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A CURIOUS EXAMPLE FROM
STATISTICAL DIFFERENTIAL GEOMETRY¹⁾

Рассмотрен пример семейства попарно сингулярных вероятностных мер на бесконечномерном пространстве. Несмотря на то, что метрика информации Фишера и ее варианты не могут быть использованы в данном случае, показано, что параметрическое многообразие имеет естественную дифференциальную структуру с ненулевой кривизной, не являющуюся римановой. Также установлено, что не существует римановой метрики, совместимой с естественной аффинной связностью, для которой кривизна отлична от нуля.

Ключевые слова и фразы: стохастические дифференциальные уравнения с частными производными, параметрическое многообразие, аффинная связность, нериманова геометрия.

§ 1. Introduction

The role of differential geometry in statistical parameter estimation was foreseen in an early paper of C. R. Rao in which the Fisher information was first used to define a Riemannian metric on the parameter manifold [6]. In his elegant paper, S. Amari has given a detailed account of the subsequent development of what may be called statistical differential geometry [1]. The Fisher information metric plays an important role in all this work.

While the theory expounded in [1] and in papers by other authors is quite general in character, to the best of our knowledge, it has not been applied to specific examples from continuous time stochastic processes. In attempting to do this, we came across the curious result alluded to in the title of this paper. Very briefly, this is what we have found: Θ is a 2-dimensional manifold of parameters $\theta = (\theta^1, \theta^2)$ from a stochastic partial differential equation. If its affine connection ∇ defines a manifold with non-zero curvature, then the manifold is non-Riemannian. More precisely, if there is *any* Riemannian metric compatible with ∇ , then the manifold is flat. In the example to be discussed, there is no Fisher information metric and the manifold is flat no matter which «estimation function» is used as a surrogate for the Fisher information to define a compatible Riemannian metric.

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We now introduce the problem in detail.

Let $(X_1, \dots, X_N)$ be observations with joint distribution $\{P_\theta, \theta \in \Theta\}$ ($= S$, say) where for $\theta = (\theta^1, \dots, \theta^n)$, P_θ are assumed to be absolutely continuous with respect to a (possibly σ -finite) measure μ . Write $l(x, \theta) = \log(dP_\theta/d\mu)(x)$. Here smoothness of $l(x, \theta)$ in θ is also assumed. Under the assumption of identifiability, S can be regarded as a differentiable manifold.

In statistical theory, various differential geometric structures on S are determined by the Fisher information Riemannian metric given by

$$\sum_{i,j=1}^n g_{ij}(\theta) d\theta^i d\theta^j$$

where $g_{ij}(\theta) = E_\theta[\partial_i l(x, \theta) \partial_j l(x, \theta)]$ and $(g_{ij}(\theta))$ is positive definite. The Riemannian curvature of S is given by the Riemann-Christoffel tensor whose components are

$$R_{jkl}^i(\theta) = \frac{\partial \Gamma_{lj}^i(\theta)}{\partial \theta^k} - \frac{\partial \Gamma_{kj}^i(\theta)}{\partial \theta^l} + \sum_{p=1}^n (\Gamma_{lj}^p(\theta) \Gamma_{kp}^i(\theta) - \Gamma_{kj}^p(\theta) \Gamma_{lp}^i(\theta)),$$

where

$$\Gamma_{ij}^k = \sum_{m=1}^n \frac{1}{2} g^{km} \left\{ \frac{\partial g_{mj}}{\partial \theta^i} + \frac{\partial g_{im}}{\partial \theta^j} - \frac{\partial g_{ij}}{\partial \theta^m} \right\}$$

are the Christoffel symbols of the second kind. Here $g^{-1} = (g^{km})$. Note that S is flat if $R_{jkl}^i = 0$ for all i, j, k, l .

When we have observations $\{X_t, 0 \leq t \leq T\}$ or $\{X^j, j = 1, 2, \dots\}$ from a stochastic process, a situation can arise that is different from finite dimensional statistics. Taking $\mu = P_{\theta_0}$, we have P_θ locally absolutely continuous with respect to P_{θ_0} but not absolutely continuous. In other words, if $\{X^j, j = 1, 2, \dots\}$ is a series representation for the process, let $\mathcal{F}_N = \sigma\{X_1, \dots, X_N\}$ and $\mathcal{F}_\infty = \sigma\{X_j, j \geq 1\}$. Then for each N , $P_\theta^N = P_{\theta|\mathcal{F}_N}$ is absolutely continuous with respect to $P_{\theta_0}^N = P_{\theta_0|\mathcal{F}_N}$ but P_θ is not absolutely continuous with respect to P_{θ_0} on $\mathcal{F}_\infty$. Let $S^N = \{P_\theta^N, \theta \in \Theta\}$ and $S = \{P_\theta, \theta \in \Theta\}$. Assume further that the family $\{P_\theta^N\}$ is identifiable for sufficiently large N and the same is true for $\{P_\theta\}$. Let $l_N(x, \theta) = \log(dP_\theta^N/dP_{\theta_0}^N)(x)$ be smooth in θ . Then, we have a Riemannian geometry for S^N defined by $g_{ij}(\theta, N) = E_\theta[\partial_i l_N(x, \theta) \partial_j l_N(x, \theta)]$ — the Fisher information metric. Hence, for each N , there is a unique Riemannian connection ∇^N on the Riemannian manifold $(S^N, g(N))$ with the Christoffel symbols of the second kind $\Gamma_{ij}^k(N)$ by Theorem 8.6 (Fundamental Lemma of Riemannian Geometry) in [4]. We show that the limit values of $\Gamma_{ij}^k(N)$ define a unique linear connection ∇ on S . The aim of this paper is to investigate whether a Riemannian metric g on S , for which ∇ is

a Riemannian connection, exists. We consider this problem for a specific example taking the number of parameters $n = 2$ and where $\{X_t, 0 \leq t \leq T\}$ is the unique solution of Stochastic Partial Differential Equations (SPDE). We begin by considering this problem when $\mathbf{P}_\theta$ is absolutely continuous with respect to $\mathbf{P}_{\theta_0}$.

§ 2. Case I: Existence of the Riemannian connection

Let H be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space with filtration $(\mathcal{F}_t)$, $t \in [0, 1]$, satisfying the usual conditions. Let (W_t) be an $(\mathcal{F}_t)$ -cylindrical Brownian motion on H .

We are going to consider the estimation problem for the following H -valued SDE:

$$\begin{aligned} dX_t &= -(\theta^1 L_1 + L_0) X_t dt + \theta^2 X_t dt + B dW_t \quad \text{for } 0 < t \leq T, \\ X_0 &= 0, \end{aligned} \quad (1)$$

where $\theta = (\theta^1, \theta^2)$ is a vector of the unknown parameters.

Let us put $\mathcal{L}_{\theta^1} := \theta^1 L_1 + L_0$. Here the operators L_1^{-1} and L_0^{-1} are positive bounded self-adjoint operators with discrete spectrum, and have the same system of eigenfunctions. Let $\{\varphi_k\}$ be the eigenfunctions of L_1^{-1} and L_0^{-1} which constitute a CONS in H , and let $\{\nu_k^{-1}\}$ and $\{\zeta_k^{-1}\}$ be the corresponding eigenvalues respectively. Then $\mathcal{L}_{\theta^1}^{-1}$ is bounded self-adjoint with discrete spectrum such that $\mathcal{L}_{\theta^1} \varphi_k = \lambda_k^{\theta^1} \varphi_k$ where $\lambda_k^{\theta^1} := \theta^1 \nu_k + \zeta_k$. We assume that $T_t := e^{-\mathcal{L}_{\theta^1} t}$ is a contraction semigroup on H .

It will be assumed that

(C1) $A^* \varphi_k = a_k \varphi_k$, $B^* \varphi_k = b_k \varphi_k$, $a_k, b_k > 0$ for all k where A^* and B^* are the adjoints of the operators A and B respectively.

