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Тогда при всех $n \geq \nu$ существует непрерывная плотность $p_n(x)$ нормированной суммы $(X_1 + \dots + X_n) n^{-1/2}$ и для всех $x \in E^k$

$$p_n(x) = \varphi(x) + \frac{1}{6\sqrt{n}} \left\{ \int ((x, u)^3 - 3(x, u)(u, u)) P(du) \right\} \varphi(x) + O\left(\frac{1}{n}\right) \quad \text{при } n \rightarrow \infty.$$

Для величины $O(1/n)$ можно выписать явную оценку, но ее запись требует дополнительных определений, поэтому мы этого делать не будем.

В заключение отметим, что результаты, приведенным, справедливы для плотности с любым ковариационным оператором.

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ON THE NORMAL APPROXIMATION TO SYMMETRIC BINOMIAL DISTRIBUTIONS

Показано, что наилучшая константа при $1/\sqrt{n}$ в оценке ошибки в центральной предельной теореме для функций распределения сумм независимых симметричных бернуллиевских случайных величин есть $1/\sqrt{2\pi}$.

Ключевые слова и фразы: биномиальное распределение, центральная предельная теорема, наилучшая оценка ошибки, симметричные бернуллиевские величины.

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1. Introduction and main result. It has long been known that the error in the central limit theorem for distribution functions of sums of n independent and identically distributed random variables is typically of the order $1/\sqrt{n}$. To be more precise, let P be a probability measure on $\mathbf{R}$ with mean μ , variance $\sigma^2 > 0$, and finite third cumulant $\kappa_3 = \int (x - \mu)^3 dP(x)$. Let h denote the lattice span of P , that is, $h := \sup \{\eta > 0: \exists a \in \mathbf{R} \text{ with } P\{a + \eta\mathbf{Z}\} = 1\}$ if P is a lattice distribution, and $h := 0$ otherwise. For $n \in \mathbf{N}$, let $F_n = F_{n,P}$ denote the distribution function of the standardized sum of n independent random variables with distribution P , let Φ denote the standard normal distribution function, and let

$$d_n = d_{n,P} = \sup_{x \in \mathbf{R}} |F_n(x) - \Phi(x)|.$$

Then, as proved in [2],

$$\lim_{n \rightarrow \infty} \sqrt{n} d_n = \frac{1}{6\sqrt{2\pi}} \left(\frac{3h}{\sigma} + \frac{|\kappa_3|}{\sigma^3} \right).$$

Hence, if P is such that $h \neq 0$ or $\kappa_3 \neq 0$, then

$$c_P := \sup_{n \in \mathbf{N}} \sqrt{n} d_n$$

is a most natural quantity for controlling the approximation errors d_n . We are not aware of a published computation of c_P in any such case. The modest aim of the present paper is to compute c_P in the classical case, where $P = B_{1/2}$ is a symmetric Bernoulli distribution: We show that $c_{B_{1/2}} = 1/\sqrt{2\pi}$ in Corollary 1.2 below by actually computing d_n for every $n \in \mathbf{N}$ in Corollary 1.1. The heart of the proof is to show in Theorem 1.1 that the supremum in the definition of d_n is attained only in the center of the respective distributions.

Notation and conventions. Throughout the rest of this paper, let $n \in \mathbf{N}$, let $B_{n,1/2}$ denote the binomial distribution with parameters n and $\frac{1}{2}$, and let for $s \in \mathbf{R}$

$$F(s) := B_{n,1/2}([-\infty, s]), \quad b_{n,1/2}(s) := B_{n,1/2}(\{s\}) = \binom{n}{s} 2^{-s},$$

$$G(s) := \Phi\left(\frac{s - n/2}{\sqrt{n/4}}\right), \quad g(s) := G(s) - G(s-1)$$

with the understanding that $\binom{n}{s} = 0$ for $s \notin \{0, 1, \dots, n\}$. Φ is the standard normal distribution function with density $\varphi(x) = \Phi'(x) = (1/\sqrt{2\pi}) \exp\{-x^2/2\}$. The left-hand limit of a distribution function H at the point s is denoted by $H(s-)$. As usual, $[x] := \sup\{s \in \mathbf{Z}: s \leq x\}$ and $\lceil x \rceil := \inf\{s \in \mathbf{Z}: s \geq x\}$ for $x \in \mathbf{R}$.

