $q(y, \cdot) = \pi(\cdot) p(\cdot, y) / \pi(y)$.

Now, we introduce the following notion of I -regularity (see [4]).

Definition 2.1. Let $O \subseteq \mathbf{R}$ be an open set. The family $\{p(\cdot, \theta); \theta \in O\}$ of probability densities is said to be I -regular (or information regular) if the map from O to $L^2(\mathbf{R}, \lambda)$, sending $\theta \in O$ to $p^{1/2}(\cdot, \theta)$ is continuously differentiable (in the sense of standard Hilbert norms on $\mathbf{R}$ and $L^2(\mathbf{R}, \lambda)$).

Remark 2.1. This definition implies that for all $\theta \in O$, $\partial p^{1/2}(\cdot, \theta) / \partial \theta \in L^2(\mathbf{R}, \lambda)$. Let us denote by $I(\theta)$ the Fisher quantity defined by

$$I(\theta) = 4 \left\| \frac{\partial}{\partial \theta} p^{1/2}(\cdot, \theta) \right\|_{L^2(\mathbf{R}, \lambda)}^2.$$

Remark 2.2. Though existence of the pointwise derivatives is not assumed by I -regularity, some consequences of that hold true. Set

$$\Psi(\cdot, \theta) = \frac{\partial}{\partial \theta} p^{1/2}(\cdot, \theta).$$

For example, Ψ is usually represented in the form

$$\Psi(\cdot, \theta) = \frac{1}{2} \frac{\partial p(\cdot, \theta) / \partial \theta}{p^{1/2}(\cdot, \theta)}. \quad (2.1)$$

Multiplying both parts by $p^{1/2}(\cdot, \theta)$, we obtain

$$\Psi(\cdot, \theta) p^{1/2}(\cdot, \theta) = \frac{1}{2} \frac{\partial}{\partial \theta} p(\cdot, \theta).$$

The last identity can be interpreted in the following way: if the map $\theta \mapsto p^{1/2}(\cdot, \theta)$ is differentiable (as a $L^2(\mathbf{R}, \lambda)$ -valued map), then $\theta \mapsto p(\cdot, \theta)$ is differentiable (as a $L^1(\mathbf{R}, \lambda)$ -valued map). In view of (2.1), $I(\theta)$ can also be written in its usual form

$$I(\theta) = \int_{\mathbf{R}} \left[\frac{\partial p(x, \theta) / \partial \theta}{p(x, \theta)} \right]^2 p(x, \theta) dx.$$

So, our regularity assumptions are the following.

(R) The family $\{\pi(\cdot + t); t \in \mathbf{R}\}$ is I -regular.

Corresponding Fisher quantity does not depend on t and can be written down as

$$I(\pi) = \int_{\mathbf{R}} \left[\frac{\pi'(x)}{\pi(x)} \right]^2 \pi(x) dx. \quad (2.2)$$

Set $J = \{x \in \mathbf{R} \mid \pi(x) > 0\}$.

(R⁺) The family $\{p(x, \cdot); x \in J\}$ is I -regular.

We denote by $I^+(x)$ the corresponding Fisher quantity defined for $x \in J$ by

$$I^+(x) = \int_{\mathbf{R}} \left[\frac{p'_x}{p}(x, y) \right]^2 p(x, y) dy. \quad (2.3)$$

(R⁻) The family $\{q(y, \cdot); y \in J\}$ is I -regular.

We denote by $I^-(y)$ the corresponding Fisher quantity defined for $y \in J$ by

$$I^-(y) = \int_{\mathbf{R}} \left[\frac{p'_y}{p}(x, y) - \frac{\pi'(y)}{\pi(y)} \right]^2 q(y, x) dx. \quad (2.4)$$

Moreover, we suppose that

$$I^+ = \int_{\mathcal{J}} I^+(x) \pi(x) dx < \infty \quad \text{and} \quad I^- = \int_{\mathcal{J}} I^-(y) \pi(y) dy < \infty. \quad (2.5)$$

Denote $I = I(\pi) + I^+ + I^-$. Finally, let $a = (a_k)_{k \in \mathbb{N}^*}$ be a real sequence. As in Section 1, we define by $\mathcal{P}$ and $\mathcal{P}^a$ respectively the distributions of the chains ξ and $\xi + a$.