(C2) The sequences $\{a_k\}$ and $\{b_k\}$ satisfy

$$\sum_{k=1}^{\infty} \frac{a_k^2}{\nu_k} < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{b_k^2}{(\nu_k)^{1/2}} < \infty.$$

(C3) As $n \rightarrow \infty$,

$$\frac{\nu_n}{\zeta_n} = o(1) \quad \text{and} \quad \frac{a_n}{\nu_n} = o(1).$$

(C4) The parameter set Θ is given by

$$\Theta = \{(\theta^1, \theta^2) \mid \theta^1 > 0 \quad \text{and} \quad \theta^1 > c\theta^2\} \quad \text{where } c = \sup_k \frac{a_k}{\nu_k} > 0.$$

(C5) The sequences $\{\zeta_k\}$ and $\{\nu_k\}$ satisfy

$$\sum_{k=1}^{\infty} \frac{\nu_k^2}{\zeta_k} < \infty.$$

Under the assumptions (C1) and (C2), there exists a unique continuous solution X to the SDE (1), i.e., $X \in C([0, T], H)$ $\mathbf{P}$ -a.s.) (see [3, Theorem 2.7]). The solution X_t to this problem can be written as $X_t = \sum_{k=1}^{\infty} \langle X_t, \varphi_k \rangle \varphi_k$ where the Fourier coefficients satisfy the equations $d\langle X_t, \varphi_k \rangle = \langle \theta^2 A X_t - \theta^2 L_0 X_t - L_1 X_t, \varphi_k \rangle dt + \langle B^* \varphi_k, dW_t \rangle$. Let $X_{t,k} := \langle X_t, \varphi_k \rangle$. Then for all k ,

$$dX_{t,k} = -(\theta^1 \nu_k + \zeta_k - \theta^2 a_k) X_{t,k} dt + b_k dW_t(\varphi_k) \quad (2)$$

where $W_t(\varphi_k)$ is a sequence of independent real valued Brownian motions. It is easy to see that SDE (2) has the explicit solution

$$X_{t,k} = b_k \int_0^t \exp \{(\theta^1 \nu_k + \zeta_k - \theta^2 a_k)(t-s)\} dW_s(\varphi_k). \quad (3)$$

We denote by $C[0, T]$ the set of all continuous functions defined on $[0, T]$, taking values in $\mathbf{R}$. Let $\tilde{\mathcal{B}}_T$ be the Borel σ -algebra on the space $C[0, T]$. Denote by $\tilde{\mathbf{P}}_{\theta,k}$ the measure on $\tilde{\mathcal{B}}_T$ generated by the solution $X_k = (X_{t,k})$, $t \in [0, T]$, to Equation (2).

Let $X_t^N := \sum_{k=1}^N X_{t,k} \varphi_k$ and $X_t := \sum_{k=1}^{\infty} X_{t,k} \varphi_k$. Then X^N and X induce the measures $\mathbf{P}_{\theta}^N$ and $\mathbf{P}_{\theta}$ on $(C([0, T], H), \mathcal{B}_T)$ respectively, where $\mathcal{B}_T$ is the Borel σ -algebra. Let us set $S^N := \{\mathbf{P}_{\theta}^N, \theta \in \Theta\}$ and $S := \{\mathbf{P}_{\theta}, \theta \in \Theta\}$.

It is well known that the measure $\mathbf{P}_{\theta}^N$ is absolutely continuous with respect to $\mathbf{P}_{\theta_0}^N$, and

$$\begin{aligned} Z_N &:= \frac{d\mathbf{P}_{\theta}^N}{d\mathbf{P}_{\theta_0}^N}(X^N) = \prod_{k=1}^N \frac{d\tilde{\mathbf{P}}_{\theta,k}}{d\tilde{\mathbf{P}}_{\theta_0,k}}(X_k) \\ &= \exp \left\{ \sum_{k=1}^N \frac{(\theta_0^1 - \theta^1) \nu_k + (\theta^2 - \theta_0^2) a_k}{b_k^2} \int_0^T X_{t,k} dX_{t,k} \right. \\ &\quad - \frac{1}{2} \sum_{k=1}^N \frac{1}{b_k^2} \left(((\theta^1)^2 - (\theta_0^1)^2) \nu_k^2 + 2(\theta_0^1 \theta_0^2 - \theta^1 \theta^2) \nu_k a_k \right. \\ &\quad \left. \left. + ((\theta^2)^2 - (\theta_0^2)^2) a_k^2 + 2(\theta^1 - \theta_0^1) \nu_k \zeta_k \right. \right. \\ &\quad \left. \left. + 2(\theta_0^2 - \theta^2) \zeta_k a_k \right) \int_0^T (X_{t,k})^2 dt \right\}. \quad (4) \end{aligned}$$

Let us set

$$l_N(X, \theta) := \log \frac{d\mathbf{P}_{\theta}^N}{d\mathbf{P}_{\theta_0}^N}(X^N).$$

Define

$$g_{ij}(\theta, N) := \mathbf{E} \left[\frac{\partial l_N(X, \theta)}{\partial \theta^i} \frac{\partial l_N(X, \theta)}{\partial \theta^j} \right], \quad i, j = 1, 2.$$

We can write the Riemannian metric $\langle \cdot, \cdot \rangle$ on S^N as

$$\langle \cdot, \cdot \rangle = \sum_{i,j} g_{ij}(N) d\theta^i \otimes d\theta^j$$

where $\otimes$ denotes the tensor product. The Fisher information as a Riemannian metric tensor was first studied by Rao (1945) (see e.g. [6] and [1]).

From (4), $(g_{ij}(\theta, N))$ is given by

$$\begin{aligned} g_{11}(\theta, N) &= \sum_{k=1}^N \frac{\nu_k^2}{b_k^2} \int_0^T \mathbf{E}(X_{t,k})^2 dt, \\ g_{12}(\theta, N) &= - \sum_{k=1}^N \frac{\nu_k a_k}{b_k^2} \int_0^T \mathbf{E}(X_{t,k})^2 dt, \\ g_{22}(\theta, N) &= \sum_{k=1}^N \frac{a_k^2}{b_k^2} \int_0^T \mathbf{E}(X_{t,k})^2 dt. \end{aligned}$$

Let us now prove that the measure $\mathbf{P}_\theta$ is absolutely continuous with respect to $\mathbf{P}_{\theta_0}$.

Note that $\tilde{\mathbf{P}}_\theta$ (respectively, $\tilde{\mathbf{P}}_{\theta_0}$) is the law of a sequence of independent random variables (X_k) with values in $C[0, T]$. Define

$$\begin{aligned} \tilde{\Omega} &= \otimes C[0, T], & X_k(\tilde{\omega}) &= k\text{th coordinate of } \tilde{\omega} \in \tilde{\Omega}, \\ \tilde{\mathcal{F}} &= \otimes \tilde{\mathcal{B}}_T, & \tilde{\mathcal{F}}_N &= \sigma\{X_1, \dots, X_N\} \quad \text{for } N \geq 1, \\ \tilde{\mathbf{P}}_\theta &= \otimes_k \tilde{\mathbf{P}}_{\theta,k}, & \tilde{\mathbf{P}}_{\theta_0} &= \otimes_k \tilde{\mathbf{P}}_{\theta_0,k}. \end{aligned}$$

Theorem 2.1. *The measure $\mathbf{P}_\theta$ is absolutely continuous with respect to $\mathbf{P}_{\theta_0}$, and the Radon-Nikodym derivative of $\mathbf{P}_\theta$ with respect to $\mathbf{P}_{\theta_0}$ is*

$$\begin{aligned} Z_\infty := \frac{d\mathbf{P}_\theta}{d\mathbf{P}_{\theta_0}}(X) &= \exp \left\{ \sum_{k=1}^{\infty} \frac{(\theta_0^1 - \theta^1) \nu_k + (\theta^2 - \theta_0^2) a_k}{b_k^2} \int_0^T X_{t,k} dX_{t,k} \right. \\ &\quad - \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{b_k^2} \left(((\theta^1)^2 - (\theta_0^2)^2) \nu_k^2 + 2(\theta_0^1 \theta_0^2 - \theta^1 \theta^2) \nu_k a_k \right. \\ &\quad \left. + ((\theta^2)^2 - (\theta_0^2)^2) a_k^2 + 2(\theta^1 - \theta_0^1) \nu_k \zeta_k \right. \\ &\quad \left. \left. + 2(\theta_0^2 - \theta^2) \zeta_k a_k \right) \int_0^T (X_{t,k})^2 dt \right\}. \end{aligned} \quad (5)$$

