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ON A FUNCTIONAL VERSION OF
THE CONVERGENCE OF A QUADRATIC FORM
IN INDEPENDENT MARTINGALES
TO A χ^2 DISTRIBUTION

Пусть $(a_{ij})_{i,j \geq 1}$ — такая бесконечная вещественная матрица, что $a_{ii} = 0$ для любого $i \geq 1$, и пусть $(X^i)_{i \geq 1}$ — такая последовательность независимых мартингалов, что $\sup_{i \geq 1} E[(X_1^i)^4] < \infty$ и для каждого $i \geq 1$ предсказуемый компенсатор квадратичной вариации X^i есть тождественная функция. В случае, когда $\sigma_n^2 = \sum_{i,j=1}^n a_{ij}^2$ для каждого $n \geq 1$, дано необходимое и достаточное условие того, что процесс, определенный формулой $\sigma_n^{-1} \sum_{i < j \leq n} a_{ij} X_t^i X_t^j$ для каждого $n \geq 1$ и $t \geq 1$, сходится по распределению к $((2\sqrt{p})^{-1} \sum_{i=1}^p ((B_t^i)^2 - t))_{t \leq 1}$, где $p \geq 1$ и $B^1, \dots, B^p$ есть p независимых стандартных броуновских движений. Кроме того, рассмотрен случай, когда $(X^i)_{i \geq 1}$ есть последовательность независимых решений «структурного уравнения».

Ключевые слова и фразы: квадратичные формы, χ^2 -распределение, функциональные предельные теоремы, мартингалы, стохастическое исчисление, броуновское движение.

Introduction

We shall assume throughout this paper that $A = (a_{ij})_{i,j \geq 1}$ is a real symmetric matrix such that $a_{ii} = 0$ for each $i \geq 1$. For each $n \geq 1$, let A_n be the $n \times n$ matrix defined by $A_n = (a_{ij})_{i,j=1,\dots,n}$ and let $(\lambda_i^{(n)})_{i=1,\dots,n}$ be the eigenvalues of A_n in decreasing order, i.e., $\lambda_1^{(n)} \geq \lambda_2^{(n)} \geq \dots \geq \lambda_n^{(n)}$.

Let $X = (X^i)_{i \geq 1}$ be a sequence of independent, centered and square integrable random variables. Define, for each $n \geq 1$, the quadratic form $Q_n(X)$ by

$$Q_n(X) = \frac{1}{\sigma_n} \sum_{i < j \leq n} a_{ij} X^i X^j,$$

if $\sigma_n^2 = \sum_{i,j=1}^n a_{ij}^2$. The study of the limiting behavior in distribution of $Q_n(x)$ began with Sevastyanov [11], who determined the class of possible

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limit distributions for quadratic forms in independent Gaussian random variables.

The central limit theorem for the quadratic form $Q_n(X)$ has already been dealt with in many papers, with various assumptions on the sequence X (see, for example, [6], [8]).

If the random variables $X^1, X^2, \dots$ are identically distributed, it is easy to find some conditions (even necessary and sufficient ones) so that the quadratic form $Q_n(X)$ converges to a recentered χ^2 distribution with p degrees of freedom. In the direction of convergence to elements of the second Wiener chaos see [9] for a slightly better result, and [2] in the case of random matrices.

In this paper, we shall deal with convergence to χ^2 distributions, but our functional approach allows us to consider cases of sequences $X = (X^i)_{i \geq 1}$ of random variables not necessarily identically distributed. Moreover, the conditions obtained are necessary and sufficient.

In [1], we study the case of Gaussian U -statistics, i.e., the sequence X is a sequence of independent random variables with common distribution $\mathcal{N}(0, 1)$, and the sum is defined for each $n \geq 1$ by $\sum_{i, j \leq n} h_{ij}^{(n)}(X^i, X^j)$, where, for each $i, j \leq n$, $h_{ij}^{(n)}: \mathbf{R}^2 \rightarrow \mathbf{R}$ is a twice continuously differentiable function, but only reaching sufficient conditions.

The first part is dedicated to fixing notation, hypotheses and the presentation of the main results (Theorems 1, 2, and 3). Then, we will expose some theoretical background, while in the third part we set out the proofs. Finally, in Part IV, we apply Theorem 1 to quadratic forms in independent solutions of the «Structure Equation».

I. Notation, hypotheses and the main results

In the following, $\mathbf{D}$ is the space of right-continuous with left-hand limits functions $f: [0, 1] \rightarrow \mathbf{R}$ so that $f(0) = 0$, endowed with the J^1 -Skorokhod topology. The Borel σ -field of $\mathbf{D}$ is denoted $\mathcal{D}$. All processes are assumed to be $\mathbf{D}$ -valued, i.e., if Z is any process on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, $Z \in \mathbf{D}$ ($\mathbf{P}$ -a.s.) (in particular, ($\mathbf{P}$ -a.s.)). Moreover, $\mathcal{L}(Z)$ is the law of Z .

Consider a sequence $(Z_n)_n$ of processes. We say that $(Z_n)_n$ converges in distribution to Z , and write $Z_n \xrightarrow{(d)} Z$, if $\mathcal{L}(Z_n) \rightarrow \mathcal{L}(Z)$ weakly in the space of all probability measures on $(\mathbf{D}, \mathcal{D})$.

Let Z be a square integrable martingale defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. As usual (c.f. [3] or [5]), we denote by $[Z, Z]$ the quadratic variation of Z , $\langle Z, Z \rangle$ the predictable compensator of $[Z, Z]$ and ΔZ_t the jump of Z at $t \leq 1$, i.e., $\Delta Z_t = Z_t - Z_{t-}$. Moreover, if ν denotes the predictable compensator of the jump measure of Z , we write $f(x) * \nu_t$ for $\int_{\mathbf{R}} f(x) \nu(dx, [0, t])$, if $t \leq 1$.

Consider a sequence $(Y_n)_n$ of random variables. The convergence of the sequence $(Y_n)_n$ to Y in distribution (respectively in probability) is denoted

(as for the case of processes) $Y_n \xrightarrow{(d)} Y$ (respectively $Y_n \xrightarrow{P} Y$).