Theorem 1.1. For every $n \in \mathbf{N}$

$$|F(s) - G(s)| < F\left(\left\lfloor \frac{n}{2} \right\rfloor\right) - G\left(\left\lfloor \frac{n}{2} \right\rfloor\right) \quad \left(s \in \mathbf{R}, s \neq \left\lfloor \frac{n}{2} \right\rfloor\right), \quad (1)$$

$$|F(s-) - G(s-)| < G\left(\left\lceil \frac{n}{2} \right\rceil -\right) - F\left(\left\lceil \frac{n}{2} \right\rceil -\right) \quad \left(s \in \mathbf{R}, s \neq \left\lceil \frac{n}{2} \right\rceil\right). \quad (2)$$

Corollary 1.1. For every $n \in \mathbf{N}$

$$\sup_{s \in \mathbf{R}} \left| B_{n,1/2}([-\infty, s]) - \Phi\left(\frac{s - n/2}{\sqrt{n/4}}\right) \right| = \begin{cases} \Phi\left(\frac{1}{\sqrt{n}}\right) - \frac{1}{2} & (n \text{ odd}), \\ \frac{1}{2} b_{n,1/2}\left(\frac{n}{2}\right) & (n \text{ even}). \end{cases} \quad (3)$$

Corollary 1.2. For every $n \in \mathbf{N}$

$$\frac{\Phi(1) - 1/2}{\sqrt{n}} \leq \sup_{s \in \mathbf{R}} \left| B_{n,1/2}([-\infty, s]) - \Phi\left(\frac{s - n/2}{\sqrt{n/4}}\right) \right| < \frac{1}{\sqrt{2\pi n}}. \quad (4)$$

The constants $\Phi(1) - \frac{1}{2} = 0.3413\dots$ and $1/\sqrt{2\pi} = 0.3989\dots$ are optimal.

Theorem 1.1 and its corollaries 1.1 and 1.2 are proved in Section 3 using the results of Section 2.

2. Auxiliary analytic inequalities. Given real numbers s, t close to zero, we often need to decide which of the two numbers $(1+s)/(1+t)$ and e^{s-t} is larger. If $t > -1$, then the following lemma, applied to either $(x, y) = (s, t)$ or $(x, y) = (t, s)$, yields a decision unless $-t < s < \min(-t + \frac{2}{3}t^2, 0)$ or $-s < t < \min(-s + \frac{2}{3}s^2, 0)$.

Lemma 2.1. *If $x, y \in \mathbf{R}$ satisfy $0 \leq y \leq |x|$ or $x \leq y \leq 0$ or $x - \frac{2}{3}x^2 \geq -y \geq 0$, then*

$$(1+x)e^{-x} \leq (1+y)e^{-y} \quad (5)$$

holds, and equality occurs if and only if $x = y$. The constant $\frac{2}{3}$ in the assumption cannot be replaced by a smaller one.

P r o o f. For $t \in \mathbf{R}$, the function

$$f(t) := (1+t)e^{-t} = 1 - \sum_{n=2}^{\infty} \frac{(-t)^n}{n(n-2)!}$$

satisfies $f'(t) = -te^{-t}$ and $f(t) - f(-t) = 2 \sum_{k=1}^{\infty} t^{2k+1}/((2k+1)(2k-1)!)$. Thus f is strictly increasing on $]-\infty, 0]$, strictly decreasing on $[0, \infty[$, and we have $f(t) - f(-t) > 0$ for $t > 0$. This obviously implies $f(x) \leq f(y)$, which is inequality (5), under each of the first two assumptions, with discussion of equality.