Theorem 2.1. *If (R) , (R^+) , (R^-) and (2.5) are satisfied and $a \in \ell_2$, then*

$$\|\mathcal{P}^a - \mathcal{P}\| \leq \sqrt{2I} \sqrt{\sum_{k \geq 1} a_k^2}. \quad (2.6)$$

Remark 2.3. If ξ is a sequence of i.i.d.r.v.'s then (2.6) is exactly the inequality (1.1) multiplied by the constant $\sqrt{2}$.

In the same way, we proved a similar inequality when the Markov chain ξ is non-homogeneous.

2.2. Random translation. In this section, $\xi = (\xi_k)_{k \in \mathbb{N}^*}$ is a sequence of i.i.d.r.v.'s defined on $(\Omega, \mathcal{A}, \mathbf{P})$ and taking values in $(\mathbf{R}, \mathcal{B}(\mathbf{R}), \lambda)$. For all $k \in \mathbb{N}^*$, ξ_k has the density p , assumed to be a.c. If p' and p'' exist, we define respectively

$$I_1(p) = \int_{\mathbf{R}} \left[\frac{p'}{p}(x) \right]^2 p(x) dx \quad \text{and} \quad I_2(p) = \int_{\mathbf{R}} \left[\frac{p''}{p}(x) \right]^2 p(x) dx.$$

Moreover, $\eta = (\eta_k)_{k \in \mathbb{N}^*}$ is a sequence of independent r.v.'s defined on $(\Omega, \mathcal{A}, \mathbf{P})$ and taking values in $(\mathbf{R}, \mathcal{B}(\mathbf{R}), \lambda)$, independent of ξ . As usual, $\mathcal{P}$ and $\mathcal{P}^\eta$ are respectively the distributions of ξ and $\xi + \eta$.

Theorem 2.2. *If $I_1(p) < \infty$ and if for $\varepsilon > 0$,*

$$\sum_{k \geq 1} \mathbf{E}[\eta_k^2 \mathbb{I}_{\{|\eta_k| \leq \varepsilon\}}] < \infty \quad \text{and} \quad \sum_{k \geq 1} \mathbf{P}\{|\eta_k| > \varepsilon\} < \infty, \quad (2.7)$$

then

$$\|\mathcal{P}^\eta - \mathcal{P}\| \leq \sqrt{I_1(p) \sum_{k \geq 1} \mathbf{E}[\eta_k^2 \mathbb{I}_{\{|\eta_k| \leq \varepsilon\}}] + 8 \sum_{k \geq 1} \mathbf{P}\{|\eta_k| > \varepsilon\}}. \quad (2.8)$$

Remark 2.4. In fact, the assumption (2.7) is equivalent to $\eta \in \ell_2$ almost surely (a.s.).

Remark 2.5. As ε tends to infinity in Theorem 2.2, we obtain a completely similar inequality to (1.1):

$$\|\mathcal{P}^\eta - \mathcal{P}\| \leq \sqrt{I_1(p)} \sqrt{\sum_{k \geq 1} \mathbf{E}(\eta_k^2)},$$

which is a direct consequence of (1.1) too.

Now, we suppose that the sequence η is symmetric.

Theorem 2.3. *Assume that η is symmetric. If $I_2(p) < \infty$ and if for $\varepsilon > 0$,*

$$\sum_{k \geq 1} \mathbf{E}[\eta_k^4 \mathbb{I}_{\{|\eta_k| \leq \varepsilon\}}] < \infty \quad \text{and} \quad \sum_{k \geq 1} \mathbf{P}\{|\eta_k| > \varepsilon\} < \infty, \quad (2.9)$$

then

$$\|\mathcal{P}^\eta - \mathcal{P}\| \leq \sqrt{\frac{25}{24} I_2(p) \sum_{k \geq 1} \mathbf{E}[\eta_k^4 \mathbb{I}_{\{|\eta_k| \leq \varepsilon\}}] + 8 \sum_{k \geq 1} \mathbf{P}\{|\eta_k| > \varepsilon\}}. \quad (2.10)$$

Remark 2.6. In this result, the assumption $I_2(p) < \infty$ is stronger than $I_1(p) < \infty$. Indeed, according to Theorem 1 in [9], $I_1(p) \leq \frac{3}{2} \sqrt{I_2(p)}$. On the other hand, we see that the assumptions concerning the convergent series are weaker in this second theorem.