Finally, let $d \in \mathbb{N}^*$. If $M = (m_{ij})_{i,j=1,\dots,d}$ is a $d \times d$ real-valued matrix, let $\|M\|$ be its Hilbert-Schmidt norm, i.e., $\|M\|^2 = \sum_{i,j=1}^d m_{ij}^2$. Moreover, I_d is the $d \times d$ identity matrix.

Now, we introduce several assumptions on A . Let (Ro) be the condition introduced by Rotar' [10]:

$$(\text{Ro}): \frac{1}{\sigma_n^2} \max_{i=1,\dots,n} \sum_{j=1}^n a_{ij}^2 \longrightarrow 0, \quad n \rightarrow \infty,$$

for $p \in \mathbb{N}^*$, (No)(p) the condition:

$$(\text{No})(p): \left\| \frac{A_n}{\|A_n\|} \left(\frac{A_n}{\|A_n\|} - \frac{1}{\sqrt{p}} I_n \right) \right\| \longrightarrow 0, \quad n \rightarrow \infty,$$

and the spectral condition (Sp)(p):

$$(\text{Sp})(p): \max_{k=1,\dots,p} \left| \frac{\lambda_k^{(n)}}{\sigma_n} - \frac{1}{\sqrt{p}} \right| \longrightarrow 0 \quad \text{and} \\ \frac{1}{\sigma_n} \max_{k=p+1,\dots,n} |\lambda_k^{(n)}| \longrightarrow 0, \quad n \rightarrow \infty.$$

R e m a r k. If there exists a sequence $(c_i)_{i \geq 1}$ such that for each $i, j \geq 1$, $a_{ij} = c_i c_j$ and $a_{ii} = 0$, then (No)(1) is satisfied.

N o t a t i o n. In the following, $p \in \mathbb{N}^*$ and $B^1, \dots, B^p$ is a sequence of independent standard Brownian motions on $(\Omega, \mathcal{F}, P)$.

Basically, our idea is to introduce a functional point of view in the quadratic form defined in introduction. We then have our main result:

Theorem 1. Let $(X^i)_{i \geq 1}$ be a sequence of independent square integrable martingales on the probability space $(\Omega, \mathcal{F}, P)$ such that $\forall i \geq 1$ and $\forall t \geq 1$, $\langle X^i, X^i \rangle_t = t$. For each $t \leq 1$, put $X_t = (X_t^i)_{i \geq 1}$ and assume that $\sup_{i \geq 1} \mathbf{E}[(X_1^i)^4] < \infty$ and (Ro) are satisfied. Then the following assertions are equivalent:

- i) (No)(p) is satisfied;
- ii) $(Q_n(X_t))_{t \leq 1}$ converges in distribution to

$$\left(\frac{1}{\sqrt{p}} \sum_{i=1}^p \int_0^t B_s^i dB_s^i \right)_{t \leq 1};$$

iii) There exists a function $f \in L^2([0, 1])$ such that $(Q_n(X_t))_{t \leq 1}$ converges in distribution to

$$\left(\frac{1}{\sqrt{p}} \sum_{i=1}^p \int_0^t f(s) dB_s^i \int_0^s f(u) dB_u^i \right)_{t \leq 1}.$$

R e m a r k. The equivalence of Assertions ii) and iii) means that if $(Q_n(X_t))_{t \leq 1}$ converges in distribution to a process of the form

$$\left(\frac{1}{\sqrt{p}} \sum_{i=1}^p \int_0^t f(s) dB_s^i \int_0^s f(u) dB_u^i \right)_{t \leq 1},$$

then, necessarily, $|f| \equiv 1$. This remark also holds for Theorem 2.

The Brownian case has its own interest: even without assuming (Ro), we obtain necessary and sufficient conditions.

Theorem 2. Let $(W^i)_{i \geq 1}$ be a sequence of independent standard Brownian motions on the probability space $(\Omega, \mathcal{F}, P)$. Put $W_t = (W_t^i)_{i \geq 1}$ if $t \leq 1$. Then the following assertions are equivalent:

- i) (No)(p) is satisfied;
- ii) $Q_n(W_1)$ converges in distribution to $(1/\sqrt{p}) \sum_{i=1}^p \int_0^1 B_s^i dB_s^i$;
- iii) $(Q_n(W_t))_{t \leq 1}$ converges in distribution to

$$\left(\frac{1}{\sqrt{p}} \sum_{i=1}^p \int_0^t B_s^i dB_s^i \right)_{t \leq 1};$$

iv) There exists a function $f \in L^2([0, 1])$ such that $(Q_n(W_t))_{t \leq 1}$ converges in distribution to

$$\left(\frac{1}{\sqrt{p}} \sum_{i=1}^p \int_0^t f(s) dB_s^i \int_0^s f(u) dB_u^i \right)_{t \leq 1}.$$

More generally, we state, without proof, the following result for quadratic forms in independent identically distributed martingales. It is an easy consequence of Theorem 2, Rotar's invariance principle (see [10]) enlarged to finite dimensional repartitions of a martingale and Aldous criterion for tightness. In contrast to Theorem 1, in Theorem 3, functional convergence is equivalent to pointwise convergence.

Theorem 3. Let $(X^i)_{i \geq 1}$ be a sequence of independent identically distributed and square integrable martingales on the probability space $(\Omega, \mathcal{F}, P)$ such that $\forall i \geq 1$ and $\forall t \geq 1$, $\langle X^i, X^i \rangle_t = t$. Put $X_t = (X_t^i)_{i \geq 1}$ if $t \leq 1$ and assume (Ro). Then the following assertions are equivalent:

- i) (No)(p) is satisfied;
- ii) $Q_n(X_1)$ converges in distribution to

$$\frac{1}{\sqrt{p}} \sum_{i=1}^p \int_0^1 B_s^i dB_s^i;$$

iii) $(Q_n(X_t))_{t \leq 1}$ converges in distribution to

$$\left(\frac{1}{\sqrt{p}} \sum_{i=1}^p \int_0^t B_s^i dB_s^i \right)_{t \leq 1}.$$

Finally, the aim of the next deterministic result is to characterize Condition (No)(p) in terms of the eigenvalues of A_n .