P r o o f. It is obvious that for all N ,

$$Z_N = \frac{d\mathbf{P}_\theta^N}{d\mathbf{P}_{\theta_0}^N}(X^N) = \frac{d\tilde{\mathbf{P}}_\theta^N}{d\tilde{\mathbf{P}}_{\theta_0}^N}(\tilde{X}^N).$$

Hence it is enough to show that $\tilde{\mathbf{P}}_\theta$ is absolutely continuous with respect to $\tilde{\mathbf{P}}_{\theta_0}$. By [7, Ch. VII, Sec. 6, Theorem 2], we need to show that

$$\mathbf{E}_{\tilde{\mathbf{P}}_{\theta_0}}(Z_\infty) = 1$$

where $\tilde{\mathbf{P}}_{\theta_0}\{Z_\infty = \lim_{N \rightarrow \infty} Z_N\} = 1$. For $M < N$, we have

$$\begin{aligned} \mathbf{E}_{\tilde{\mathbf{P}}_{\theta_0}}|Z_N - Z_M| &\leq \left(\mathbf{E}_{\tilde{\mathbf{P}}_{\theta_0}} \left| \sqrt{Z_N} + \sqrt{Z_M} \right|^2 \right)^{1/2} \\ &\times \left(\mathbf{E}_{\tilde{\mathbf{P}}_{\theta_0}} \left| \sqrt{Z_N} - \sqrt{Z_M} \right|^2 \right)^{1/2} \leq 2 \left(\int_{\tilde{\Omega}} \left| \sqrt{Z_N} - \sqrt{Z_M} \right|^2 d\tilde{\mathbf{P}}_{\theta_0} \right)^{1/2} \\ &= 2 \left(\int_{\tilde{\Omega}} \left[1 - \sqrt{Z_N Z_M} \right] d\tilde{\mathbf{P}}_{\theta_0} \right)^{1/2} \\ &= 2 \left(\int_{\tilde{\Omega}} \left[1 - \prod_{k=M+1}^N \sqrt{q_k(X_k)} \right] d\tilde{\mathbf{P}}_{\theta_0} \right)^{1/2} \\ &= 2 \left[1 - \prod_{k=M+1}^N \int_{C[0,T]} \sqrt{q_k(X_k)} d\tilde{\mathbf{P}}_{\theta_0,k} \right]^{1/2}, \end{aligned}$$

the meaning of $q(X_k)$ being self evident. (6)

If we prove that (6) converges to 0 as $M, N \rightarrow \infty$, Z_N converges in L^1 . Since Z_N converges to Z_∞ ($\tilde{\mathbf{P}}_{\theta_0}$ -a.s.) and $\mathbf{E}_{\tilde{\mathbf{P}}_{\theta_0}}(Z_N) = 1$ for all N , we have

$$\mathbf{E}_{\tilde{\mathbf{P}}_{\theta_0}}(Z_\infty) = 1.$$

To prove that (6) converges to 0 as $M, N \rightarrow \infty$, we only need to show that the product

$$\prod_{k=1}^{\infty} \mathbf{E}_{\tilde{\mathbf{P}}_{\theta_0,k}} \sqrt{q_k(X_k)}$$

converges, which is equivalent to the following condition

$$\sum_{k=1}^{\infty} \left[1 - \mathbf{E}_{\tilde{\mathbf{P}}_{\theta_0,k}} \sqrt{q_k(X_k)} \right] < \infty. \quad (7)$$

First notice that

$$\mathbf{E}_{\tilde{\mathbf{P}}_{\theta_0,k}} \sqrt{q_k(X_k)} \leq (\mathbf{E}_{\tilde{\mathbf{P}}_{\theta_0,k}} q_k(X_k))^{1/2} = 1. \quad (8)$$

Hence the series

$$\sum_{k=1}^{\infty} \left[1 - \mathbf{E}_{\tilde{\mathbf{P}}_{\theta_0, k}} \sqrt{q_k(X_k)} \right]$$

converges or diverges with the series

$$\sum_{k=1}^{\infty} \left| \log \mathbf{E}_{\tilde{\mathbf{P}}_{\theta_0, k}} \sqrt{q_k(X_k)} \right|.$$

Substituting (2) into (4), we have

$$\begin{aligned} Z_N &:= \frac{d\tilde{\mathbf{P}}_{\theta}^N}{d\tilde{\mathbf{P}}_{\theta_0}^N}(\tilde{X}^N) \\ &= \exp \left\{ \sum_{k=1}^N \frac{(\theta_0^1 - \theta^1) \nu_k + (\theta^2 - \theta_0^2) a_k}{b_k} \int_0^T X_{t,k} dW_t(\varphi_k) \right. \\ &\quad \left. - \frac{1}{2} \sum_{k=1}^N \frac{((\theta_0^1 - \theta^1) \nu_k + (\theta^2 - \theta_0^2) a_k)^2}{b_k^2} \int_0^T (X_{t,k})^2 dt \right\}. \quad (9) \end{aligned}$$

By Jensen's inequality, (8) and (9), we have

$$\begin{aligned} \sum_{k=1}^{\infty} \left| \log \mathbf{E}_{\tilde{\mathbf{P}}_{\theta_0, k}} \sqrt{q_k(X_k)} \right| &\leq \frac{1}{2} \sum_{k=1}^{\infty} \left| \mathbf{E}_{\tilde{\mathbf{P}}_{\theta_0, k}} \log q_k(X_k) \right| \\ &= \frac{1}{4} \sum_{k=1}^{\infty} \frac{((\theta_0^1 - \theta^1) \nu_k + (\theta^2 - \theta_0^2) a_k)^2}{b_k^2} \int_0^T \mathbf{E}_{\tilde{\mathbf{P}}_{\theta_0, k}} (X_{t,k})^2 dt. \quad (10) \end{aligned}$$

For the proof, from (C3), it is enough to show that

$$\sum_{k=1}^{\infty} \frac{\nu_k^2}{b_k^2} \int_0^T \mathbf{E}_{\tilde{\mathbf{P}}_{\theta_0, k}} (X_{t,k})^2 dt < \infty. \quad (11)$$

It follows from (C5) that (11) is satisfied. This completes the proof of the theorem.

Let us now set

$$l(X, \theta) := \log \frac{d\mathbf{P}_{\theta}}{d\mathbf{P}_{\theta_0}}(X),$$

and define

$$g_{ij}(\theta) := \mathbf{E} \left[\frac{\partial l(X, \theta)}{\partial \theta^i} \frac{\partial l(X, \theta)}{\partial \theta^j} \right], \quad i, j = 1, 2.$$

Then we can write the Riemannian metric $\langle \cdot, \cdot \rangle = \sum_{i,j}^2 g_{ij} d\theta^i \otimes d\theta^j$ where

$$g_{11}(\theta) = \sum_{k=1}^{\infty} \frac{\nu_k^2}{b_k^2} \int_0^T \mathbf{E} (X_{t,k})^2 dt,$$

$$g_{12}(\theta) = - \sum_{k=1}^{\infty} \frac{\nu_k a_k}{b_k^2} \int_0^T \mathbf{E}(X_{t,k})^2 dt,$$

$$g_{22}(\theta) = \sum_{k=1}^{\infty} \frac{a_k^2}{b_k^2} \int_0^T \mathbf{E}(X_{t,k})^2 dt.$$

From the assumptions (C3), (C4) and (C5), we have

$$\lim_{N \rightarrow \infty} g_{ij}(\theta, N) = g_{ij}(\theta) < \infty \quad \text{uniformly in } \theta \in \Theta \text{ and } i, j = 1, 2, \quad (12)$$

and

$$\lim_{N \rightarrow \infty} \frac{\partial g_{ij}(\theta, N)}{\partial \theta^k} = \frac{\partial g_{ij}(\theta)}{\partial \theta^k} < \infty \quad \text{uniformly in } \theta \in \Theta \text{ for all } i, j, k = 1, 2. \quad (13)$$

