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FOURIER-HERMITE EXPANSIONS FOR NONLINEAR FILTERING¹⁾

Цель статьи — предложить подход к нелинейной фильтрации, который позволил бы разделить требующие большого времени вычисления, включающие коэффициенты системы, и вычисления, связанные только с наблюдаемыми данными. Такой подход основывается на разложении Камерона-Мартина для ненормированных оптимальных фильтров.

Ключевые слова и фразы: нелинейная фильтрация, разложение Фурье-Эрмита.

1. Introduction. Since the celebrated work of Kalman and Bucy (1961), recursive filters described by stochastic differential equations have become predominant in filtering of randomly perturbed dynamical systems. The Kalman-Bucy approach was extended to nonlinear systems and substantially developed especially in the case of diffusion processes (see, e.g., [5], [8], [14], [15]). According to this theory, the nonlinear filtering problem boils down to solving the Zakai equation for the (a posteriori) filtering distribution of the state process. In spite of substantial progress made in the analysis of the Zakai equation and other related equations practical applications of nonlinear filtering remains marginal as compared to Kalman's type linear filtering and extended Kalman filters.

A lukewarm attitude to nonlinear filtering in the engineering community relates to the computational complexity of the Zakai equation. Note that this equation is the second order stochastic parabolic partial differential equations (PDE). Even deterministic PDE's of this type are subject to the «curse of dimensionability»; in high dimensions (more than 3D) a numerical solution of such equations in real time becomes unrealistic. This is of course unacceptable in many important problems (e.g., tracking) where filtering has to be performed on line.

To compute the optimal nonlinear (or linear) filter one needs the coefficients of the system and the path of the observation process. The latter is normally available only on line while the coefficients are often known a priori. Unfortunately the standard numerical algorithms for nonlinear filtering do not separate observations and parameters and could not possibly take advantage of an a priori knowledge regarding the mean dynamics of the system.

In this article we suggest an approach to nonlinear filtering which allows the separation of time consuming computations involving the coefficients of the system from those dealing with the observation data only. This approach is based on a Cameron-Martin type expansion for the non-normalized optimal filter. An additional advantage of our approach is that it does not require any assumptions on the law of the state process.

Other publications related to this approach include [7], [9], [10], [12], [13], [16], [17].

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2. Fourier-Hermite expansions for optimal filters. Let $(\Omega, \mathcal{F}, \mathbf{P}_0)$ be a complete probability space and $(w_t)_{t \geq 0}$ be an r -dimensional Brownian motion on this space. Let $(x_t)_{t \geq 0}$ be a stochastic process with values in a metric space S independent of the Brownian motion w . The σ -algebra generated by x_s for $s \leq t$ is denoted by $\mathcal{F}_t^x$. Let $(h_t(x))_{t \geq 0}$ be an $\mathcal{F}_t^x$ -adapted stochastic process with values in $\mathbf{R}^r$. We assume that

$$\mathbf{E}_0 \exp \left\{ \int_0^T |h_s(x)|^2 ds \right\} < \infty \quad \text{for some } T > 0. \quad (1)$$

Let f_T be an $\mathcal{F}_T^x$ -measurable random variable such that $\mathbf{E}_0 f_T^2 < \infty$ and $(y_t)_{t \geq 0}$ be an r -dimensional process of the form

$$y_t = y + \int_0^t h_s(x) ds + w_t, \quad t \geq 0.$$

Note that due to (1) the integral above is defined at least for $t \leq T$.

In this work we consider a nonlinear filtering problem with x_t as a state (unobservable) process, y_t as the observation process and $f_T(x)$ as the functional subject to estimation. In other words, we want to find the optimal in the mean square estimator (filter) $\hat{f}_T$ for the random variable f_T based on the observations $\{y_s, s \leq T\}$. It is a standard fact that $\hat{f}_T = \mathbf{E}_0[f_T | \mathcal{F}_T^y]$, where $\mathcal{F}_T^y$ is the σ -algebra generated by $y_s, x \leq t$. Set

$$\rho^{-1} = \exp \left\{ - \int_0^T h_s(x) dw_s - \frac{1}{2} \int_0^T |h_s(x)|^2 ds \right\}.$$

The measure $\mathbf{P}$ defined by $d\mathbf{P} = \rho^{-1} d\mathbf{P}_0$ is a probability measure on $(\Omega, \mathcal{F})$. Moreover the observation process y_t is a Brownian motion on $(\Omega, \mathcal{F}, \mathbf{P})$ independent of x_t , and the latter process has the same law on $(\Omega, \mathcal{F}, \mathbf{P})$ as on $(\Omega, \mathcal{F}, \mathbf{P}_0)$ (see, e.g., [8]). A basic result of nonlinear filtering (Kallianpur-Striebel formula) says that

$$\hat{f}_T = \frac{\mathbf{E}[f_T \rho | \mathcal{F}_T^y]}{\mathbf{E}[\rho | \mathcal{F}_T^y]} \quad (2)$$

(see, e.g., [6], [8], [15]).