Proposition 1. *Conditions (No)(p) and (Sp)(p) are equivalent.*

II. Theoretical background on the convergence of martingales

Let $\mathbf{C} = \mathbf{C}([0, 1]; \mathbf{R})$ be the space of continuous functions $f: [0, 1] \rightarrow \mathbf{R}$ with $f(0) = 0$, endowed with the uniform convergence topology, and $\mathcal{C}$ its Borel σ -field. For $a \geq 0$, let S_a be the function $S_a: \mathbf{D} \rightarrow [0, 1] \cup \{+\infty\}$ defined, for each $\alpha \in \mathbf{D}$, by

$$S_a(\alpha) = \begin{cases} \inf \left\{ t \leq 1: |\alpha(t)| \geq a \text{ or } |\alpha(t^-)| \geq a \right\}, \\ +\infty \text{ if } |\alpha(t)| < a \text{ for each } t \leq 1. \end{cases}$$

Let $(Z_n)_n$ be a sequence of martingales defined on the filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \leq 1}, \mathbf{P})$. For each $n \geq 1$, let ν^n be the predictable compensator of the jump measure of Z_n . Denote by Z the canonical process on the filtered space $(\mathbf{C}, \mathcal{C}, \bar{\mathbf{P}})$, $(\mathcal{C}_t)_{t \leq 1}$ the canonical filtration and let $V: \mathbf{D} \rightarrow \mathbf{D}$ be a functional.

The following result can be obtained from [5, Ch. IX, Theorem 4.47] by localization (see [5, Ch. IX, Theorem 3.39] for a similar generalization).

Theorem 4. *Assume that*

i) $\bar{\mathbf{P}}$ is a unique probability measure such that for each stopping time T , Z^T is a continuous martingale, vanishing at 0 and with quadratic variation V^T ;

ii) For each $a > 0$ and $\alpha \in \mathbf{D}$, there is a non-decreasing, continuous and deterministic (i.e., independent of α) function F_a such that $(F_a(t) - (V_{t \wedge S_a}) \circ \alpha)_{t \leq 1}$ is non-decreasing;

iii) For each $t \leq 1$, the application $\alpha \mapsto V_t(\alpha)$ is continuous;

iv) For each $\varepsilon > 0$

$$\lim_{a \nearrow \infty} \limsup_n \mathbf{P}\{|x| I_{\{|x| > a\}} * \nu_1^n \geq \varepsilon\} = 0;$$

v) For each $t \leq 1$, $a > 0$:

$$[Z_n, Z_n]_{t \wedge S_a(Z_n)} - (V_{t \wedge S_a}) \circ Z_n \xrightarrow{\mathbf{P}} 0, \quad n \rightarrow \infty.$$

Then, the laws $\mathcal{L}(Z_n)$ converge weakly towards $\bar{\mathbf{P}}$.

The following result, due to [5, Ch. IX, Theorem 4.47b], is a converse of Theorem 4:

Theorem 5. *Assume that the laws $\mathcal{L}(Z_n)$ converge weakly towards $\bar{\mathbf{P}}$, and that Assertions i), iii) and iv) of Theorem 4 are satisfied. Then, for each $t \leq 1$:*

$$[Z_n, Z_n]_t - V_t(Z_n) \xrightarrow{\mathbf{P}} 0, \quad n \rightarrow \infty.$$

III. Proofs

In this section, we adopt the notation of Part II, with

$$\bar{\mathbf{P}} = \mathbf{P} \circ \left(\left(\frac{1}{\sqrt{p}} \sum_{i=1}^p \int_0^t B_s^i dB_s^i \right)_{t \leq 1} \right)^{-1}.$$

The process Z will be the canonical process on the filtered space $(\mathbf{C}, \mathcal{C}, (\mathcal{C}_t)_{t \leq 1}, \bar{\mathbf{P}})$. For each $n \geq 1$, the martingale Z_n will be $(Q_n(X_t))_{t \leq 1}$ for the proof of Theorem 1, and $(Q_n(W_t))_{t \leq 1}$ for the proof of Theorem 2. For the sake of convenience, we put

$$Q_n(X) = (Q_n(X_t))_{t \leq 1} \quad \text{and} \quad Q_n(W) = (Q_n(W_t))_{t \leq 1}.$$

Finally, V will be the functional

$$V: \mathbf{D} \rightarrow \mathbf{C}, \quad \alpha \mapsto \int_0^1 \left(\frac{2}{\sqrt{p}} \alpha(s) + s \right) ds.$$

First of all, we state two lemmas, delaying their proofs to the end of this section.

Lemma 1. *Assertions i), ii) and iii) of Theorem 4 are satisfied by the functional V .*

Lemma 2. *Let X be a square integrable martingale on the space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \leq 1}, \mathbf{P})$ such that $\mathbf{E}[X_1^2] = 1$, and ν the predictable compensator of the jump measure of X . Then, for $a > 0$ and $\varepsilon > 0$:*

$$\mathbf{P}\{|x| I_{\{|x| > a\}} * \nu_1 \geq \varepsilon\} \leq \left(1 + \frac{1}{\varepsilon}\right) \left(\mathbf{P}\left\{\sup_{t \leq 1} |\Delta X_t| > a\right\}\right)^{1/2}.$$

The next lemma is derived from [8, Lemma 1.3].

Lemma 3. *Let $(\eta^i)_{i \leq 1}$ be a sequence of centered independent random variables, and for each $n \geq 1$, $B_n = (b_{ij})_{i,j=1,\dots,n}$ a real symmetric matrix with zeros on the diagonal. Then, if $q \geq 2$, there exists a constant c such that for each $n \geq 1$:*

$$\mathbf{E} \left[\left\| \sum_{i,j=1}^n b_{ij} \eta^i \eta^j \right\|^q \right] \leq c \sup_{i \geq 1} \mathbf{E} [|\eta^i|^q]^2 \|B_n\|^q.$$

Proof of Theorem 1. We first prove that i) $\Rightarrow$ ii). According to Lemma 1, one only needs to show that Assertions iv) and v) of Theorem 4 are satisfied.