It remains to be proved that the implication

$$x - ax^2 \geq -y > 0 \implies f(x) < f(y) \quad (6)$$

is true if $a = \frac{2}{3}$, and false if $a < \frac{2}{3}$. To this end, we may assume that $\frac{1}{4} < a \leq \frac{2}{3}$. By the isotonicity of f on $]-\infty, 0]$, implication (6) is true if and only if $x - ax^2 > 0$ implies $f(x) < f(ax^2 - x)$. Let us consider $x \in \mathbf{R}$ with $x - ax^2 \geq 0$, that is, $0 \leq x \leq a^{-1}$. Since $\min_{x \in \mathbf{R}}(1 - x + ax^2) = 1 - 1/(4a) > 0$, we may put

$$h(x) := \ln \frac{f(ax^2 - x)}{f(x)} = \ln(1 - x + ax^2) - \ln(1 + x) + 2x - ax^2$$

and observe that $h(0) = 0$ and

$$(1 - x + ax^2)(1 + x)h'(x) = (3a - 2)x^2 + (4a - 2a^2)x^3 - 2a^2x^4. \quad (7)$$

If now $a < \frac{2}{3}$, then the right-hand side of (7) is negative for sufficiently small $x > 0$, hence so is $h'(x)$, implying that (6) is false in this case. If, on the other hand, $a = \frac{2}{3}$, then, for all $x > 0$ under consideration, the right-hand side of (7) is

$$\frac{8}{9}x^3(2 - x) \geq \frac{8}{9}x^3\left(2 - \frac{3}{2}\right) > 0,$$

yielding $h'(x) > 0$ and hence the truth of (6). Lemma 2.1 is proved.

Lemma 2.2. *Let $x, y, z \in \mathbf{R}$ with $y \neq z$. Then*

$$\begin{aligned} & \operatorname{sign} \left(\frac{\Phi(z+x) - \Phi(y+x)}{\Phi(z) - \Phi(y)} - \exp \left\{ -\frac{x}{2}(x+y+z) \right\} \right) \\ &= \operatorname{sign}(x(x+y+z)) \end{aligned} \quad (8)$$

and

$$\frac{\Phi(z+x) - \Phi(y+x)}{\Phi(z) - \Phi(y)} \leq \exp \left\{ -\frac{x^2}{2} - \min(xy, xz) \right\}. \quad (9)$$

In particular, if $x \geq 0$, $y < z$, and $x + y + z \geq 0$, then

$$\exp \left\{ -\frac{x}{2}(x+y+z) \right\} \leq \frac{\Phi(z+x) - \Phi(y+x)}{\Phi(z) - \Phi(y)} \leq \exp \left\{ -\frac{x}{2}(x+2y) \right\}. \quad (10)$$

P r o o f. For $x, h \in \mathbf{R}$ with $h \neq 0$

$$\frac{\Phi(x + h/2) - \Phi(x - h/2)}{h\varphi(x)} = \frac{2}{h} \int_0^{h/2} \operatorname{ch}(xt) e^{-t^2/2} dt \quad (11)$$

is obviously a strictly increasing function of $|x|$. Writing $f(x, h)$ for the left-hand side of (11), we get, for arbitrary $x, y, z \in \mathbf{R}$ with $y \neq z$,

$$\operatorname{sign} \left(\frac{f((z+y)/2, x, z-y)}{f((z+y)/2, z-y)} - 1 \right) = \operatorname{sign} \left(\left| \frac{z+y}{2} + x \right| - \left| \frac{z+y}{2} \right| \right) = \operatorname{sign}(x(x+y+z)).$$

Inserting the definition of f yields (8).

To prove (9), we may assume that $y < z$. Then

$$\begin{aligned} \Phi(z+x) - \Phi(y+x) &= e^{-x^2/2} \int_y^z e^{-tx} \varphi(t) dt \leq e^{-x^2/2} e^{-\min(yx, zx)} \int_y^z \varphi(t) dt \\ &= \exp \left\{ -\frac{x^2}{2} - \min(xy, xz) \right\} (\Phi(z) - \Phi(y)). \end{aligned}$$

Lemma 2.2 is proved.