Remark 2.7. We can assume that $\eta \in \ell_4$ a.s. instead of (2.9).

Remark 2.8. As ε tends to infinity in Theorem 2.3, we obtain the following result.

Corollary 2.1. *Assume that η is symmetric. If we have*

$$I_2(p) < \infty \quad \text{and} \quad \sum_{k \geq 1} \mathbf{E}[\eta_k^4] < \infty, \quad (2.11)$$

then

$$\|\mathcal{P}^\eta - \mathcal{P}\| \leq \sqrt{\frac{25}{24} I_2(p) \sum_{k \geq 1} \mathbf{E}[\eta_k^4]}. \quad (2.12)$$

Of course, we have proved similar results when the r.v.'s ξ_k are not identically distributed.

3. Absolute continuity

Those inequalities allow us to show directly absolute continuity. Of course, this problem is well known. For example, L. A. Shepp gives a result for the case of a sequence of i.i.d.r.v.'s in [10]. H. Sato has proved a result in [7] for locally equivalent Markov chains. On the other hand, H. Sato, M. Tamashiro and C. Watari, [8] and [9], have proved results for the case of a sequence of i.i.d.r.v.'s and random translation. Here, our method is different and seems easier, in particular, we do not use the alternative of Kakutani [5], as it is the case of the researches made until now.

For example, in the case of a sequence of i.i.d.r.v.'s defined in Section 1, if we use the inequality (1.1), we easily obtain the equivalence between $\mathcal{P}$ and $\mathcal{P}^a$, which is a part of the result of L. A. Shepp [10].

3.1. Absolute continuity and Markov chains. We use the notation of Section 2.1, that is ξ is a stationary time-homogeneous Markov chain.

Theorem 3.1. Assume that $\pi > 0$ λ -almost everywhere (a.e.) and $p > 0$ λ^2 -a.e. and (R), (R^+) , (R^-) , (2.5) are satisfied, then for any sequence $a \in \ell_2$, we have $\mathcal{P} \sim \mathcal{P}^a$.

Remark 3.1. In [7], H. Sato has proved absolute continuity of locally equivalent Markov chains. In particular, Theorem 4.6 in [7] looks like our result and the assumptions are similar.

3.2. Absolute continuity and random translation. Here, we use the notation of Theorems 2.2 and 2.3. ξ is a sequence of i.i.d.r.v.'s with the distribution $\mathcal{P}$ and η is a sequence of independent r.v.'s, independent of ξ . As usual, $\mathcal{P}^\eta$ is the distribution of $\xi + \eta$. We obtain two results corresponding to Theorems 2.2 and 2.3.

Theorem 3.2. Assume that $p > 0$ λ -a.e. and $I_1(p) < \infty$, then for any sequence η such that for some $\varepsilon > 0$, (2.7) is satisfied, we have $\mathcal{P} \sim \mathcal{P}^\eta$.

Remark 3.2. H. Sato and M. Tamashiro have studied [8] the particular case when ξ is a standard Gaussian sequence and η is non-negative.

Theorem 3.3. Assume that $p > 0$ λ -a.e. and $I_2(p) < \infty$, then for any symmetric sequence η such that for some $\varepsilon > 0$, (2.9) is satisfied, we have $\mathcal{P} \sim \mathcal{P}^\eta$.

Corollary 3.1. Assume that $p > 0$ λ -a.e. and $I_2(p) < \infty$, then for all symmetric sequence η such that (2.11) is satisfied, we have $\mathcal{P} \sim \mathcal{P}^\eta$.

Remark 3.3. This result is proved by H. Sato and C. Watari in [9].

4. Proofs

First, we prove finite-dimensional inequalities, that is we study the total variation between the distributions of a vector and its translate. Next, we prove that we can pass to the limit. Finally, we apply those inequalities to absolute continuity.