We begin with v). For each $a > 0$ and $n \geq 1$, put

$$S_a^n = S_a(Q_n(X)) \quad \text{and} \quad M^n = ([X^n, X^n]_t - t)_{t \leq 1}.$$

Note that M^n is a martingale. Using the symmetry of the matrix A_n , we have for each $n \geq 1$, $t \leq 1$ and $a > 0$

$$\begin{aligned}
 & \left| [Q_n(X), Q_n(X)]_{t \wedge S_a^n} - V_{t \wedge S_a^n}(Q_n(X)) \right| \\
 &= \left| \sum_{i=1}^n \int_0^{t \wedge S_a^n} \frac{1}{\sigma_n^2} \left(\sum_{j=1}^n a_{ij} X_{s-}^j \right)^2 d[X^i, X^i]_s \right. \\
 &\quad \left. - \int_0^{t \wedge S_a^n} \left(\frac{2}{\sigma_n \sqrt{p}} \sum_{i < j \leq n} a_{ij} X_s^i X_s^j + s \right) ds \right| \\
 &= \left| \sum_{i=1}^n \int_0^{t \wedge S_a^n} \left(\frac{1}{\sigma_n^2} \left(\sum_{j=1}^n a_{ij} X_{s-}^j \right)^2 - \left(\frac{1}{\sigma_n \sqrt{p}} \sum_{i,j=1}^n a_{ij} X_s^i X_s^j + s \right) \right) ds \right| \\
 &\quad + \frac{1}{\sigma_n^2} \left| \sum_{i=1}^n \int_0^{t \wedge S_a^n} \left(\sum_{j=1}^n a_{ij} X_{s-}^j \right)^2 dM_s^i \right| \\
 &\leq \int_0^1 \left| \frac{1}{\sigma_n^2} \sum_{i,j=1}^n a_{ij}^2 (X_{s-}^j)^2 - s \right| ds \\
 &\quad + \int_0^1 \left| \frac{1}{\sigma_n^2} \sum_{i=1}^n \sum_{1 \leq j \neq j' \leq n} a_{ij} a_{ij'} X_{s-}^j X_{s-}^{j'} - \frac{1}{\sigma_n \sqrt{p}} \sum_{i,j=1}^n a_{ij} X_s^i X_s^j \right| ds \\
 &\quad + \sup_{t \leq 1} \left| \frac{1}{\sigma_n^2} \int_0^t \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij} X_{s-}^j \right)^2 dM_s^i \right|. \tag{1}
 \end{aligned}$$

Taking the expectation of the square of the first term of (1), we obtain:

$$\begin{aligned}
 & \mathbf{E} \left[\left(\int_0^1 \frac{1}{\sigma_n^2} \sum_{i,j=1}^n a_{ij}^2 (X_{s-}^j)^2 - s \right) ds \right]^2 \\
 &\leq \frac{1}{\sigma_n^4} \int_0^1 ds \mathbf{E} \left[\left(\sum_{j=1}^n ((X_{s-}^j)^2 - s) \sum_{i=1}^n a_{ij}^2 \right)^2 \right] \\
 &= \frac{1}{\sigma_n^4} \int_0^1 \sum_{j=1}^n \mathbf{E} [((X_{s-}^j)^2 - s)^2] ds \left(\sum_{i=1}^n a_{ij}^2 \right)^2 \\
 &\leq \left(\sup_{j \geq 1} \mathbf{E} [(X_1^j)^4] + 1 \right) \frac{1}{\sigma_n^2} \max_{j \leq n} \sum_{i=1}^n a_{ij}^2,
 \end{aligned}$$

the rightmost term vanishing, as $n \rightarrow \infty$, according to (Ro). Taking the

expectation of the square of the second term of (1), we get

$$\begin{aligned} & \mathbf{E} \left[\left(\int_0^1 \left| \frac{1}{\sigma_n^2} \sum_{i=1}^n \sum_{1 \leq j \neq j' \leq n} a_{ij} a_{ij'} X_s^j X_{s-}^{j'} - \frac{1}{\sigma_n \sqrt{p}} \sum_{i,j=1}^n a_{ij} X_s^j X_s^j \right| ds \right)^2 \right] \\ & \leq \int_0^1 \mathbf{E} \left[\left(\frac{1}{\sigma_n^2} \sum_{i=1}^n \sum_{1 \leq j \neq j' \leq n} a_{ij} a_{ij'} X_s^j X_s^{j'} - \frac{1}{\sigma_n \sqrt{p}} \sum_{i,j=1}^n a_{ij} X_s^j X_s^j \right)^2 \right] ds, \end{aligned}$$

since $\Delta X_s^i = 0$ (P-a.s.) for each $i \geq 1$ and $s \leq 1$. Then, the last term equals

$$\begin{aligned} & \int_0^1 \mathbf{E} \left[\left(\frac{1}{\sigma_n^2} \sum_{1 \leq j \neq j' \leq n} X_s^j X_s^{j'} \left(\sum_{i=1}^n a_{ji} a_{ij'} - \frac{\sigma_n}{\sqrt{p}} a_{jj'} \right) \right)^2 \right] ds \\ & = \int_0^1 s^2 ds \frac{2}{\sigma_n^4} \sum_{1 \leq j \neq j' \leq n} \left(\sum_{i=1}^n a_{ji} a_{ij'} - \frac{\sigma_n}{\sqrt{p}} a_{jj'} \right)^2. \end{aligned}$$

Assuming (Ro), this converges to 0 if and only if

$$\frac{1}{\sigma_n^4} \sum_{j,j'=1}^n \left(\sum_{i=1}^n a_{ji} a_{ij'} - \frac{\sigma_n}{\sqrt{p}} a_{jj'} \right)^2 = \left\| \frac{A_n}{\|A_n\|} \left(\frac{A_n}{\|A_n\|} \frac{1}{\sqrt{p}} I_n \right) \right\|^2$$

converges to 0, which is satisfied under (No)(p). We now turn to the last term of (1). According to the Burkholder inequality, there exists a constant c_1 such that for each $n \geq 1$

$$\begin{aligned} & \mathbf{E} \left[\sup_{t \leq 1} \left(\frac{1}{\sigma_n^2} \int_0^t \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij} X_{s-}^j \right)^2 dM_s^i \right)^2 \right] \\ & \leq \frac{c_1}{\sigma_n^4} \sum_{i=1}^n \mathbf{E} \left[\sup_{s \leq 1} \left(\sum_{j=1}^n a_{ij} X_s^j \right)^4 [M^i, M^i]_1 \right] \\ & = \frac{c_1}{\sigma_n^4} \sum_{i=1}^n \mathbf{E} \left[\sup_{s \leq 1} \left(\sum_{j=1}^n a_{ij} X_s^j \right)^4 \right] \mathbf{E}[[M^i, M^i]_1]. \end{aligned}$$