Lemma 2.3. For $x > 0$ we have

$$x - \frac{x^3}{6} < x e^{-x^2/6} < \sqrt{2\pi} \left(\Phi(x) - \frac{1}{2} \right) < \min \left(x, x - \frac{x^3}{6} + \frac{x^5}{40} \right). \quad (12)$$

P r o o f. Put $f(x) := \sqrt{2\pi}(\Phi(x) - \frac{1}{2}) - x \exp\{-x^2/6\}$. Then $f(0) = 0$ and $f'(x) = e^{-x^2/6}(e^{-x^2/3} - (1 - x^2/3)) > 0$, which implies the central inequality. The others are more obvious.

Lemma 2.4. Let $x, h \in \mathbf{R}$ with $|x| \leq 1$ and $h > 0$. Then

$$\Phi\left(x + \frac{h}{2}\right) - \Phi\left(x - \frac{h}{2}\right) < h\varphi(x).$$

P r o o f. We have $\operatorname{ch} t < e^{t^2/2}$ for $t \neq 0$, by termwise comparison of power series. Hence, using $|x| \leq 1$, the right-hand side of (11) is $\leq (2/h) \int_0^{h/2} \operatorname{ch}(t) e^{-t^2/2} dt < 1$.

Lemma 2.5. If $k \in \mathbf{N}$, then

$$\frac{e^{-1/(8k)}}{\sqrt{\pi k}} < b_{2k, 1/2}(k) = b_{2k-1, 1/2}(k) = \binom{2k}{k} 2^{-2k} < \frac{1}{\sqrt{\pi k}}.$$

P r o o f. The identities are obvious and the right-hand inequality is well known. For a proof of the left-hand inequality, see [3, the lower bound in inequality (10)].

3. Proof of the main result. Theorem 1.1 will be proved by combining Propositions 3.1–3.3.

Lemma 3.1. Let $s \in \mathbf{R}$ with $s \geq n/2$. Then

$$\exp \left\{ -\frac{s - n/2}{n/4} \right\} \leq \frac{g(s+1)}{g(s)} \leq \exp \left\{ -\frac{s - n/2 - 1/2}{n/4} \right\}. \quad (13)$$

P r o o f. Let $x := 1/\sqrt{n/4}$, $y := (s-1-n/2)/\sqrt{n/4}$, and $z := (s-n/2)/\sqrt{n/4}$. Then $x > 0$, $y < z$, and $x+y+z = 2(s-n/2)/\sqrt{n/4} \geq 0$, so that (10) from Lemma 2.2 applies and yields (13).

Proposition 3.1. The following statements hold:

$$\frac{b_{n, 1/2}(s)}{g(s)} \text{ is strictly decreasing on } \left\{ s \in \mathbf{Z}: \frac{n}{2} \leq s \leq n+1 \right\}, \quad (14)$$

$$b_{n, 1/2}(s) < g(s) \quad \left(s \in \mathbf{Z}, s > \left\lfloor \frac{n}{2} \right\rfloor \right), \quad (15)$$

$$0 < F(s) - G(s) < F\left(\left\lfloor \frac{n}{2} \right\rfloor\right) - G\left(\left\lfloor \frac{n}{2} \right\rfloor\right) \quad \left(s \in \mathbf{Z}, s > \left\lfloor \frac{n}{2} \right\rfloor \right). \quad (16)$$

Remark. For $s \in \mathbf{Z}$ with $s \geq n/2 - 1$, the first inequality in (16) is known as part (ii) of Tusnády's lemma from [1, p. 250]. In [4, Theorem 1.2] the improvement $F(s) \geq \Phi((s - n/2 + \frac{1}{4})/\sqrt{n/4})$ is presented. Our proof of (16) is independent of these references.