4.1. Finite-dimensional inequalities.

4.1.1. Markov chains. Let us come back to the case of stationary homogeneous Markov chains. Let $n \in \mathbb{N}^*$. We define the random vectors $\bar{\xi} = (\xi_1, \dots, \xi_n)$ with distribution P_n and $\bar{\xi} + \bar{a} = (\xi_1 + a_1, \dots, \xi_n + a_n)$ with distribution P_n^a . Then, the problem is to estimate the total variation between P_n and P_n^a , that is the following quantity

$$\|P_n^a - P_n\| = \int_{\mathbb{R}^n} \left| \pi(x_1 + a_1) \prod_{k=2}^n p(x_{k-1} + a_{k-1}, x_k + a_k) - \pi(x_1) \prod_{k=2}^n p(x_{k-1}, x_k) \right| d\bar{x}. \quad (4.1)$$

Theorem 4.1. Under the assumptions (R), (R⁺), (R⁻) and (2.5), we have

$$\|P_n^a - P_n\| \leq \sqrt{2I} \sqrt{\sum_{k=1}^n a_k^2}. \quad (4.2)$$

Sketched proof. The main idea of the proof came from [4]. Set

$$\begin{aligned} \Psi: [0; 1] &\longrightarrow \mathbf{R} \\ t &\longmapsto \Psi(t) = \pi(x_1 + ta_1) \prod_{k=2}^n p(x_{k-1} + ta_{k-1}, x_k + ta_k). \end{aligned}$$

Then, (4.1) can be written as

$$\|P_n^a - P_n\| = \int_{\mathbf{R}^n} |\Psi(1) - \Psi(0)| d\bar{x} \leq \int_0^1 \int_{\mathbf{R}^n} |\Psi'(t)| d\bar{x} dt. \quad (4.3)$$

We calculate $\Psi'(t)$ and making a change of variables, we obtain

$$\|P_n^a - P_n\| \leq \mathbf{E} \left| \sum_{k=1}^n a_k X_k \right| \leq \left[\mathbf{E} \left(\sum_{k=1}^n a_k X_k \right)^2 \right]^{1/2}, \quad (4.4)$$

where

$$\begin{cases} X_1 = \frac{\pi'}{\pi}(\xi_1) + \frac{p'_x}{p}(\xi_1, \xi_2), \\ X_k = \frac{p'_y}{p}(\xi_{k-1}, \xi_k) + \frac{p'_x}{p}(\xi_k, \xi_{k+1}) \quad \text{for all } 2 \leq k \leq n-1, \\ X_n = \frac{p'_y}{p}(\xi_{n-1}, \xi_n). \end{cases} \quad (4.5)$$

According to the regularity assumptions, we can easily prove the following lemma, which is a consequence of Remark 2.2.

Lemma 4.1. If the probability densities family $\{p(\cdot, \theta); \theta \in O\}$ is *I-regular*, then for all $\theta \in O$

$$\int_{\mathbf{R}} \frac{\partial}{\partial \theta} p(x, \theta) dx = 0.$$

Then, we can apply this lemma to (R), (R⁺) and (R⁻) and we obtain

$$\int_{\mathbf{R}} \pi'(x) dx = 0, \quad (4.6)$$

$$\int_{\mathbf{R}} p'_x(x, y) dy = 0 \quad \forall x \in J \quad \text{and} \quad \int_{\mathbf{R}} \pi(x) p'_y(x, y) dx = \pi'(y) \quad \forall y \in J. \quad (4.7)$$

Lemma 4.2. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be a probability density assumed to be absolutely continuous and such that $I(f) < \infty$, then

$$\int_{\mathbf{R}} f'(t) dt = 0.$$

The proof of this lemma is obvious and we omit it.

Applying this result to the conditional density of $\xi_2/\xi_1 = x$, we have for almost every $x \in J$

$$\int_{\mathbf{R}} p'_y(x, y) dy = 0. \quad (4.8)$$

Then, we study the moments of X_k and in view of (4.6)-(4.8) and (2.2)-(2.5), we prove that for all $1 \leq k \leq n$, $\mathbf{E}(X_k) = 0$ and $\mathbf{E}(X_k^2) \leq I$. Moreover, we prove that for all