The nominator of the right-hand side of (2) is usually referred to as the non-normalized (optimal) filter for f and will be denoted by $\phi_T[f]$. So formula (2) reduces the nonlinear filtering problem to computation of the non-normalized filter $\phi_T[f]$. Below we develop a Fourier-Hermite expansion for this filter and estimate the rate of convergence for the expansion. This expansion serves us as a basis for the development of an efficient recursive algorithm for nonlinear filtering of Markov processes (see [12]). Let us fix an arbitrary orthonormal system $(m_k)_{k \geq 1}$ in the space $L_2(0, T)$ of square integrable functions on $(0, T)$. Introduce random variables

$$\xi_{k,l} = \int_0^T m_k(s) dy^l(s), \quad : \xi_{k,l}^n := H_n(\xi_{k,l}),$$

where H_n is the n th Hermite polynomial defined by

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}.$$

A random variable $: \xi_{k,l}^n :$ is usually referred to as the Wick product (polynomial). This term and the notation originate in physical literature (see, e.g., [2], [11]; etc.). Let $\alpha = (\alpha_k)$ be a sequence of r -dimensional multi-indices, i.e., $\alpha_k = (\alpha_k^l)_{1 \leq l \leq r}$, $\alpha_k^l \in \mathbf{N} = \{0, 1, 2, \dots\}$, $1 \leq l \leq r$, $k \geq 1$. We shall consider only α such that $|\alpha| = \sum_{k,l} \alpha_k^l < \infty$, i.e., only a finite number of α_k^l is not zero. We denote by J the set of all such multi-indices. If $\alpha \in J$, the number $\alpha! = \prod_{k,l} (\alpha_k^l!)$ is well defined.

The celebrated Cameron-Martin result (see Theorem 1 below) says that $\xi_\alpha(y) = \prod_{k,l} : \xi_{k,l}^{\alpha_k^l} : / \alpha!$, $\alpha \in J$, is a complete orthonormal system (CONS) in $L_2(\Omega, \mathcal{F}_T^y, \mathbf{P})$.

Theorem 1 ([1]). Let η be an $\mathcal{F}_T^y$ -measurable random variable and $E\eta^2 < \infty$. Then

$$\eta = \sum_{\alpha \in J} E[\eta \xi_\alpha(y)] \xi_\alpha(y) \quad (3)$$

and

$$E\eta^2 = \sum E[\eta \xi_\alpha(y)]^2. \quad (4)$$

Series (3) converges in $L_2(\Omega, \mathbf{P})$.

Note that

$$\begin{aligned} E|\phi_T[f]|^2 &\leq E f^2 E \rho^2 \leq c E \left[\exp \left\{ \int_0^t 2h_s^2(x) ds \right\} \right. \\ &\quad \times E \left[\exp \left\{ \int_0^t h_s(x) dy_s - \frac{1}{2} \int_0^t |2h_s(x)|^2 ds \right\} \middle| \mathcal{F}_t^x \right] \Big]. \end{aligned}$$

Since

$$E \left[\exp \left\{ \int_0^t 2h_s(x) dy_s - \frac{1}{2} \int_0^t |2h_s(x)|^2 ds \right\} \middle| \mathcal{F}_t^x \right] \leq 1,$$

we conclude that $E|\phi_T[f]|^2 < \infty$. So by the Cameron-Martin theorem,

$$\phi_T[f] = \sum_{\alpha \in J} \phi_T^\alpha[f] \xi_\alpha(y) \quad (\text{in } L_2(\Omega, \mathbf{P})), \quad (5)$$

where $\phi_T^\alpha[f] = E[f_T \rho \xi_\alpha(y)]$. Of course to assess the importance of this expansion for computational purposes, one has to address two basic questions:

- (i) What is the rate of convergence of the above series?
- (ii) How does one compute the Fourier-Hermite coefficients $\phi_T^\alpha[f]$?