This last equality comes from the fact that for each $i \geq 1$, the sum $\sum_{j=1}^n a_{ij} X^j$ does not depend on X^i , so that the processes $[M^i, M^i]$ and $\sum_{j=1}^n a_{ij} X^j$ are independent. Then, according to the Burkholder and Minkowski inequalities, there exists a constant c_2 such that $\forall n \geq 1$:

$$\mathbf{E} \left[\sup_{s \leq 1} \left(\sum_{j=1}^n a_{ij} X_s^j \right)^4 \right] \leq c_2 \mathbf{E} \left[\left(\sum_{j=1}^n a_{ij}^2 (X_1^j)^2 \right)^2 \right]$$

$$\leq c_2 \left(\sum_{j=1}^n a_{ij}^2 \mathbf{E}[(X_1^j)^4]^{1/2} \right)^2 \leq c_2 \sup_{j \geq 1} \mathbf{E}[(X_1^j)^4] \left(\sum_{j=1}^n a_{ij}^2 \right)^2.$$

Now, for each $i \geq 1$, by the very definition of M^i :

$$\mathbf{E}[[M^i, M^i]_1] = \mathbf{E}[(X^i, X^i)_1 - 1]^2 \leq c_3 (1 + \mathbf{E}[(X_1^i)^4]),$$

for some constant c_3 independent of i . Consequently, the last term in (1) is bounded by

$$\begin{aligned} c_1 c_2 c_3 \left(1 + \sup_{j \geq 1} \mathbf{E}[(X_1^j)^4] \right) \sup_{j \geq 1} \mathbf{E}[(X_1^j)^4] \frac{1}{\sigma_n^4} \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij}^2 \right)^2 \\ \leq c_1 c_2 c_3 \left(1 + \sup_{j \geq 1} \mathbf{E}[(X_1^j)^4] \right) \sup_{j \geq 1} \mathbf{E}[(X_1^j)^4] \frac{1}{\sigma_n^4} \max_{j \leq n} \sum_{i=1}^n a_{ij}^2, \end{aligned}$$

which vanishes, as $n \rightarrow \infty$, under (Ro).

We now show that Assertion iv) of Theorem 4 is fulfilled. For each $n \geq 1$, call ν^n the predictable compensator of the jump measure of $Q_n(X)$. According to Lemma 2, for each $n \geq 1$, $\varepsilon > 0$ and $a > 0$:

$$\mathbf{P}\{|x| I_{\{|x|>a\}} * \nu_1^n \geq \varepsilon\} \leq \left(1 + \frac{1}{\varepsilon} \right) \left(\mathbf{P}\left\{ \sup_{t \leq 1} |\Delta Q_n(X_t)| > a \right\} \right)^{1/2}.$$

But for each $n \geq 1$, $\sup_{t \leq 1} (\Delta Q_n(X_t))^2 \leq [Q_n(X), Q_n(X)]_1$ (P-a.s.) so that, according to the Markov inequality:

$$\mathbf{P}\{|x| I_{\{|x|>a\}} * \nu_1^n \geq \varepsilon\} \leq \left(1 + \frac{1}{\varepsilon} \right) \frac{\mathbf{E}[[Q_n(X), Q_n(X)]_1]^{1/2}}{a} = \left(1 + \frac{1}{\varepsilon} \right) \frac{1}{a}.$$

Clearly, this implies iv).

We now prove that ii) $\Rightarrow$ i). The process $Q_n(X)$ converges in distribution to a continuous process, so that the sequence $(Q_n(X))_n$ is C-tight and

$$\mathbf{P}\left\{ \sup_{t \leq 1} |\Delta Q_n(X_t)| \geq a \right\} \rightarrow 0 \quad (\forall a > 0), \quad n \rightarrow \infty,$$

according to [5, Ch. VI, Proposition 3.26]. Then, according to Lemma 2 this implies, that Assertion iv) of Theorem 4 is satisfied. Moreover, by Lemma 1, Assertions i) and iii) are satisfied as well, so that, using Theorem 5:

$$[Q_n(X), Q_n(X)]_1 - V_1(Q_n(X)) \xrightarrow{\mathbf{P}} 0, \quad n \rightarrow \infty.$$

We have already proved that, assuming $\sup_{j \leq 1} \mathbf{E}[(X_1^j)^4] < \infty$ and (Ro), this convergence implies as $n \rightarrow \infty$

$$\int_0^1 \left(\frac{1}{\sigma_n^2} \sum_{i=1}^n \sum_{1 \leq j \neq j' \leq n} a_{ij} a_{ij'} X_s^j X_s^{j'} - \frac{1}{\sigma_n \sqrt{p}} \sum_{i,j=1}^n a_{ij} X_s^i X_s^j \right) ds \xrightarrow{\mathbf{P}} 0. \quad (2)$$

By Lemma 3 and the Cauchy-Schwarz Inequality, there exists a constant c_4 such that for each $n \geq 1$ and $s \leq 1$:

$$\begin{aligned} & \mathbf{E} \left[\left(\frac{1}{\sigma_n^2} \sum_{1 \leq j \neq j' \leq n} X_s^j X_s^{j'} \sum_{i=1}^n a_{ji} a_{ij'} \right)^2 \right] \\ & \leq \frac{c_4}{\sigma_n^8} \sup_{j \geq 1} \mathbf{E}[(X_1^i)^4]^2 \sum_{1 \leq j \neq j' \leq n} \left(\sum_{i=1}^n a_{ji} a_{ij'} \right)^2 \leq c_4 \sup_{j \geq 1} \mathbf{E}[(X_1^j)^4]^2. \end{aligned}$$