$1 \leq k \leq n$, $\text{Cov}(X_k, X_{k+1}) \leq I/2$ and for all $1 \leq k \leq n-2$ and for all $2 \leq i \leq n-k$, $\text{Cov}(X_k, X_{k+i}) = 0$. Well, let us come back to (4.4), we have

$$\begin{aligned} \|P_n^a - P_n\| &\leq \left[D \left(\sum_{k=1}^n a_k X_k \right) \right]^{1/2} = \left[\sum_{k=1}^n a_k^2 D(X_k) + 2 \sum_{1 \leq i < j \leq n} |a_i| |a_j| \text{Cov}(X_i, X_j) \right]^{1/2} \\ &\leq \left[I \sum_{k=1}^n a_k^2 + I \sum_{i=1}^{n-1} |a_i| |a_{i+1}| \right]^{1/2} \\ &\leq \sqrt{I} \left[\sum_{k=1}^n a_k^2 + \left(\sum_{i=1}^{n-1} |a_i|^2 \right)^{1/2} \left(\sum_{i=1}^{n-1} |a_{i+1}|^2 \right)^{1/2} \right]^{1/2}, \end{aligned}$$

which proves Theorem 4.1.

4.1.2. Random translation. We use the notation of Section 2.2. Let $n \in \mathbb{N}^*$, P_n and P_n^η are respectively the distributions of $\bar{\xi} = (\xi_1, \dots, \xi_n)$ and $\bar{\xi} + \bar{\eta} = (\xi_1 + \eta_1, \dots, \xi_n + \eta_n)$.

We shall estimate

$$\|P_n^\eta - P_n\| = \int_{\mathbb{R}^n} \left| \prod_{k=1}^n \mathbb{E}[p(u_k - \eta_k)] - \prod_{k=1}^n p(u_k) \right| d\bar{u}. \quad (4.9)$$

Theorem 4.2. If $I_1(p) < \infty$, then for $\varepsilon > 0$ we have

$$\|P_n^\eta - P_n\| \leq \sqrt{I_1(p) \sum_{k=1}^n \mathbb{E}[\eta_k^2 \mathbb{I}_{\{|\eta_k| \leq \varepsilon\}}] + 8 \sum_{k=1}^n \mathbb{P}\{|\eta_k| > \varepsilon\}}. \quad (4.10)$$

Sketched proof. It is based on ideas similar to those used by H. Sato and C. Watari [9]. In view of [11] (Ch. 3, § 9, Theorem 1), we know that

$$\|P_n^\eta - P_n\| \leq 2\sqrt{2} \rho(P_n^\eta, P_n), \quad (4.11)$$

where ρ is the Hellinger metric

$$\rho^2(P_n^\eta, P_n) = \frac{1}{2} \int_{\mathbb{R}^n} \left[\sqrt{\prod_{k=1}^n \mathbb{E}[p(u_k - \eta_k)]} - \sqrt{\prod_{k=1}^n p(u_k)} \right]^2 d\bar{u} = 1 - H(P_n^\eta, P_n),$$

where, in view of independence $H(P_n^\eta, P_n) = \prod_{k=1}^n \int_{\mathbb{R}} \sqrt{\mathbb{E}[p(u_k - \eta_k)]} \sqrt{p(u_k)} du_k$. Therefore

$$\begin{aligned} \rho^2(P_n^\eta, P_n) &= 1 - \prod_{k=1}^n \left[1 - \frac{1}{2} \int_{\mathbb{R}} \left(\sqrt{\mathbb{E}[p(x - \eta_k)]} - \sqrt{p(x)} \right)^2 dx \right] \\ &\leq \frac{1}{2} \sum_{k=1}^n \int_{\mathbb{R}} \left(\sqrt{\mathbb{E}[p(x - \eta_k)]} - \sqrt{p(x)} \right)^2 dx. \end{aligned} \quad (4.12)$$

Let $1 \leq k \leq n$ and $\varepsilon > 0$, we have

$$\int_{\mathbb{R}} \left(\sqrt{\mathbb{E}[p(x - \eta_k)]} - \sqrt{p(x)} \right)^2 dx \leq A_1 + A_2, \quad (4.13)$$

where

$$\begin{aligned} A_1 &= \int_{\mathbb{R}} \left(\sqrt{\mathbb{E}[p(x - \eta_k) \mathbb{I}_{\{|\eta_k| \leq \varepsilon\}}]} - \sqrt{p(x)} \mathbb{P}\{|\eta_k| \leq \varepsilon\} \right)^2 dx, \\ A_2 &= \int_{\mathbb{R}} \left(\sqrt{\mathbb{E}[p(x - \eta_k) \mathbb{I}_{\{|\eta_k| > \varepsilon\}}]} - \sqrt{p(x)} \mathbb{P}\{|\eta_k| > \varepsilon\} \right)^2 dx. \end{aligned}$$