Proof. Claim (14) is equivalent to

$$\frac{b_{n,1/2}(s+1)}{b_{n,1/2}(s)} < \frac{g(s+1)}{g(s)} \quad \left(s \in \mathbf{Z}, \frac{n}{2} \leq s \leq n \right). \quad (17)$$

So let $s \in \mathbf{Z}$ with $n/2 \leq s \leq n$. Then

$$\begin{aligned} \frac{b_{n,1/2}(s+1)}{b_{n,1/2}(s)} &= \frac{n-s}{s+1} \leq \frac{n-s}{s} = \frac{1-2z}{1+2z} \quad \left[\text{with } z := \frac{s-n/2}{n} \right] \\ &\leq \exp\{-4z\} \quad [\text{by Lemma 2.1 with } x = -2z, y = 2z, 0 \leq y \leq |x|] \\ &= \exp\left\{ -\frac{s-n/2}{n/4} \right\} \end{aligned}$$

and in fact at least one of the above two inequalities is strict. Thus an application of the left-hand inequality in (13) from Lemma 3.1 yields (17) and hence (14).

By (14) and by $b_{n,1/2}(s) = 0$ for $s > n$, it suffices to prove (15) for $s = \lfloor n/2 \rfloor + 1$.

Let us assume first that $n = 2k$ is even. Then $s = k + 1$. By Lemma 2.3 with $x = 2/\sqrt{n}$, we obtain

$$g(s) = \Phi\left(\frac{2}{\sqrt{n}}\right) - \frac{1}{2} > \frac{1}{\sqrt{2\pi}} \left(\frac{2}{\sqrt{n}} - \frac{1}{6} \left(\frac{2}{\sqrt{n}} \right)^3 \right) = \frac{1}{\sqrt{\pi k}} \left(1 - \frac{1}{3k} \right).$$

On the other hand, Lemma 2.5 yields

$$b_{n,1/2}(s) = b_{2k,1/2}(k) \frac{k}{k+1} < \frac{1}{\sqrt{\pi k}} \left(1 - \frac{1}{3k} \right).$$

Thus (15) holds in the present case.

Let us assume now that $n = 2k - 1$ is odd. Then $s = k$. By Lemma 2.3 with $x = 1/\sqrt{n}$, we obtain

$$\begin{aligned} g(s) &= \Phi\left(\frac{1}{\sqrt{n}}\right) - \Phi\left(-\frac{1}{\sqrt{n}}\right) = 2\left(\Phi\left(\frac{1}{\sqrt{n}}\right) - \frac{1}{2}\right) \\ &> \frac{2}{\sqrt{2\pi}} \left(\frac{1}{\sqrt{n}} - \frac{1}{6} \left(\frac{1}{\sqrt{n}} \right)^3 \right) = \frac{2}{\sqrt{2\pi n}} \left(1 - \frac{1}{6n} \right). \end{aligned}$$

On the other hand, Lemma 2.5 yields

$$b_{n,1/2}(s) = b_{2k-1,1/2}(k) < \frac{1}{\sqrt{\pi k}} = \frac{2}{\sqrt{2\pi n}} \sqrt{\frac{2k-1}{2k}}$$

and we have

$$\sqrt{\frac{2k-1}{2k}} = \sqrt{1 - \frac{1}{2k}} < 1 - \frac{1}{4k} < 1 - \frac{1}{6n},$$

proving (15) for the present case as well.

By (15), $F(s) - G(s)$ is strictly decreasing on $\{s \in \mathbf{Z}: s \geq \lfloor n/2 \rfloor\}$. Hence the first inequality in (16) needs only to be proved for $s \geq n$, which cases are trivial since $F(s) = 1$. Also the second inequality is obvious. Proposition 3.1 is proved.