Similarly, for another constant c_5 independent of n ,

$$\mathbf{E} \left[\left(\frac{1}{\sigma_n} \sum_{i,j=1}^n a_{ij} X_s^i X_s^j \right)^4 \right] \leq c_5 \sup_{j \geq 1} \mathbf{E}[(X_1^j)^4]^2.$$

Hence the convergence in (2) holds in L^2 . For each $n \geq 1$, call

$$b_{jj'} = \sum_{i=1}^n a_{ji} a_{ij'} \quad \text{if } j, j' \geq 1.$$

We have

$$\begin{aligned} & \mathbf{E} \left[\left(\int_0^1 \left(\frac{1}{\sigma_n^2} \sum_{i=1}^n \sum_{1 \leq j \neq j' \leq n} a_{ij} a_{ij'} X_s^j X_s^{j'} - \frac{1}{\sigma_n \sqrt{p}} \sum_{i,j=1}^n a_{ij} X_s^j X_s^j \right) ds \right)^2 \right] \\ & = \int_0^1 \int_0^1 \mathbf{E} \left[\left(\frac{1}{\sigma_n^2} \sum_{1 \leq j \neq j' \leq n} X_s^j X_s^{j'} \left(b_{jj'} - \frac{\sigma_n}{\sqrt{p}} a_{jj'} \right) \right) \right. \\ & \quad \left. \times \left(\frac{1}{\sigma_n^2} \sum_{1 \leq j \neq j' \leq n} X_u^j X_u^{j'} \left(b_{jj'} - \frac{\sigma_n}{\sqrt{p}} a_{jj'} \right) \right) \right] du ds \\ & = \int_0^1 \int_0^1 \mathbf{E} \left[\left(\frac{1}{\sigma_n^2} \sum_{1 \leq j \neq j' \leq n} X_{s \wedge u}^j X_{s \wedge u}^{j'} \left(b_{jj'} - \frac{\sigma_n}{\sqrt{p}} a_{jj'} \right) \right)^2 \right] du ds \\ & = 2 \int_0^1 \int_0^1 (s \wedge u)^2 du ds \frac{1}{\sigma_n^4} \sum_{1 \leq j \neq j' \leq n} \left(b_{jj'} - \frac{\sigma_n}{\sqrt{p}} a_{jj'} \right)^2. \end{aligned}$$

We showed in the first part of the proof, that, assuming (Ro), this last term vanishes, as $n \rightarrow \infty$, if and only if (No)(p) is satisfied, which implies i).

Now we prove ii) $\iff$ iii). Clearly, ii) $\Rightarrow$ iii) with $f \equiv 1$ and one only needs to prove that iii) $\Rightarrow$ ii). According to Lemma 3, there exists a constant

c_6 such that for each $n \geq 1$ and $t \leq 1$:

$$\mathbf{E} \left[\left(\sum_{i < j \leq n} a_{ij} X_t^i X_t^j \right)^4 \right] \leq c_6 \sigma_n^4.$$

Then, the sequence $(Q_n(X_t)^2)_n$ is uniformly integrable for each $t \leq 1$, from which we deduce that

$$\begin{aligned} \frac{t^2}{2} &= \mathbf{E}[Q_n(X_t)^2] \longrightarrow \frac{1}{p} \mathbf{E} \left[\left(\sum_{i=1}^p \int_0^t f(s) dB_s^i \int_0^s f(u) dB_u^i \right)^2 \right] \\ &= \int_0^t f^2(s) ds \int_0^s f^2(u) du, \quad n \rightarrow \infty, \end{aligned}$$

so that $t = \int_0^t f^2(s) ds \quad \forall t \leq 1$ and $f^2 \equiv 1$. Consequently, $(Q_n(X_t))_{t \leq 1}$ converges in distribution to $(1/\sqrt{p}) (\sum_{i=1}^p \int_0^t \bar{B}_s^i d\bar{B}_s^i)_{t \leq 1}$, where for each $i = 1, \dots, p$, $\bar{B}^i$ is the standard Brownian motion defined for each $s \leq 1$ by $\bar{B}_s^i = \int_0^s f(u) dB_u^i$, proving ii).

We now prove Theorem 2.

P r o o f o f T h e o r e m 2. The equivalence between iv) and iii) was established in Theorem 1, and iii) $\Rightarrow$ ii) is clear.

We show that i) $\Rightarrow$ iii). Standard arguments imply that for each $n \geq 1$ and $t \leq 1$,

$$Q_n(W_t) = \frac{1}{2\sigma_n} \sum_{i=1}^n \lambda_i^{(n)} \left((Y_i^{(n)}(t))^2 - t \right), \quad (3)$$

where $Y_1^{(n)}, \dots, Y_n^{(n)}$ are independent standard Brownian motions. According to Lemma 1 and Proposition 1, one only needs to prove that (Sp)(p) implies Assertion v) of Theorem 4. Using relation (3), we have for each $a > 0$, $t \leq 1$ and $n \geq 1$, if $S_a^n = S_a(Q_n(W))$:

$$\begin{aligned} &[Q_n(W), Q_n(W)]_{t \wedge S_a^n} - V_{t \wedge S_a^n}(Q_n(W)) \\ &= \int_0^{t \wedge S_a^n} \frac{1}{\sigma_n^2} \sum_{i=1}^n \left((\lambda_i^{(n)})^2 - \frac{\sigma_n}{\sqrt{p}} \lambda_i^{(n)} \right) \left((Y_i^{(n)}(s))^2 - s \right) ds. \end{aligned}$$