It is clear that

$$A_2 \leq 2P\{|\eta_k| > \varepsilon\}. \quad (4.14)$$

To estimate A_1 , set

$$\begin{aligned} \Phi_k: [0; 1] &\longrightarrow \mathbf{R} \\ t &\longmapsto \Phi_k(t) = \mathbf{E}^{1/2} [p(x - t\eta_k) \mathbb{I}_{\{|\eta_k| \leq \varepsilon\}}]. \end{aligned}$$

Using $I_1(p) < \infty$, we prove that Φ_k is differentiable and

$$A_1 \leq \int_0^1 \int_{\mathbf{R}} \frac{\mathbf{E}^2[-\eta_k p'(x - t\eta_k) \mathbb{I}_{\{|\eta_k| \leq \varepsilon\}}]}{4\mathbf{E}[p(x - t\eta_k) \mathbb{I}_{\{|\eta_k| \leq \varepsilon\}}]} dx dt.$$

By Schwarz's inequality and using a change of variables, we have

$$A_1 \leq \frac{1}{4} \int_{\mathbf{R}} \frac{p'^2}{p}(u) du \int_{\mathbf{R}} v^2 \mathbb{I}_{\{|v| \leq \varepsilon\}} Q_k(dv) = \frac{1}{4} I_1(p) \mathbf{E}[\eta_k^2 \mathbb{I}_{\{|\eta_k| \leq \varepsilon\}}]. \quad (4.15)$$

At last, (4.12)–(4.15) prove the announced inequality.

Now, we study the symmetric case.

Theorem 4.3. Assume that η is symmetric. If $I_2(p) < \infty$, then for $\varepsilon > 0$, we have

$$\|P_n^\eta - P_n\| \leq \sqrt{\frac{25}{24} I_2(p) \sum_{k=1}^n \mathbf{E}[\eta_k^4 \mathbb{I}_{\{|\eta_k| \leq \varepsilon\}}] + 8 \sum_{k=1}^n P\{|\eta_k| > \varepsilon\}}. \quad (4.16)$$

Sketch of proof. In view of the symmetry of the distributions of η_k , we have

$$\|P_n^\eta - P_n\| \leq 2 \sqrt{\sum_{k=1}^n \int_{\mathbf{R}} \left(\sqrt{\mathbf{E}[p(x + \eta_k)]} - \sqrt{p(x)} \right)^2 dx}$$

and

$$\int_{\mathbf{R}} \left(\sqrt{\mathbf{E}[p(x + \eta_k)]} - \sqrt{p(x)} \right)^2 dx \leq A_1 + 2P\{|\eta_k| > \varepsilon\}, \quad (4.17)$$

where

$$A_1 = \int_{\mathbf{R}} \left(\sqrt{\mathbf{E}[p(x + \eta_k) \mathbb{I}_{\{|\eta_k| \leq \varepsilon\}}]} - \sqrt{p(x)} P\{|\eta_k| \leq \varepsilon\} \right)^2 dx. \quad (4.18)$$

Using $I_2(p) < \infty$, we show that Φ_k and Φ'_k are differentiable and

$$\Phi_k(1) - \Phi_k(0) = \int_0^1 \Phi'_k(t) dt = \Phi'_k(0) + \int_0^1 (1-t) \Phi''_k(t) dt. \quad (4.19)$$