To answer these questions in a reasonably complete way, further specifications are needed regarding the properties of the state process x_t and the observation function h . However, considerable insight into the problem can be obtained even in our pretty general setting (see Theorem 2 below). Refinements of this theorem for particular important cases are given in [12], [10].

Let $z = (z_k)_{k=1}^\infty$ be a sequence in $\mathbf{R}^n$ such that $\sum_k |z_k|^2 < \infty$. Set

$$\mathcal{D}_z^\alpha = \prod_{k,l} \frac{\partial^{\alpha_k^l}}{(\partial z_k^l)^{\alpha_k^l}}, \quad m(s, z) = (m^l(s, z))_{r \geq l \geq 1}, \quad m^l(s, z) = \sum_k z_k^l m_k(s).$$

Theorem 2. (i) For $\alpha = (\alpha_k^l) \in J$,

$$\phi_T^\alpha[f] = \frac{1}{\sqrt{\alpha!}} \mathcal{D}_z^\alpha E \left[f_T \exp \left\{ \int_0^T h_s(x) m(s, z) ds \right\} \right] \Big|_{z=0}. \quad (6)$$

(ii) Suppose $E[f_T^2 \exp\{\int_0^T |h_s(x)|^2 ds\}] < \infty$, then

$$E \left| \hat{f}_T - \sum_{|\alpha| \leq N} \phi_T^\alpha[f] \xi_\alpha(y) \right|^2 \leq \sum_{k=N+1}^\infty E \left[f_T^2 \frac{(\int_0^T |h(s, x_s)|^2 ds)^k}{k!} \right].$$

The expression for the Fourier-Hermite coefficient $\phi_T^\alpha[f]$ as given by Theorem 2 is not overly specific and it should not be expected to be such since no assumptions were made regarding the law of x . However in a number of important particular cases it is possible to derive from (6) much more explicit formulas for $\phi_T^\alpha[f]$ (see [10], [16]). In particular, a more explicit formula for the Fourier-Hermite coefficients in (6) can be given if $(x_t)_{t \geq 0}$ is a Markov process.

To each multi-index $\alpha \in J$ of the length n (i.e., $|\alpha| = n$), we relate a set K_α . The elements of K_α are pairs of positive integers (k, i) such that each pair (k, i) is represented by α_k^i copies. This of course implies that if $\alpha_k^i = 0$, then the corresponding pair (k, i) is

not included in K_α . Note that since $|\alpha| = n$, K_α consists of n such pairs if each pair is counted according to its multiplicity (α_k^i).

Let $\mathcal{P}^n$ be a permutation group of the set $\{1, \dots, n\}$, $s^n = (s_1, \dots, s_n)$ and $ds^n = ds_1 \dots ds_n$.

Theorem 3. Let $f: S \rightarrow \mathbf{R}^1$ and $h: [0, T] \times S \rightarrow \mathbf{R}^r$ be measurable functions. Assume that $(x_t)_{t \geq 0}$ is a Markov process with the transition function $P(s, y, t, \Gamma)$ and $f_T = f(x_T)$, $h_s(x) = h(s, x_s)$. Then for $|\alpha| = n$, $K_\alpha = \{(k_1, i_1), \dots, (k_n, i_n)\}$

$$\begin{aligned} \phi_T^\alpha[f] &= \int_0^T \int_0^{s_1} \dots \int_0^{s_{n-1}} \sum_{\sigma \in \mathcal{P}^n} \int_S T_{0, s_n} h^{i_{\sigma(n)}}(s_n, \cdot) \\ &\quad \times T_{s_n, s_{n-1}} h^{i_{\sigma(n-1)}}(s_{n-1}, \cdot) \dots h^{i_{\sigma(1)}}(s_1, \cdot) \\ &\quad \times T_{s_1, T} f(y) P^0(dy) m_{k_{\sigma(1)}}(s_1) \dots m_{k_{\sigma(n)}}(s_n) ds^n, \\ \phi_T^0[f] &= \int_S T_{0, T} f(y) P^0(dy), \end{aligned} \quad (7)$$

where $T_{s,t}g(y) = \int_S P(s, y, t, dz)g(z)$ and $P^0(dy) = P(x_0 \in dy)$.

Before proceeding with the proof of Theorems 2 and 3 we give several simple corollaries of these results.