The processes $Y_1^{(n)}, \dots, Y_n^{(n)}$ being independent, we deduce that for each $n \geq p$:

$$\begin{aligned} &\mathbf{E} \left[\left(\int_0^{t \wedge S_a^n} \frac{1}{\sigma_n^2} \sum_{i=1}^n \left((\lambda_i^{(n)})^2 - \frac{\sigma_n}{\sqrt{p}} \lambda_i^{(n)} \right) \left((Y_i^{(n)}(s))^2 - s \right) ds \right)^2 \right] \\ &\leq \frac{1}{\sigma_n^4} \mathbf{E} \left[\left(\sum_{i=1}^n \left((\lambda_i^{(n)})^2 - \frac{\sigma_n}{\sqrt{p}} \lambda_i^{(n)} \right) \left((Y_i^{(n)}(1))^2 - 1 \right) \right)^2 \right] \end{aligned}$$

$$\begin{aligned}
 &= \frac{2}{\sigma_n} \sum_{i=1}^n (\lambda_i^{(n)})^2 - \left(\lambda_i^{(n)} - \frac{\sigma_n}{\sqrt{p}} \right)^2 \\
 &\leq 2 \sum_{i=1}^p \left(\frac{\lambda_i^{(n)}}{\sigma_n} \right)^2 \left(\frac{\lambda_i^{(n)}}{\sigma_n} - \frac{1}{\sqrt{p}} \right)^2 + 2 \left(\frac{1}{\sqrt{p}} + \frac{1}{\sigma_n} \max_{i=p+1, \dots, n} |\lambda_i^{(n)}| \right)^2 \\
 &\quad \times \left(1 - \sum_{i=1}^p \left(\frac{\lambda_i^{(n)}}{\sigma_n} \right)^2 \right) \quad (4)
 \end{aligned}$$

because $\sum_{i=1}^n (\lambda_i^{(n)})^2 = \sigma_n^2$. It is clear that under $(\text{Sp})(p)$, the last term vanishes as $n \rightarrow \infty$, implying iii).

Only ii) $\Rightarrow$ i) is left to show. It is a classical exercise to prove, using representation (3), that if $Q_n(W_1)$ converges in distribution to

$$\frac{1}{\sqrt{p}} \sum_{i=1}^p \int_0^1 B_s^i dB_s^i,$$

then

$$\max_{i=1, \dots, n} \left| \frac{\lambda_i^{(n)}}{\sigma_n} - \lambda_i \right| \rightarrow 0, \quad n \rightarrow \infty,$$

where $\lambda_1 = \dots = \lambda_p = (1/\sqrt{p})$ and $\lambda_i = 0$ if $i \geq p+1$, the numbers $(\lambda_i)_{i \geq 1}$ being arranged in decreasing order. Now, $(\text{No})(p)$ is satisfied according to Proposition 1.

Proof of Proposition 1. Note that for each $n \geq 1$:

$$\left\| \frac{A_n}{\|A_n\|} \left(\frac{A_n}{\|A_n\|} - \frac{1}{\sqrt{p}} I_n \right) \right\|^2 = \sum_{i=1}^n \left(\frac{\lambda_i^{(n)}}{\sigma_n} \right)^2 \left(\frac{\lambda_i^{(n)}}{\sigma_n} - \frac{1}{\sqrt{p}} \right)^2. \quad (5)$$

The last term already appeared in (4): we proved that, under $(\text{Sp})(p)$, this term vanishes, as $n \rightarrow \infty$, hence $(\text{Sp})(p) \Rightarrow (\text{No})(p)$.

Now, assume $(\text{No})(p)$. Then, according to (5) as $n \rightarrow \infty$

$$\max_{i=1, \dots, n} \left(\frac{\lambda_i^{(n)}}{\sigma_n} \right)^2 \left(\frac{1}{\sqrt{p}} - \frac{\lambda_i^{(n)}}{\sigma_n} \right)^2 \leq \sum_{i=1}^n \left(\frac{\lambda_i^{(n)}}{\sigma_n} \right)^2 \left(\frac{1}{\sqrt{p}} - \frac{\lambda_i^{(n)}}{\sigma_n} \right)^2 \rightarrow 0,$$

so that for each $i \geq 1$, the sequence $\lambda_i^{(n)}/\sigma_n$ converges uniformly to 0 or to $1/\sqrt{p}$. But for each $n \geq 1$, the eigenvalues $(\lambda_i^{(n)})_{i=1, \dots, n}$ are arranged in the decreasing order and $\sum_{i=1}^n (\lambda_i^{(n)})^2 = \sigma_n^2$, implying only two possibilities: either $\lambda_i^{(n)}/\sigma_n$ converges to $1/\sqrt{p}$ for each $i = 1, \dots, p$ and $\max_{i=p+1, \dots, n} \lambda_i^{(n)}/\sigma_n$ converges to 0 as $n \rightarrow \infty$, or $\max_{i=1, \dots, n} \lambda_i^{(n)}/\sigma_n$ van-

ishes as $n \rightarrow \infty$. Suppose the second case occurs. Then, as $n \rightarrow \infty$

$$\begin{aligned} & \sum_{i=1}^n \left(\frac{\lambda_i^{(n)}}{\sigma_n} \right)^2 \left(\frac{1}{\sqrt{p}} - \frac{\lambda_i^{(n)}}{\sigma_n} \right)^2 \\ &= \frac{1}{p} \sum_{i=1}^n \left(\frac{\lambda_i^{(n)}}{\sigma_n} \right)^2 + \sum_{i=1}^n \left(\frac{\lambda_i^{(n)}}{\sigma_n} \right)^4 - \frac{2}{\sqrt{p}} \sum_{i=1}^n \left(\frac{\lambda_i^{(n)}}{\sigma_n} \right)^3 \end{aligned}$$

converges to $1/p$, contradicting (No)(p): therefore the eigenvalues satisfy the first possibility, i.e., (Sp)(p) is valid.

We now prove Lemma 1. In [1, Lemma 5], we proved the following result:

Lemma 4. *Let T be a stopping time and $p \in \mathbf{N}^*$. The following assertions are equivalent:*

i) X^T is a square integrable continuous martingale such that $X_0^T = 0$ and for each $t \leq 1$,

$$\langle X^T, X^T \rangle_t = \int_0^{t \wedge T} \left(\frac{2}{\sqrt{p}} X_s + s \right) ds;$$

ii) There exist p independent standard Brownian motions $B^1, \dots, B^p$ such that for each $t \leq 1$,

$$X_{t \wedge T} = \frac{1}{\sqrt{p}} \sum_{i=1}^p \int_0^{t \wedge T} B_s^i dB_s^i.$$

Lemma 1 is a straightforward consequence of Lemma 4.