Christoffel symbols of the first kind: The Riemannian metric $g_{ij}(\theta, N)$ yields a natural affine connection, whose coefficients are given by

$$\Gamma_{ijk}(\theta, N) = \frac{1}{2} \left\{ \frac{\partial g_{jk}(\theta, N)}{\partial \theta^i} + \frac{\partial g_{ik}(\theta, N)}{\partial \theta^j} - \frac{\partial g_{ij}(\theta, N)}{\partial \theta^k} \right\}.$$

This connection is called the Levi-Civita connection. Christoffel symbols of the first kind Γ_{ijk} can be expressed in terms of

$$\frac{\partial g_{11}(\theta, N)}{\partial \theta^1}, \quad \frac{\partial g_{11}(\theta, N)}{\partial \theta^2}, \quad \frac{\partial g_{22}(\theta, N)}{\partial \theta^1} \quad \text{and} \quad \frac{\partial g_{22}(\theta, N)}{\partial \theta^2}.$$

Proposition 2.1.

$$\begin{aligned} \Gamma_{111}(\theta, N) &= \frac{1}{2} \frac{\partial g_{11}(\theta, N)}{\partial \theta^1}, \\ \Gamma_{112}(\theta, N) &= \Gamma_{121}(\theta, N) = \Gamma_{211}(\theta, N) = \frac{1}{2} \frac{\partial g_{12}(\theta, N)}{\partial \theta^1} = \frac{1}{2} \frac{\partial g_{11}(\theta, N)}{\partial \theta^2}, \\ \Gamma_{221}(\theta, N) &= \Gamma_{122}(\theta, N) = \Gamma_{212}(\theta, N) = \frac{1}{2} \frac{\partial g_{22}(\theta, N)}{\partial \theta^1} = \frac{1}{2} \frac{\partial g_{12}(\theta, N)}{\partial \theta^2}, \\ \Gamma_{222}(\theta, N) &= \frac{1}{2} \frac{\partial g_{22}(\theta, N)}{\partial \theta^2}. \end{aligned}$$

From Proposition 2.1 and (13), we have

$$\begin{aligned} \lim_{N \rightarrow \infty} \Gamma_{ijk}(\theta, N) &= \Gamma_{ijk}(\theta) < \infty \quad \text{uniformly in } \theta \in \Theta \\ &\text{for all } i, j, k = 1, 2, \end{aligned} \quad (14)$$

where

$$\Gamma_{ijk}(\theta) = \frac{1}{2} \left\{ \frac{\partial g_{jk}(\theta)}{\partial \theta^i} + \frac{\partial g_{ik}(\theta)}{\partial \theta^j} - \frac{\partial g_{ij}(\theta)}{\partial \theta^k} \right\}.$$

Christoffel symbols of the second kind: It is easy to compute the limit value $\Gamma_{ij}^k(\theta, \infty)$ of Christoffel symbols of the second kind $\Gamma_{ij}^k(\theta, N)$, which is given by

$$\Gamma_{ij}^k(\theta, N) = g^{k1}(\theta, N) \Gamma_{ij1}(\theta, N) + g^{k2}(\theta, N) \Gamma_{ij2}(\theta, N), \quad (15)$$

where $(g^{ij}(N)) = (g_{ij}(N))^{-1}$. It follows from (12) and (13) that

$$\lim_{N \rightarrow \infty} \Gamma_{ij}^k(\theta, N) = \Gamma_{ij}^k(\theta) < \infty \quad \text{uniformly in } \theta \in \Theta$$

for all $i, j, k = 1, 2,$ (16)

where $\Gamma_{ij}^k(\theta) = g^{k1}(\theta) \Gamma_{ij1}(\theta) + g^{k2}(\theta) \Gamma_{ij2}(\theta)$.

The Riemann (or Riemann–Christoffel) curvature tensor: Now we calculate the Riemann curvature tensor components which are given by

$$R_{jkl}^i(\theta, N) = \frac{\partial \Gamma_{lj}^i(\theta, N)}{\partial \theta^k} - \frac{\partial \Gamma_{kj}^i(\theta, N)}{\partial \theta^l} + \sum_{p=1}^2 (\Gamma_{lj}^p(\theta, N) \Gamma_{kp}^i(\theta, N) - \Gamma_{kj}^p(\theta, N) \Gamma_{lp}^i(\theta, N)) \quad (17)$$

(see [8; Vol. 2, 4D-6]). From (14), we have

$$\lim_{N \rightarrow \infty} R_{jkl}^i(\theta, N) = R_{jkl}^i(\theta) < \infty \quad \text{for all } \theta \in \Theta \text{ for all } i, j, k = 1, 2, \quad (18)$$

where

$$R_{jkl}^i(\theta) = \frac{\partial \Gamma_{lj}^i(\theta)}{\partial \theta^k} - \frac{\partial \Gamma_{kj}^i(\theta)}{\partial \theta^l} + \sum_{p=1}^2 (\Gamma_{lj}^p(\theta) \Gamma_{kp}^i(\theta) - \Gamma_{kj}^p(\theta) \Gamma_{lp}^i(\theta)).$$

Hence we have the following theorem.

Theorem 2.2. *The Riemannian metric (g_{ij}) on a manifold $S = \{\mathbf{P}_\theta, \theta \in \Theta\}$ is the same as the limit value of $g_{ij}(N)$. Therefore, geometry quantities on S corresponding to g_{ij} are the same as the limit values of geometry quantities corresponding to $g_{ij}(N)$ on a manifold $S^N := \{\mathbf{P}_\theta^N, \theta \in \Theta\}$.*

§ 3. Case II: Nonexistence of the Riemannian connection

3.1. The preliminaries. In this section, we consider the estimation problem for the following H -valued SDE:

$$\begin{aligned} dX_t &= -\theta^1 LX_t dt + \theta^2 AX_t dt + B dW_t \quad \text{for } 0 < t \leq T, \\ X_0 &= 0. \end{aligned} \quad (19)$$

Here the operator L^{-1} is a bounded self-adjoint with discrete spectrum. Let $\{\varphi_k\}$ be the eigenfunctions of L^{-1} which constitute a CONS in H and let $\{\lambda_k^{-1}\}$ be the corresponding eigenvalues. Throughout what follows, it will be assumed that

(C1) $A^* \varphi_k = a_k \varphi_k$, $B^* \varphi_k = b_k \varphi_k$, $a_k, b_k > 0$ for all k where A^* and B^* are the adjoint of the operator A and B respectively.

(C2) The sequences $\{a_k\}$ and $\{b_k\}$ satisfy

$$\sum_{k=1}^{\infty} \frac{a_k^2}{\lambda_k} < \infty, \quad \sum_{k=1}^{\infty} \frac{b_k^2}{\lambda_k^{1/2}} < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{a_k}{\lambda_k} < \infty.$$

(C3) As $N \rightarrow \infty$,

$$\frac{(\sum_{k=1}^N a_k)^2}{\sum_{k=1}^N \lambda_k} = o(1) \quad \text{and} \quad \frac{\sum_{k=1}^N a_k}{\sum_{k=1}^N \lambda_k} = o(N^{-1}).$$

(C4) The parameter set Θ is given by

$$\Theta = \{(\theta^1, \theta^2) \mid \theta^1 > 0 \quad \text{and} \quad \theta^1 > c\theta^2\} \quad \text{where } c = \sup_k \frac{a_k}{\lambda_k} > 0.$$

As in the case of Section 2, it follows by the assumptions (C1) and (C2) that there exists a unique continuous solution X to the SDE (19). The solution X_t is given by

$$X_t = \sum_{k=1}^{\infty} X_{t,k} \varphi_k \quad (20)$$

where for all k ,

$$dX_{t,k} = -(\theta^1 \lambda_k - \theta^2 a_k) X_{t,k} dt + b_k dW_t(\varphi_k). \quad (21)$$

The solution of (20) is

$$X_{t,k} = b_k \int_0^t e^{-(\theta^1 \lambda_k - \theta^2 a_k)(t-s)} dW_s(\varphi_k).$$

We use the same notation as in Section 2.

Let us now show that for sufficiently large N , $\{\mathbf{P}_\theta^N, \theta \in \Theta\}$ is identifiable, and also that $\{\mathbf{P}_\theta, \theta \in \Theta\}$ is identifiable.