Corollary 1. Assume that $\sup_{s,x} |h(s, x)|^2 = c < \infty$. Then

$$\mathbf{E} \left| \hat{f}_T - \sum_{|\alpha| \leq N} \phi^\alpha[f] \xi_\alpha(y) \right|^2 \leq \frac{(cT)^{N+1}}{(N+1)!} e^{cT} \mathbf{E} f_T^2.$$

Corollary 2. Assume that the law of x_T is absolutely continuous with respect to the Lebesgue measure on S . Then the non-normalized filtering measure $\Phi_T(dx)$ and filtering measure $\Pi_T(dx)$ generated by the functionals $\mathbf{E}_0[1_{\{f_T \in \cdot\}} | \mathcal{F}_T^y]$ and $\mathbf{E}[1_{\{f_T \in \cdot\}} \rho | \mathcal{F}_T^y]$ respectively are also absolutely continuous with respect to the Lebesgue measure. Their respective densities $\varphi_T(x)$ and $\pi_T(x)$ are given by the formulas

$$\begin{aligned} \varphi_T(x) &= \sum_\alpha \frac{1}{\sqrt{\alpha!}} P_T(x) \mathcal{D}_x^\alpha \mathbf{E} \left[\exp \left\{ \int_0^T h_s(x) m(s, z) ds \right\} \middle| x_T = x \right] \quad (\mathbf{P}\text{-a.s.}), \\ \pi_T(x) &= \varphi_T(x) \left(\int_{\mathbf{R}^d} \varphi_T(x) dx \right)^{-1}, \quad \text{where } P_T(x) = \frac{d\mathbf{P}(x_T \in dx)}{dx}. \end{aligned} \quad (8)$$

It was shown by K. Itô [4] that a Fourier-Hermite expansion can always be rewritten in terms of multiple Wiener integrals (and vice versa). Applying this celebrated result of Itô we arrive at the following statement.

Corollary 3 (cf. [13]). Under the assumptions of Theorem 3

$$\begin{aligned} \phi_T[f] &= \int_S T_{0, T} f d\mathbf{P}^0 + \sum_{n=1}^{\infty} \sum_{i_1, \dots, i_n=1}^r \int_0^T \int_0^{s_1} \dots \int_0^{s_{n-1}} \int_S T_{0, s_n} h^{i_n}(s_n, \cdot) \dots h^{i_1}(s_1, \cdot) \\ &\quad \times T_{s_1, T} f d\mathbf{P}^0 dy_{s_n}^{i_n} \dots dy_{s_1}^{i_1}. \end{aligned}$$

Proof of Theorem 2. We split the proof of (i) into two simple lemmas.

Lemma 1 (see [3]). Set

$$\gamma_z = \exp \left\{ \int_0^T m(u, z) dy_u - \frac{1}{2} \int_0^T |m(u, z)|^2 du \right\}.$$

Then for $\alpha \in J$,

$$\xi_\alpha(y) = \frac{1}{\sqrt{\alpha!}} \mathcal{D}_x^\alpha \gamma_z \Big|_{z=0}. \quad (9)$$

Proof. Note that since $(m_k)_{k \geq 1}$ is a CONS, and y_t is a Brownian motion on $(\Omega, \mathcal{F}, \mathbf{P})$,

$$\gamma_z(t) = \prod_{k,l} \exp \left\{ z_k^l \xi_{k,l} - \frac{1}{2} (z_k^l)^2 \right\}. \quad (10)$$

Formula (9) yields (10), since for each s , the function $\exp\{sv - v^2/2\}$ is analytic in v and can be expanded in the series

$$\exp\left\{sv - \frac{v^2}{2}\right\} = \sum_{n \geq 0} \frac{v^n}{n!} H_n(s).$$

Lemma 2. For every $z = (z_k)_{k \geq 1}$

$$\mathbf{E}[f_T \rho \gamma_z] = \mathbf{E}\left[f_T \exp \int_0^T h_s(x) m(s, z) ds\right]. \quad (11)$$

Proof. By assumption (1) and standard properties of exponential martingales

$$\mathbf{E}\left[\exp\left\{\int_0^T (h_s(x) + m(s, z)) dy_s - \frac{1}{2} \int_0^T |h_s(x) + m(s, z)|^2 ds\right\} \middle| \mathcal{F}_T^x\right] = 1.$$

Making use of this equality we derive

$$\begin{aligned} \mathbf{E}[f_T \rho \gamma_z] &= \mathbf{E}\left[f_T \exp\left\{\int_0^T h_s(x) m(s, z) ds\right\}\right. \\ &\quad \times \mathbf{E}\left[\exp\left\{\int_0^T (h_s(x) + m(s, z)) dy_s - \frac{1}{2} \int_0^T |h_s(x) + m(s, z)|^2 ds\right\} \middle| \mathcal{F}_T^x\right] \\ &= \mathbf{E}\left[f(x) \exp\left\{\int_0^T h_s(x) m(s, z) ds\right\}\right]. \end{aligned}$$

This completes the proof of the lemma.