Proposition 3.2. *The following statements hold:*

$$\frac{g(s)}{b_{n,1/2}(s-1)} \quad \text{is strictly decreasing on} \quad \left\{ s \in \mathbf{Z}: \frac{n+1}{2} \leq s \leq \frac{n}{2} + \sqrt{\frac{3n}{4}} + 1 \right\}, \quad (18)$$

$$g(s) < b_{n,1/2}(s-1) \quad \left(s \in \mathbf{Z}, \left\lfloor \frac{n}{2} \right\rfloor < s \leq \frac{n}{2} + \sqrt{\frac{3n}{4}} + 1 \right), \quad (19)$$

$$G(s) - F(s-1) < G\left(\left\lceil \frac{n}{2} \right\rceil\right) - F\left(\left\lceil \frac{n}{2} \right\rceil - 1\right) \\ \left(s \in \mathbf{Z}, \left\lceil \frac{n}{2} \right\rceil < s \leq \frac{n}{2} + \sqrt{\frac{3n}{4}} + 1\right). \quad (20)$$

P r o o f. For (18), we have to prove

$$\frac{b_{n,1/2}(s)}{b_{n,1/2}(s-1)} > \frac{g(s+1)}{g(s)} \quad (21)$$

under the assumption

$$s \in \mathbf{Z}, \quad \frac{n+1}{2} \leq s \leq \frac{n}{2} + \sqrt{\frac{3n}{4}}. \quad (22)$$

So let us assume (22). Then $s \leq n$ (since otherwise we would have $n+1 \leq n/2 + \sqrt{3n/4}$, which is false) and

$$\frac{b_{n,1/2}(s)}{b_{n,1/2}(s-1)} = \frac{n-s+1}{s} = \frac{1+y}{1+x}$$

with $x := (s - n/2)/(n/2)$, $y := -(s - n/2 - 1)/(n/2)$ satisfying the assumptions of Lemma 2.1: If $-y \geq 0$, then we observe that

$$x + y - \frac{2}{3}x^2 = \frac{2}{n} - \frac{2}{3}\left(\frac{s - n/2}{n/2}\right)^2 \geq 0$$

so that $x - 2x^2/3 \geq -y \geq 0$. If, on the other hand, $-y < 0$, then $s = (n+1)/2$ and hence $y = 1/n = x$, so that we have a trivial case, where $0 \leq y \leq |x|$. Hence Lemma 2.1 yields

$$\frac{b_{n,1/2}(s)}{b_{n,1/2}(s-1)} = \frac{1+y}{1+x} \geq e^{y-x} = \exp\left\{-\frac{s - n/2 - 1/2}{n/4}\right\}.$$

The right-hand inequality in (13) from Lemma 3.1 now yields (21) and hence (18).

By (18), it suffices to prove (19) for $s = \lceil n/2 \rceil + 1$.

If $n = 2k$ is even, then $s = k + 1$ and we get

$$g(s) = \Phi\left(\frac{2}{\sqrt{n}}\right) - \frac{1}{2} \stackrel{(a)}{<} \frac{1}{\sqrt{2\pi}} \left(\frac{2}{\sqrt{n}} - \frac{1}{6}\left(\frac{2}{\sqrt{n}}\right)^3 + \frac{1}{40}\left(\frac{2}{\sqrt{n}}\right)^5\right) \\ = \frac{1}{\sqrt{\pi k}} \left(1 - \frac{1}{3k} + \frac{1}{10k^2}\right) \stackrel{(b)}{<} \frac{1}{\sqrt{\pi k}} \left(1 - \frac{1}{8k}\right) \\ < \frac{1}{\sqrt{\pi k}} e^{-1/(8k)} \stackrel{(c)}{<} b_{2k,1/2}(k) = b_{n,1/2}(s-1),$$

where (a) holds by Lemma 2.3, (c) holds by Lemma 2.5, and in (b) we used $k \geq 1$.