Lemma 3.1. $\{\mathbf{P}_\theta^N, \theta \in \Theta\}$, for sufficiently large N , and $\{\mathbf{P}_\theta, \theta \in \Theta\}$ are identifiable.

P r o o f. Since $\sum_{k=1}^{\infty} a_k/\lambda_k < \infty$, there exists an N_0 such that

$$\frac{a_{N_0}}{\lambda_{N_0}} < \frac{a_{N_0-1}}{\lambda_{N_0-1}}. \quad (22)$$

Suppose that for $N \geq N_0$, $\mathbf{P}_\theta^N = \mathbf{P}_{\theta_*}^N$. Then

$$\begin{aligned} \mathbf{E}_{\mathbf{P}_\theta^N} \|\Pi^i \pi_t X^N\|^2 &= \mathbf{E}_{\mathbf{P}_{\theta_*}^N} \|\Pi^i \pi_t X^N\|^2 \\ \text{for } i &= 1, \dots, N \text{ and } 0 < t \leq T \end{aligned} \quad (23)$$

where Π^i is the orthogonal projection of H onto the subspace spanned by $\{\varphi_k, k = 1, \dots, i\}$, and π_t is the evaluation map at t . From (23),

$$b_k^2 \frac{1 - \exp\{-2(\theta^1 \lambda_k - \theta^2 a_k)t\}}{2(\theta^1 \lambda_k - \theta^2 a_k)} = b_k^2 \frac{1 - \exp\{-2(\theta_*^1 \lambda_k - \theta_*^2 a_k)t\}}{2(\theta_*^1 \lambda_k - \theta_*^2 a_k)} \quad (24)$$

for $k = 1, \dots, N$ and $0 < t \leq T$. Let us set $\alpha_k(\theta) = (\theta^1 \lambda_k - \theta^2 a_k)$ and $\alpha_k(\theta_*) = (\theta_*^1 \lambda_k - \theta_*^2 a_k)$. Then

$$\frac{1 - \exp\{-2\alpha_k(\theta_*)t\}}{2\alpha_k(\theta_*)} - \frac{1 - \exp\{-2\alpha_k(\theta)t\}}{2\alpha_k(\theta)} = 0. \quad (25)$$

Hence (25) can be written as

$$\int_0^t (\exp\{-2\alpha_k(\theta_*)s\} - \exp\{-2\alpha_k(\theta)s\}) ds = 0 \quad \text{for } 0 \leq t \leq T,$$

which implies that $\alpha_k(\theta) = \alpha_k(\theta_*)$ for $k = 1, \dots, N$, i.e.,

$$(\theta^1 - \theta_*^1) \lambda_k = (\theta^2 - \theta_*^2) a_k \quad \text{for } k = 1, \dots, N.$$

From this if $\theta^1 \neq \theta_*^1$ and $\theta^2 \neq \theta_*^2$,

$$\frac{a_{N_0}}{\lambda_{N_0}} = \frac{a_{N_0} - 1}{\lambda_{N_0} - 1} = \frac{\theta^1 - \theta_*^1}{\theta^2 - \theta_*^2}$$

which is a contradiction in view of (22). Hence $\theta^1 = \theta_*^1$ or $\theta^2 = \theta_*^2$. Since $a_k > 0$ and $\lambda_k > 0$, $\theta^1 = \theta_*^1$ implies $\theta^2 = \theta_*^2$, and vice-versa. Thus we have $\theta^1 = \theta_*^1$ and $\theta^2 = \theta_*^2$. Therefore, $\{\mathbf{P}_\theta^N, \theta \in \Theta\}$ is identifiable for $N \geq N_0$. Similarly, we can prove that $\{\mathbf{P}_\theta, \theta \in \Theta\}$ is identifiable.

R e m a r k 3.1. Suppose that $\{a_k/\lambda_k\}$ is a strictly decreasing sequence. According to the above lemma, $\{\mathbf{P}_\theta^N, \theta \in \Theta\}$ is identifiable for $N \geq 2$. But for $N = 1$, we can choose $\theta_* (\neq \theta) \in \Theta$ such that $\mathbf{P}_\theta^1 = \mathbf{P}_{\theta_*}^1$, i.e., θ_* is on a line through θ with slope $\lambda_1/a_1 = 1/c$.

As in the case of Section 2, for each N , the measure $\mathbf{P}_\theta^N$ is absolutely continuous with respect to $\mathbf{P}_{\theta_0}^N$, and

$$\begin{aligned} Z_N &:= \frac{d\mathbf{P}_\theta^N}{d\mathbf{P}_{\theta_0}^N}(X^N) = \frac{d\tilde{\mathbf{P}}_\theta^N}{d\tilde{\mathbf{P}}_{\theta_0}^N}(\tilde{X}^N) \\ &= \exp \left\{ \sum_{k=1}^N \frac{(\theta_0^1 - \theta^1) \lambda_k + (\theta^2 - \theta_0^2) a_k}{b_k^2} \int_0^T X_{t,k} dX_{t,k} \right. \\ &\quad \left. - \frac{1}{2} \sum_{k=1}^N \left[((\theta^1)^2 - (\theta_0^1)^2) \lambda_k^2 + 2(\theta_0^1 \theta_0^2 - \theta^1 \theta^2) \lambda_k a_k \right. \right. \\ &\quad \left. \left. + ((\theta^2)^2 - (\theta_0^2)^2) a_k^2 \right] b_k^{-2} \int_0^T (X_{t,k})^2 dt \right\}. \end{aligned}$$

But the measures $\mathbf{P}_\theta$ and $\mathbf{P}_{\theta_0}$ corresponding to the solutions of Equation (19) are not absolutely continuous.

Lemma 3.2. $\mathbf{P}_\theta \perp \mathbf{P}_{\theta_0}$, i.e., $\mathbf{P}_\theta$ and $\mathbf{P}_{\theta_0}$ are singular.

P r o o f. Since $\mathbf{P}_\theta$ is locally absolutely continuous with respect to $\mathbf{P}_{\theta_0}$, we have $\mathbf{E}_{\mathbf{P}_{\theta_0}} Z_N = 1$, and hence $\lim_{N \rightarrow \infty} Z_N$ exists $\mathbf{P}_{\theta_0}$ -a.s. Let $Z_\infty := \lim_{N \rightarrow \infty} Z_N$. To prove $\mathbf{P}_\theta \perp \mathbf{P}_{\theta_0}$, we only need to show that $\mathbf{E}_{\mathbf{P}_{\theta_0}} Z_\infty = 0$. It is clear from (20) that

$$\begin{aligned} Z_N &= \frac{d\mathbf{P}_\theta^N}{d\mathbf{P}_{\theta_0}^N}(X^N) = \frac{d\tilde{\mathbf{P}}_\theta^N}{d\tilde{\mathbf{P}}_{\theta_0}^N}(\tilde{X}^N) \\ &= \exp \left\{ \sum_{k=1}^N \frac{(\theta_0^1 - \theta^1) \lambda_k + (\theta^2 - \theta_0^2) a_k}{b_k} \int_0^T X_{t,k} dW_t(\varphi_k) \right. \\ &\quad \left. - \frac{1}{2} \sum_{k=1}^N \left(\frac{(\theta_0^1 - \theta^1) \lambda_k + (\theta^2 - \theta_0^2) a_k}{b_k} \right)^2 \int_0^T (X_{t,k})^2 dt \right\}. \end{aligned}$$

Now consider

$$\sum_{k=1}^N \left(\frac{(\theta_0^1 - \theta^1) \lambda_k + (\theta^2 - \theta_0^2) a_k}{b_k} \right)^2 (X_{t,k})^2 := \sum_{k=1}^N \beta_k^{(3)}(t), \quad \text{say,} \quad (26)$$

where

$$\begin{aligned} \beta_k^{(3)}(t) &= \left(\frac{(\theta_0^1 - \theta^1) \lambda_k + (\theta^2 - \theta_0^2) a_k}{b_k} \right)^2 \exp \{ -2(\theta_0^1 \lambda_k - \theta_0^2 a_k) t \} b_k^2 \\ &\quad \times \left(\int_0^t \exp \{ (\theta_0^1 \lambda_k - \theta_0^2 a_k) s \} dW_s(\varphi_k) \right)^2. \end{aligned}$$