Combining (9) and (11) we arrive at (i). Now we proceed with the proof of (ii).

By Parseval's equality and (i) we have that the error of approximation

$$\begin{aligned} \delta &= \mathbf{E}\left[f_T(x) - \sum_{|\alpha| \leq N} |\phi_T^\alpha[f] \xi_\alpha(y)|^2\right] = \mathbf{E} \sum_{|\alpha| \geq N+1} |\phi_T^\alpha[f]|^2 \\ &= \sum_{|\alpha| \geq N+1} \frac{1}{\alpha!} \left(\mathbf{E}\left[f_T \mathcal{D}_z^\alpha \exp\left\{\int_0^T h_s(x) m(s, z) ds\right\}\right]_{z=0}\right)^2. \end{aligned}$$

On the other hand,

$$\left(\mathcal{D}_z^{|\alpha|} \exp\left\{\int_0^T h_s(x) m(s, z) ds\right\}\right) \Big|_{z=0} = \prod_{k,i} \left(\int_0^T h_s^i(x) m_k^i(s) ds\right)^{\alpha_k^i}.$$

So the error of approximation

$$\begin{aligned} \delta &= \sum_{|\alpha| \geq N+1} \frac{1}{\alpha!} \left(\mathbf{E}\left[f_T \prod_{k,i} \left(\int_0^T h_s^i(x) m_k^i(s) ds\right)^{\alpha_k^i}\right]\right)^2 \\ &\leq \sum_{|\alpha| \geq N+1} \mathbf{E}\left[f_T^2 \prod_{k,i} \frac{\left(\int_0^T h_s^i(x) m_k^i(s) ds\right)^{2\alpha_k^i}}{\alpha_k^i!}\right] \\ &= \mathbf{E}\left[f_T^2 \sum_{j=N+1}^{\infty} \frac{1}{j!} \left(\sum_{k=1}^{\infty} \sum_{i=1}^r \left(\int_0^T h_s^i(x) m_k^i(s) ds\right)^2\right)^j\right] \\ &= \mathbf{E}\left[f_T^2 \sum_{j=N+1}^{\infty} \frac{1}{j!} \left(\int_0^T |h_s(x)|^2 ds\right)^j\right]. \end{aligned}$$

This completes the proof of Theorem 2.

Proof of Theorem 3. It follows from (6) that for $|\alpha| \neq 0$

$$\begin{aligned}\phi_T^\alpha[f] &= \mathbf{E} \left[f(x_T) \prod_{i,k} \left(\left(\int_0^T h^i(x) m_k(s) ds \right)^{\alpha_k} / \sqrt{\alpha_k!} \right) \right] \\ &= \mathbf{E} \left[f(x_T) \int_0^T \cdots \int_0^T h^i(x_{s_1}) m_{k_1}(s_1) \cdots h^{i_n}(x_{s_n}) m_{k_n}(s_n) ds^n \right] \\ &= \sum_{\sigma \in \mathcal{P}^n} \mathbf{E} \left[f(x_T) \int_0^T \int_0^{s_1} \cdots \int_0^{s_{n-1}} h^{i_{\sigma(1)}}(s_1, x_{s_1}) \cdots h^{i_{\sigma(n)}}(s_n, x_{s_n}) \right. \\ &\quad \left. \times m_{k_{\sigma(1)}}(s_1) \cdots m_{k_{\sigma(n)}}(s_n) ds^n \right]. \quad (12)\end{aligned}$$

Now making use of the Markov property we derive from (12)

$$\begin{aligned}\phi_T^\alpha[f] &= \int_0^T \int_0^{s_1} \cdots \int_0^{s_{n-1}} \sum_{\sigma \in \mathcal{P}^n} \int_S T_{0,s_n} h^{i_{\sigma(n)}}(s_n, \cdot) T_{s_n, s_{n-1}} h^{i_{\sigma(n-1)}}(s_{n-1}, \cdot) \\ &\quad \times \cdots \times h^{i_{\sigma(1)}}(s_1, \cdot) T_{s_1, T} f dP^0 m_{k_{\sigma(1)}}(s_1) \cdots m_{k_{\sigma(n)}}(s_n) ds^n.\end{aligned}$$

This completes the proof.

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