If $n = 2k - 1$ is odd, then $s = k + 1$ and we have for $n \notin \{1, 3\}$

$$g(s) = \Phi\left(\frac{3}{\sqrt{n}}\right) - \Phi\left(\frac{1}{\sqrt{n}}\right) \stackrel{(a)}{<} \frac{2}{\sqrt{n}} \varphi\left(\frac{2}{\sqrt{n}}\right) \\ = \frac{1}{\sqrt{\pi k}} \sqrt{\frac{k}{k-1/2}} \exp\left\{-\frac{1}{k-1/2}\right\} \\ < \frac{1}{\sqrt{\pi k}} \sqrt{1 + \frac{1}{k}} \exp\left\{-\frac{1}{k}\right\} \stackrel{(b)}{<} \frac{1}{\sqrt{\pi k}} e^{-1/(2k)} \\ &\stackrel{(c)}{<} b_{2k-1,1/2}(k) = b_{n,1/2}(s-1),$$

where (a) holds by Lemma 2.4 with $x = 2/\sqrt{n} \leq 1$, (c) holds by Lemma 2.5, and in (b) we used the inequality $\sqrt{1+x} < e^{x/2}$.

Finally, for $n = 1$ we have

$$g(s) = \Phi(3) - \Phi(1) < \frac{1}{2} = b_{1,1/2}(1) = b_{n,1/2}(s-1)$$

and for $n = 3$ we have

$$g(s) = \Phi(\sqrt{3}) - \Phi\left(\frac{1}{\sqrt{3}}\right) = 0.2402 \dots < \frac{3}{8} = b_{3,1/2}(2) = b_{n,1/2}(s-1)$$

which completes the proof of (19).

By (19), $G(s) - F(s-1)$ is strictly decreasing on $\{s \in \mathbf{Z}: n/2 \leq s \leq n/2 + \sqrt{3n/4} + 1\}$. Thus (20) holds. Proposition 3.2 is proved.

Proposition 3.3. For $s \in \mathbf{Z}$ with $(s-1-n/2)/\sqrt{n/4} \geq \sqrt{3}$, we have

$$G(s) - F(s-1) < \frac{2 \exp\{-3/2\}}{\sqrt{2\pi n}} = \frac{0.1780 \dots}{\sqrt{n}}.$$

P r o o f. We have

$$\begin{aligned} G(s) - F(s-1) &= G(s-1) - F(s-1) + G(s) - G(s-1) \\ &\stackrel{(a)}{<} G(s) - G(s-1) \\ &\stackrel{(b)}{<} \frac{1}{\sqrt{n/4}} \varphi\left(\frac{s-1-n/2}{\sqrt{n/4}}\right) \leq \frac{1}{\sqrt{n/4}} \varphi(\sqrt{3}), \end{aligned}$$

where (a) holds by (16), and (b) holds, since φ decreases strictly on $[0, \infty[$. Proposition 3.3 is proved.

We will use the symmetries

$$F(s) = 1 - F(n-s-1) \quad (s \in \mathbf{Z}), \quad (23)$$

$$G(s) = 1 - G(n-s) \quad (s \in \mathbf{R}). \quad (24)$$

By (23), we have, in particular, $F(\lfloor n/2 \rfloor) = F(\lceil n/2 \rceil -) = \frac{1}{2}$ if n is odd, and $F(\lfloor n/2 \rfloor) = \frac{1}{2} + (\frac{1}{2}) b_{n,1/2}(n/2)$ and $F(\lceil n/2 \rceil -) = \frac{1}{2} - (\frac{1}{2}) b_{n,1/2}(n/2)$ if n is even. Hence, using (24), we get

$$\begin{aligned} d_n &:= F\left(\left\lfloor \frac{n}{2} \right\rfloor\right) - G\left(\left\lfloor \frac{n}{2} \right\rfloor\right) = G\left(\left\lceil \frac{n}{2} \right\rceil -\right) - F\left(\left\lceil \frac{n}{2} \right\rceil -\right) \\ &= \begin{cases} \Phi\left(\frac{1}{\sqrt{n}}\right) - \frac{1}{2} & (n \text{ odd}), \\ \frac{1}{2} b_{n,1/2}\left(\frac{n}{2}\right) & (n \text{ even}). \end{cases} \end{aligned} \quad (25)$$