Since $\sum_{k=1}^{\infty} \lambda_k = \infty$, we have

$$\begin{aligned} \sum_{k=1}^N \mathbf{E}_{\mathbf{P}_{\varphi_0}} \beta_k^{(3)}(t) &= \sum_{k=1}^N \left(\frac{(\theta_0^1 - \theta^1) \lambda_k + (\theta^2 - \theta_0^2) a_k}{b_k} \right)^2 \\ &\quad \times \exp \{ -2(\theta_0^1 \lambda_k - \theta_0^2 a_k) t \} b_k^2 \mathbf{E}_{\mathbf{P}_{\theta_0}} \\ &\quad \times \left(\int_0^t \exp \{ (\theta_0^1 \lambda_k - \theta_0^2 a_k) s \} dW_s(\varphi_k) \right)^2 \\ &= \sum_{k=1}^N \frac{((\theta_0^1 - \theta^1) \lambda_k + (\theta^2 - \theta_0^2) a_k)^2}{2(\theta_0^1 \lambda_k - \theta_0^2 a_k)} \\ &\quad \times \left(1 - \exp \{ -2(\theta_0^1 \lambda_k - \theta_0^2 a_k) t \} \right) \longrightarrow \infty \quad (27) \end{aligned}$$

as $N \rightarrow \infty$ for $0 < t \leq T$. Hence $\sum_{k=1}^{\infty} \beta_k^{(3)}(t) = \infty$ ($\tilde{\mathbf{P}}_{\theta_0}$ -a.s.) for $0 < t \leq T$ because $\{(\beta_k^{(3)}(t))^{1/2}\}_k$ is a Gaussian sequence (see [7, p. 501, Lemma]). Since $\sum_{k=1}^{\infty} \beta_k^{(3)}(t) = \infty$ ($\mathbf{P}_{\theta_0}$ -a.s.) for $0 < t \leq T$,

$$\sum_{k=1}^N \left(\frac{(\theta_0^1 - \theta^1) \lambda_k + (\theta^2 - \theta_0^2) a_k}{b_k} \right)^2 (X_{t,k})^2 \longrightarrow \infty \quad (\tilde{\mathbf{P}}_{\theta_0}\text{-a.s.}) \text{ as } N \rightarrow \infty.$$

Therefore, by the monotone convergence theorem,

$$\sum_{k=1}^N \left(\frac{(\theta_0^1 - \theta^1) \lambda_k + (\theta^2 - \theta_0^2) a_k}{b_k} \right)^2 \int_0^T (X_{t,k})^2 dt \longrightarrow \infty \quad (28)$$

($\tilde{\mathbf{P}}_{\theta_0}$ -a.s.) as $N \rightarrow \infty$. Let us set

$$M_T^N := \sum_{k=1}^N \frac{(\theta_0^1 - \theta^1) \lambda_k + (\theta^2 - \theta_0^2) a_k}{b_k} \int_0^T X_{t,k} dW_t(\varphi_k).$$

Then it follows that, with respect to $\tilde{\mathbf{P}}_{\theta_0}$, the stochastic sequence $M_T = (M_T^N, \tilde{\mathcal{F}}_N)$ is a square integrable martingale. Using the notation M_T^N , we write Z_N as

$$Z_N := \frac{d\mathbf{P}_{\theta}^N}{d\mathbf{P}_{\theta_0}^N}(X^N) = \exp \left\{ M_T^N - \frac{1}{2} \langle M_T \rangle_N \right\} \quad (29)$$

where

$$\langle M_T \rangle_N = \sum_{n=1}^N \mathbf{E}[(M_T^n - M_T^{n-1})^2 \mid \tilde{\mathcal{F}}_{n-1}],$$

i.e., the quadratic variation of the square integrable martingale with $M_T^0 = 0$ and $\tilde{\mathcal{F}}_0 = \{\emptyset, \otimes C[0, T]\}$. By (28),

$$\lim_{N \rightarrow \infty} \langle M_T \rangle_N = \infty \quad (\tilde{\mathbf{P}}_{\theta_0}\text{-a.s.}) \quad (30)$$

and, hence according to a version of the strong law of large numbers for square integrable martingales (see [7, Theorem 4, p. 487]), we have

$$\lim_{N \rightarrow \infty} \frac{M_T^N}{\langle M_T \rangle_N} = 0 \quad (\mathbf{P}_{\theta_0}\text{-a.s.}) \quad (31)$$

From (30) and (31), we obtain

$$\begin{aligned} \mathbf{E}_{\mathbf{P}_{\theta_0}} Z_\infty &= \mathbf{E}_{\tilde{\mathbf{P}}_{\theta_0}} Z_\infty = \mathbf{E}_{\tilde{\mathbf{P}}_{\theta_0}} \left(\lim_{N \rightarrow \infty} Z_N \right) \\ &= \mathbf{E}_{\tilde{\mathbf{P}}_{\theta_0}} \lim_{N \rightarrow \infty} \exp \left\{ M_T^N - \frac{1}{2} \langle M_T \rangle_N \right\} = 0. \end{aligned}$$

This completes the proof of the lemma.

3.2. Differential geometric quantities. Let

$$S^N = \{\mathbf{P}_\theta^N, \theta \in \Theta\} \quad \text{and} \quad S = \{\mathbf{P}_\theta, \theta \in \Theta\}.$$

A family of probability measures ($= \mathcal{P}$, say) on $C([0, T], H)$ will be endowed with the topology of weak convergence. We give C the uniform topology by defining the distance between two points x and y as

$$\rho(x, y) = \sup_{t \in [0, T]} \|x(t) - y(t)\|.$$

We will define a topology for the sets S^N and S , called the relative topology on S^N and S , respectively. Since $C([0, T], H)$ is separable, S^N and S are metrizable and separable (see e.g. [2, p. 239]). Let us now define a mapping

$$\psi_N: S^N \longrightarrow \Theta \subseteq \mathbf{R}^2 \quad \text{by} \quad \psi_N(\mathbf{P}_\theta^N) = \theta$$

for each N , and

$$\psi: S \longrightarrow \Theta \subseteq \mathbf{R}^2 \quad \text{by} \quad \psi(\mathbf{P}_\theta) = \theta.$$

Let us now show that these functions ψ_N and ψ play the role of coordinate functions in S^N and S , respectively. If this can be proved, differential structures can be introduced in S^N and S by these coordinate functions. In the following lemma, the notation $\Rightarrow$ denotes the weak convergence of probability measures, and we write

$$X_n \xrightarrow{\mathcal{D}} X$$

if the distributions $\mathbf{P}_n$ of X_n converge weakly to the distribution $\mathbf{P}$ of X .

Lemma 3.3. *For sufficiently large N_0 , S^N is a sequence of manifolds for $N \geq N_0$, and further S is a manifold.*

P r o o f. It is obvious that S^N and S are Hausdorff spaces. First we consider the case of S . For the proof, we need to show that $\theta_n \rightarrow \theta$ as $n \rightarrow \infty$ implies $\mathbf{P}_{\theta_n} \Rightarrow \mathbf{P}_\theta$ as $n \rightarrow \infty$, and vice-versa. Suppose that $\theta_n \rightarrow \theta$ as $n \rightarrow \infty$. It is easy to see that for each N , $X^{N,n} \xrightarrow{\mathcal{D}} X^N$ as $n \rightarrow \infty$. Here

$$X_t^{N,n} = \sum_{k=1}^N X_{t,k}^n \varphi_k$$

where for each n , $(X_{t,k}^n)$ is the solution to the SDE (20) when $\theta^1 = \theta_n^1$ and $\theta^2 = \theta_n^2$.