By the symmetry of the distribution of η_k , we have $\Phi'_k(0) = 0$. By Schwarz's and Hölder's inequalities and $(a+b)^2 \leq 2a^2 + 2b^2$, we obtain

$$\begin{aligned} A_1 &\leq \int_0^1 (1-t)^2 \int_{\mathbf{R}} \frac{1}{2} \mathbf{E} \left[\eta_k^4 \frac{p'^2}{p}(x + t\eta_k) \mathbb{I}_{\{|\eta_k| \leq \varepsilon\}} \right] dx dt \\ &\quad + \int_0^1 (1-t)^2 \int_{\mathbf{R}} \frac{1}{8} \mathbf{E} \left[\eta_k^4 \frac{p'^4}{p^3}(x + t\eta_k) \mathbb{I}_{\{|\eta_k| \leq \varepsilon\}} \right] dx dt \\ &= \frac{1}{3} \mathbf{E}[\eta_k^4 \mathbb{I}_{\{|\eta_k| \leq \varepsilon\}}] \left[\frac{1}{2} I_2(p) + \frac{1}{8} K(p) \right], \end{aligned}$$

where, in view of Proposition in [9], we have

$$K(p) = \int_{\mathbf{R}} \frac{p'^4}{p^3}(u) du \leq \frac{9}{4} I_2(p). \quad (4.20)$$

This and (4.17) prove the announced inequality.

4.1.3. Passage to the limit. Let $\mathbf{R}^\infty$ be the space of infinite sequences $x = (x_k)_{k \in \mathbf{N}^*}$ of reals numbers. We denote $\mathcal{B}^\infty$ the σ -field of Borel subsets in $\mathbf{R}^\infty$.

Let $X = (X_k)_{k \in \mathbb{N}^*}$ and $Y = (Y_k)_{k \in \mathbb{N}^*}$ be two sequences of real-valued r.v.'s, defined on $(\Omega, \mathcal{A})$. Let us denote by $\mathcal{P}$ and $\mathcal{Q}$ respectively the distributions of X and Y in $(\mathbb{R}^\infty, \mathcal{B}^\infty)$. The distributions of $\bar{X} = (X_1, \dots, X_n)$ and $\bar{Y} = (Y_1, \dots, Y_n)$ are P_n and Q_n respectively.

Lemma 4.3. *With the notation, we have*

$$\|\mathcal{P} - \mathcal{Q}\| = \lim_{n \rightarrow \infty} \|P_n - Q_n\|.$$

The proof follows easily from Theorem 2.7 in [3].

Then, the passage to the limit in the inequalities proved in Subsections 4.1.1 and 4.1.2 can easily be made in view of Lemma 4.3 and prove all the announced results.

4.2. Absolute continuity. Here, we just give only the outline of the proofs. We introduce the following notation for all $n \in \mathbb{N}^*$,

$$\begin{aligned} \xi^n &= (\xi_1, \dots, \xi_{n-1}, \xi_n + a_n, \xi_{n+1} + a_{n+1}, \dots) \text{ with distribution } \mathcal{P}_n, \\ \xi^{n+1} &= (\xi_1, \dots, \xi_{n-1}, \xi_n, \xi_{n+1} + a_{n+1}, \dots) \text{ with distribution } \mathcal{P}_{n+1}. \end{aligned}$$

The first stage is to show that for all $n \in \mathbb{N}^*$, $\mathcal{P}_n \sim \mathcal{P}_{n+1}$. By induction, for all $n \geq 2$

$$\mathcal{P}_n \sim \mathcal{P}_1 = \mathcal{P}^a. \quad (4.21)$$

Then, set $a^n = (0, \dots, 0, a_n, a_{n+1}, \dots)$. We see that the distributions of ξ and $\xi + a^n$ are $\mathcal{P}$ and $\mathcal{P}_n$ respectively. With the usual notation, $\mathcal{P}_n = \mathcal{P}^{a^n}$.

Thus, we can use Theorem 2.1, in particular we can use inequalities to show that

$$\lim_{n \rightarrow \infty} \|\mathcal{P}_n - \mathcal{P}\| = 0 \quad \text{and} \quad \mathcal{P}_n \xrightarrow[n \rightarrow \infty]{\text{var}} \mathcal{P}. \quad (4.22)$$

At last, we can prove the following lemma.

Lemma 4.4. *If $\mu_n \xrightarrow{\text{var}} \mu$, $n \rightarrow \infty$, and if for all $n \geq 2$, $\mu_n \sim \mu_1$, then $\mu \ll \mu_1$.*

Applying this lemma to (4.21) and (4.22), we have $\mathcal{P} \ll \mathcal{P}_1 = \mathcal{P}^a$. A similar method proves $\mathcal{P}^a \ll \mathcal{P}$.

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