Proof of Lemma 1. Lemma 4 implies Assertion i) of Theorem 4. If $a > 0$, let F_a be the function defined for each $t \leq 1$ by

$$F_a(t) = \int_0^t (2p^{-1/2}a + s) ds.$$

Then $(F_a(t) - V_{t \wedge S_a^n} \circ \alpha)_{t \leq 1}$ is non-decreasing for each $\alpha \in \mathbf{D}$ which shows that Assertion ii) of Theorem 4 is satisfied for the function F_a . Finally, it is clear that for each $t \leq 1$, the function $\alpha \mapsto V_t(\alpha)$ is continuous, implying iii).

Proof of Lemma 2. According to the Lengart-Rebolledo inequality (c.f. [5, Ch. I, Lemma 3.30]), for each $a > 0$, $\varepsilon > 0$ and $\eta > 0$:

$$\begin{aligned} & \mathbf{P}\{|x| I_{\{|x| > a\}} * \nu_1 \geq \varepsilon\} \leq \frac{\eta}{\varepsilon} + \frac{1}{\varepsilon} \mathbf{E} \left[\sup_{s \leq 1} |\Delta X_s| I_{\{\sup_{s \leq 1} |\Delta X_s| > a\}} \right] \\ & + \mathbf{P} \left\{ \sum_{s \leq 1} |\Delta X_s| I_{\{|\Delta X_s| > a\}} > \eta \right\} \\ & \leq \frac{\eta}{\varepsilon} + \frac{1}{\varepsilon} \mathbf{E} \left[\sup_{s \leq 1} |\Delta X_s|^2 \right]^{1/2} \left(\mathbf{P} \left\{ \sup_{s \leq 1} |\Delta X_s| > a \right\} \right)^{1/2} \end{aligned}$$

$$\begin{aligned}
 & + \mathbf{P} \left\{ \sum_{s \leq 1} |\Delta X_s| I_{\{|\Delta X_s| > a\}} > 0 \right\} \\
 & \leq \frac{\eta}{\varepsilon} + \frac{1}{\varepsilon} \mathbf{E}[(X_1)^2]^{1/2} \left(\mathbf{P} \left\{ \sup_{s \leq 1} |\Delta X_s| > a \right\} \right)^{1/2} + \mathbf{P} \left\{ \sup_{s \leq 1} |\Delta X_s| > a \right\} \\
 & \leq \frac{\eta}{\varepsilon} + \left(1 + \frac{1}{\varepsilon} \right) \left(\mathbf{P} \left\{ \sup_{s \leq 1} |\Delta X_s| > a \right\} \right)^{1/2}.
 \end{aligned}$$

Now, letting $\eta \rightarrow 0$, we obtain the lemma.

IV. Application of Theorem 1

Let $f: \mathbf{R} \rightarrow \mathbf{R}$. We shall say that the martingale X defined on $(\Omega, \mathcal{F}, \mathbf{P})$ is a solution to the structure equation (SE)(f) if for each $t \leq 1$:

$$[X, X]_t = t + \int_0^t f(X_{s-}) dX_s \quad (\mathbf{P}\text{-a.s.}).$$

Note that $\forall t \leq 1, \langle X, X \rangle_t = t$. Equation (SE)(f) has been studied by many authors. Meyer [7] showed that (SE)(f) admits a solution if f is continuous, but in general, this solution is not unique (c.f. [4]).

If $p \in \mathbf{N}^*$ and $B^1, \dots, B^p$ is a sequence of independent standard Brownian motions of $(\Omega, \mathcal{F}, \mathbf{P})$, we show the following corollary.

Corollary 1. *For each $i \geq 1$, let $f_i: \mathbf{R} \rightarrow \mathbf{R}$ be a function such that there exists $K_i \geq 0$ such that $\forall x \in \mathbf{R}: |f_i(x)| \leq K_i x^2$. We suppose that $(X^i)_{i \geq 1}$ is a sequence of independent martingales on $(\Omega, \mathcal{F}, \mathbf{P})$ such that for each $i \geq 1$, X^i is a solution to (SE)(f_i). Call $X_t = (X_t^i)_{i \geq 1}$ for each $t \leq 1$ and assume that $\sup_{i \geq 1} K_i < \infty$ and (Ro) are satisfied. Then the following assertions are equivalent:*

- i) (No)(p) is satisfied;
- ii) $(Q_n(X_t))_{t \leq 1}$ converges in distribution to

$$\left(\frac{1}{\sqrt{p}} \sum_{i=1}^p \int_0^t B_s^i dB_s^i \right)_{t \leq 1}$$

R e m a r k. We can apply this result to quadratic forms in mixing of Brownian motions ($f_i \equiv 0$), compensated Poisson processes ($f_i \equiv \alpha \neq 0$) of Azéma martingales (case $f_i(x) = \alpha x + \beta, \forall x \in \mathbf{R}$, if $\alpha \neq 0, \beta \in \mathbf{R}$).

P r o o f. According to Theorem 1, one only needs to prove that $\sup_{i \geq 1} K_i < \infty$ implies $\sup_{i \geq 1} \mathbf{E}[(X_1^i)^4] < \infty$. By the very definition of

X^i , $i \geq 1$, we have for each $t \leq 1$:

$$\begin{aligned} \mathbf{E} \left[([X^i, X^i]_t)^2 \right] &\leq 2 + 2 \int_0^t \mathbf{E} [f_i(X_s^i)^2] ds = 2 + 2 \int_0^t \mathbf{E} [f_i(X_s^i)^2 I_{\{|X_s^i| \geq 1\}}] ds \\ &\quad + 2 \int_0^t \mathbf{E} [f_i(X_s^i)^2 I_{\{|X_s^i| < 1\}}] ds \\ &\leq 2(1 + K_i^2) + 2K_i^2 \int_0^t \mathbf{E} [(X_s^i)^4] ds. \end{aligned}$$

Hence, according to the Burkholder inequality, there is a constant c such that for each $t \leq 1$ and $i \geq 1$:

$$\mathbf{E} [(X_t^i)^4] \leq 2c(1 + K_i^2) + 2cK_i^2 \int_0^t \mathbf{E} [(X_s^i)^4] ds.$$

Finally, Gronwall lemma implies that for each $i \geq 1$,

$$\mathbf{E} [(X_1^i)^4] \leq 2c(1 + K_i^2) \exp\{2cK_i^2\},$$

hence the result.

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