From Chebyshev's inequality and (21), for each $\varepsilon > 0$,

$$\begin{aligned} \mathbf{P}\{\rho(X^N, X) \geq \varepsilon\} &= \mathbf{P}\left\{\sup_{t \in [0, T]} \sum_{k=N}^{\infty} (X_{t,k})^2 \geq \varepsilon^2\right\} \\ &\leq \frac{1}{\varepsilon^2} \sum_{k=N+1}^{\infty} \mathbf{E}\left[\sup_{t \in [0, T]} (X_{t,k})^2\right] \\ &\leq \frac{2}{\varepsilon^2} \sum_{k=N+1}^{\infty} b_k^2 \mathbf{E}\left[\sup_{t \in [0, T]} \exp\{-2(\theta^1 \lambda_k - \theta^2 a_k)t\} \right. \\ &\quad \left. \times \left(\int_0^t \exp\{(\theta^1 \lambda_k - \theta^2 a_k)s\} dW_s(\varphi_k)\right)^2\right]. \quad (32) \end{aligned}$$

Now let

$$B_t := \exp\{-2(\theta^1 \lambda_k - \theta^2 a_k)t\}$$

and

$$M_t := \int_0^t \exp\{(\theta^1 \lambda_k - \theta^2 a_k)s\} dW_s(\varphi_k).$$

Applying Itô's formula to $F(x, y) = xy^2$, we obtain

$$\begin{aligned} B_t(M_t)^2 &= 2 \int_0^t \exp\{-2(\theta^1 \lambda_k - \theta^2 a_k)s\} \\ &\quad \times \int_0^s \exp\{(\theta^1 \lambda_k - \theta^2 a_k)u\} dW_u(\varphi_k) dW_s(\varphi_k) \\ &\quad + \int_0^t \left(\int_0^s \exp\{(\theta^1 \lambda_k - \theta^2 a_k)u\} dW_u(\varphi_k)\right)^2 \\ &\quad \times \exp\{-2(\theta^1 \lambda_k - \theta^2 a_k)s\} ds. \end{aligned}$$

Hence, it follows by Doob's martingale inequality that

$$\begin{aligned}
 \mathbf{E} \left[\sup_{t \in [0, T]} B_t(M_t)^2 \right] &\leq 2 \left\{ \mathbf{E} \left[\sup_{t \in [0, T]} \left(\int_0^t \exp \{ -2(\theta^1 \lambda_k - \theta^2 a_k) s \} \right. \right. \right. \\
 &\quad \times \left. \left. \left. \int_0^s \exp \{ (\theta^1 \lambda_k - \theta^2 a_k) u \} dW_u(\varphi_k) dW_s(\varphi_k) \right)^2 \right] \right\}^{1/2} \\
 &\quad + \int_0^T \int_0^t \exp \{ -2(\theta^1 \lambda_k - \theta^2 a_k) (t-s) \} ds dt \\
 &\leq 4 \left\{ \int_0^T \int_0^t \exp \{ -2(\theta^1 \lambda_k - \theta^2 a_k) (t-s) \} ds dt \right\}^{1/2} \\
 &\quad + \int_0^T \int_0^t \exp \{ -2(\theta^1 \lambda_k - \theta^2 a_k) (t-s) \} ds dt \\
 &\leq \frac{2\sqrt{2T}}{\sqrt{\lambda_k(\theta^1 - \theta^2 a_k/\lambda_k)}} + \frac{T}{2\lambda_k(\theta^1 - \theta^2 a_k/\lambda_k)}. \tag{33}
 \end{aligned}$$

From this, the second term of the right-hand side in (32) is dominated by

$$\frac{4\sqrt{2T}}{\varepsilon^2} \sum_{k=N}^{\infty} \frac{b_k^2}{\lambda_k^{1/2}} \frac{1}{\sqrt{\theta^1 - \theta^2 a_k/\lambda_k}} + \frac{T+1}{\varepsilon^2} \sum_{k=N}^{\infty} \frac{b_k^2}{\lambda_k} \frac{1}{(\theta^1 - \theta^2 a_k/\lambda_k)}. \tag{34}$$

By assumption (C2), we obtain

$$\lim_{N \rightarrow \infty} \mathbf{P}\{\rho(X^N, X) \geq \varepsilon\} = 0,$$

which implies $X^N \xrightarrow{\mathcal{D}} X$ as $N \rightarrow \infty$. Let $X_t^n = \sum_{k=1}^{\infty} X_{t,k}^n \varphi_k$. Substituting θ_n for θ in (34), we can show that for each $\varepsilon > 0$,

$$\begin{aligned}
 \mathbf{P}\{\rho(X^{N,n}, X) \geq \varepsilon\} &\leq \frac{4\sqrt{2T}}{\varepsilon^2} \sum_{k=N}^{\infty} \frac{b_k^2}{\lambda_k^{1/2}} \frac{1}{\sqrt{\theta_n^1 - \theta_n^2 a_k/\lambda_k}} \\
 &\quad + \frac{T+1}{\varepsilon^2} \sum_{k=N}^{\infty} \frac{b_k^2}{\lambda_k} \frac{1}{(\theta_n^1 - \theta_n^2 a_k/\lambda_k)} \\
 &\leq \frac{4\sqrt{2T}}{\varepsilon^2} \frac{1}{\sqrt{\theta_n^1 - \theta_n^2 \mathbf{1}_{\{\theta_n^2 \geq 0\}}^c}} \sum_{k=N}^{\infty} \frac{b_k^2}{\lambda_k^{1/2}} \\
 &\quad + \frac{T+1}{\varepsilon^2} \frac{1}{\theta_n^1 - \theta_n^2 \mathbf{1}_{\{\theta_n^2 \geq 0\}}^c} \sum_{k=N}^{\infty} \frac{b_k^2}{\lambda_k},
 \end{aligned}$$

where $c = \sup a_k / \lambda_k$. It follows from the condition (C2) that we have

$$\lim_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P}\{\rho(X^{N,n}, X) \geq \varepsilon\} = 0$$

for each $\varepsilon > 0$. Hence we have $X^n \xrightarrow{\mathcal{D}} X$ as $n \rightarrow \infty$, i.e., $\mathbf{P}_{\theta_n} \Rightarrow \mathbf{P}_\theta$ as $\theta_n \rightarrow \theta$.

Now suppose that $\mathbf{P}_{\theta_n} \Rightarrow \mathbf{P}_\theta$. Then for every N , $\mathbf{P}_{\theta_n}^N \Rightarrow \mathbf{P}_\theta^N$. Hence for fixed $t \in [0, T]$, we have

$$X_{t,k}^n \xrightarrow{\mathcal{D}} X_{t,k} \quad \text{as } n \rightarrow \infty,$$

for $k = 1, 2, \dots$. For every $\varepsilon > 0$, we have

$$\sup_n \mathbf{E}(X_{t,k}^n)^{2+\varepsilon} < \infty \quad \text{for every } k \text{ and } t \in [0, T].$$

Hence for every k and $t \in [0, T]$, $(|X_{t,k}^n|^2)$ is uniformly integrable, which in turn implies that

$$\mathbf{E}|X_{t,k}^n|^2 \rightarrow \mathbf{E}|X_{t,k}|^2 \quad \text{as } n \rightarrow \infty.$$

Thus for every k and $t \in [0, T]$,

$$\frac{1 - \exp\{-2(\theta_n^1 \lambda_k - \theta_n^2 a_k) t\}}{2(\theta_n^1 \lambda_k - \theta_n^2 a_k)} \rightarrow \frac{1 - \exp\{-2(\theta^1 \lambda_k - \theta^2 a_k) t\}}{2(\theta^1 \lambda_k - \theta^2 a_k)} \quad (35)$$

as $n \rightarrow \infty$. Let N_0 be given as in (22). From (35), we have

$$\theta_n^1 \lambda_{N_0-1} - \theta_n^2 a_{N_0-1} \rightarrow \theta^1 \lambda_{N_0-1} - \theta^2 a_{N_0-1}, \quad (36)$$

$$\theta_n^1 \lambda_{N_0} - \theta_n^2 a_{N_0} \rightarrow \theta^1 \lambda_{N_0} - \theta^2 a_{N_0}, \quad (37)$$

as $n \rightarrow \infty$. From (36) and (37), we obtain

$$\theta_n^1 (\lambda_{N_0-1} a_{N_0} - \lambda_{N_0} a_{N_0-1}) \rightarrow \theta^1 (\lambda_{N_0-1} a_{N_0} - \lambda_{N_0} a_{N_0-1}),$$

as $n \rightarrow \infty$. It follows from (22) that

$$\lambda_{N_0-1} a_{N_0} - \lambda_{N_0} a_{N_0-1} \neq 0.$$

Therefore, we have $\lim_{n \rightarrow \infty} \theta_n^1 = \theta^1$. Similarly, $\lim_{n \rightarrow \infty} \theta_n^2 = \theta^2$. As in the case of S , we can show that S^N is homeomorphic to $\Theta \subseteq \mathbf{R}^2$ under a homeomorphism ψ_N .