Lemmas 2.3 and 2.5 yield

$$d_n < \frac{1}{\sqrt{2\pi n}} \quad (n \in \mathbf{N}), \quad (26)$$

$$\lim_{n \rightarrow \infty} \sqrt{n} d_n = \frac{1}{\sqrt{2\pi}}. \quad (27)$$

Put $a_k := \sqrt{2k-1} d_{2k-1}$ and $b_k := \sqrt{2k} d_{2k}$ for $k \in \mathbf{N}$. The sequence $(a_k)_{k \in \mathbf{N}}$ is increasing due to the concavity of Φ on $[0, \infty[$. The sequence $(b_k)_{k \in \mathbf{N}}$ is increasing, because $b_{k+1}/b_k = (2k+1)/\sqrt{2k(2k+2)} > 1$. Since $a_1 = \Phi(1) - \frac{1}{2} = 0.3413 \dots < 0.3535 \dots = \sqrt{2}/4 = b_1$, it follows that

$$\sqrt{n} d_n \geq \sqrt{1} d_1 = \Phi(1) - \frac{1}{2} = 0.3413 \dots \quad (n \in \mathbf{N}). \quad (28)$$

P r o o f o f T h e o r e m 1.1. Since F is the distribution function of a probability measure concentrated in $\mathbf{Z}$ and G is continuous and strictly increasing, we have for $s \in \mathbf{R} \setminus \mathbf{Z}$

$$\begin{aligned} F(s) - G(s) &= F(s-) - G(s-) < F(\lfloor s \rfloor) - G(\lfloor s \rfloor), \\ G(s) - F(s) &= G(s-) - F(s-) < G(\lceil s \rceil-) - F(\lceil s \rceil-). \end{aligned}$$

Hence, taking into account the central equality in (25), we may assume that $s \in \mathbf{Z}$ when proving (1) and (2).

If $s \in \mathbf{Z}$ with $s < \lfloor n/2 \rfloor$, then $t := n - s > \lceil n/2 \rceil$ and, using (23) and (24),

$$F(s) - G(s) = G(n - s) - F(n - s - 1) = G(t-) - F(t-).$$

Similarly, if $s \in \mathbf{Z}$ with $s < \lceil n/2 \rceil$, then $t := n - s > \lfloor n/2 \rfloor$ and

$$F(s-) - G(s-) = G(n - s) - F(n - s) = G(t) - F(t).$$

Hence it is enough to prove (1) and (2) for $s \in \mathbf{Z}$ and with « $\neq$ » replaced by « $>$ ».

If $s \in \mathbf{Z}$ with $s > \lfloor n/2 \rfloor$, then the inequality in (1) is contained in (16).

So it remains to prove the inequality in (2) assuming $s \in \mathbf{Z}$ and $s > \lceil n/2 \rceil$. If $F(s - 1) - G(s) > 0$, then

$$|F(s-) - G(s-)| = F(s - 1) - G(s) \leq F(s) - G(s) < d_n$$

by (1) with $s > \lfloor n/2 \rfloor$. So let $|F(s-) - G(s-)| = G(s) - F(s - 1)$. If $s \leq n/2 + \sqrt{3n/4} + 1$, then $G(s) - F(s - 1) < d_n$ by (20). Finally, if $s > n/2 + \sqrt{3n/4} + 1$, then Proposition 3.3 and inequality (28) yield

$$G(s) - F(s - 1) < \frac{2 \exp\{-3/2\}}{\sqrt{2\pi n}} < \frac{\Phi(1) - \frac{1}{2}}{\sqrt{n}} \leq d_n.$$

P r o o f o f C o r o l l a r y 1.1. Obvious by (1) and (25).

P r o o f o f C o r o l l a r y 1.2. Obvious by (3), (26), (27), and (28).

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