Remark 3.2. Since S can be covered with one chart (S, φ) , $\mathcal{A} = \{(S, \psi)\}$ is an C^∞ -related atlas of S , i.e., this atlas $\mathcal{A}$ consists of one chart. There is then a uniquely determined maximal atlas $\mathcal{A}'$ which includes the given atlas $\mathcal{A} = \{(S, \psi)\}$ (see [8, Vol. 1, 2-4, Lemma 1]). Hence we can define a differentiable manifold S to be a pair $(S, \mathcal{A}')$. Since S is metrizable,

it is paracompact which in turn implies that a Riemannian metric exists on S (see e.g. [5, Ch. 1, Sec. 3, 3.2]).

Let us set

$$l_N(X, \theta) := \log \frac{d\mathbf{P}_\theta^N}{d\mathbf{P}_{\theta_0}^N}(X^N),$$

and consider two partial derivatives $\partial l_N(X, \theta)/\partial \theta_i$, $i = 1, 2$. If we use a coordinate system $\theta = \psi_N(\mathbf{P}_\theta^N)$ in S^N , then the tangent space $T_\theta S^N$ is spanned by

$$\left. \frac{\partial}{\partial \theta^1} \right|_\theta \quad \text{and} \quad \left. \frac{\partial}{\partial \theta^2} \right|_\theta$$

where $(\partial/\partial \theta^i)|_\theta$ denotes an operator taking the function $f: S^N \rightarrow \mathbf{R}$ to $(\partial f/\partial \theta^i)(\theta)$. In particular, the inner product of the two basis vectors is

$$g_{ij}(\theta, N) = \left\langle \left. \frac{\partial}{\partial \theta^1} \right|_\theta, \left. \frac{\partial}{\partial \theta^2} \right|_\theta \right\rangle = \mathbf{E} \left[\frac{\partial l_N(X, \theta)}{\partial \theta^i} \frac{\partial l_N(X, \theta)}{\partial \theta^j} \right], \quad i, j = 1, 2.$$

It is easy to see that $(g_{ij}(\theta, N))$ is given by

$$\begin{aligned} g_{11}(\theta, N) &= \sum_{k=1}^N \frac{\lambda_k^2}{b_k^2} \int_0^T \mathbf{E}(X_{t,k})^2 dt, \\ g_{12}(\theta, N) &= - \sum_{k=1}^N \frac{\lambda_k a_k}{b_k^2} \int_0^T \mathbf{E}(X_{t,k})^2 dt, \\ g_{22}(\theta, N) &= \sum_{k=1}^N \frac{a_k^2}{b_k^2} \int_0^T \mathbf{E}(X_{t,k})^2 dt. \end{aligned}$$

As in the case of Section 2, the Christoffel symbols of the first kind $\Gamma_{ijk}(\theta, N)$ can be expressed in terms of

$$\frac{\partial g_{11}(\theta, N)}{\partial \theta^1}, \quad \frac{\partial g_{11}(\theta, N)}{\partial \theta^2}, \quad \frac{\partial g_{22}(\theta, N)}{\partial \theta^1}, \quad \text{and} \quad \frac{\partial g_{22}(\theta, N)}{\partial \theta^2}.$$

Proposition 3.1.

$$\begin{aligned} \Gamma_{111}(\theta, N) &= \frac{1}{2} \frac{\partial g_{11}(\theta, N)}{\partial \theta^1}, \\ \Gamma_{112}(\theta, N) &= \Gamma_{121}(\theta, N) = \Gamma_{211}(\theta, N) = \frac{1}{2} \frac{\partial g_{12}(\theta, N)}{\partial \theta^1} = \frac{1}{2} \frac{\partial g_{11}(\theta, N)}{\partial \theta^2}, \\ \Gamma_{221}(\theta, N) &= \Gamma_{122}(\theta, N) = \Gamma_{212}(\theta, N) = \frac{1}{2} \frac{\partial g_{22}(\theta, N)}{\partial \theta^1} = \frac{1}{2} \frac{\partial g_{12}(\theta, N)}{\partial \theta^2}, \\ \Gamma_{222}(\theta, N) &= \frac{1}{2} \frac{\partial g_{22}(\theta, N)}{\partial \theta^2}. \end{aligned}$$

We compute the limit value $\Gamma_{ij}^k(\theta, \infty)$ of Christoffel symbols of the second kind $\Gamma_{ij}^k(\theta, N)$, using the assumptions (C2) and (C3).

Proposition 3.2.

$$\begin{aligned}\Gamma_{12}^1(\theta) &= \Gamma_{21}^1(\theta) = \Gamma_{22}^1(\theta) = 0, & \Gamma_{11}^1(\theta) &= -\frac{1}{2\theta^1}, \\ \Gamma_{11}^2(\theta) &= \frac{1}{2g_{22}(\theta)} \left\{ \frac{T}{2} \left(\frac{\theta^2}{\theta^1} \right) \sum_{k=1}^{\infty} \frac{\lambda_k a_k^2}{(\theta^1 \lambda_k - \theta^2 a_k)^2} - \frac{1}{4} \sum_{k=1}^{\infty} \frac{\lambda_k^2 a_k}{(\theta^1 \lambda_k - \theta^2 a_k)^3} \right. \\ &\quad - \frac{1}{4} \left(\frac{\theta^2}{\theta^1} \right) \sum_{k=1}^{\infty} \frac{\lambda_k a_k^2}{(\theta^1 \lambda_k - \theta^2 a_k)^3} \\ &\quad + \frac{T}{2} \sum_{k=1}^{\infty} \frac{\lambda_k^2 a_k \exp\{-2(\theta^1 \lambda_k - \theta^2 a_k) T\}}{(\theta^1 \lambda_k - \theta^2 a_k)^2} \\ &\quad + \frac{1}{4} \sum_{k=1}^{\infty} \frac{\lambda_k^2 a_k \exp\{-2(\theta^1 \lambda_k - \theta^2 a_k) T\}}{(\theta^1 \lambda_k - \theta^2 a_k)^3} \\ &\quad \left. + \frac{1}{4} \left(\frac{\theta^2}{\theta^1} \right) \sum_{k=1}^{\infty} \frac{\lambda_k a_k^2 \exp\{-2(\theta^1 \lambda_k - \theta^2 a_k) T\}}{(\theta^1 \lambda_k - \theta^2 a_k)^3} \right\}, \\ \Gamma_{22}^2(\theta) &= \Gamma_{21}^2(\theta) = \frac{1}{2g_{22}(\theta)} \frac{\partial g_{22}(\theta)}{\partial \theta^2}, & \Gamma_{22}^2(\theta) &= \frac{1}{2g_{22}(\theta)} \frac{\partial g_{22}(\theta)}{\partial \theta^2}.\end{aligned}$$

Remark 3.3. Since for all $\theta \in \Theta$, $g_{11}(\theta, N) \rightarrow \infty$ and $g_{12}(\theta, N) \rightarrow \infty$, we can't define the Riemannian metric $\langle \cdot, \cdot \rangle$ with $(g_{ij}(\infty))$ on a manifold S where $(g_{ij}(\infty))$ is the limit value of $(g_{ij}(\theta, N))$. From Proposition 3.2, it is possible that we can obtain a general connection on a manifold S .

It is convenient to use different indices to denote the components with respect to different coordinate systems. Thus we use $i, j, k, \dots$ for representing quantities corresponding to $\theta = \psi_N(\mathbf{P}_\theta^N)$ or $\theta = \psi(\mathbf{P}_\theta)$ and $\alpha, \beta, \gamma, \dots$ for representing quantities corresponding to $\theta' = \psi'_N(\mathbf{P}_\theta^N)$ or $\theta' = \psi'(\mathbf{P}_\theta^N)$. Let θ and θ' be two coordinate systems, with corresponding g_{ij} and g'_{ij} for the expression of a Riemannian metric. Then for the corresponding Christoffel symbols of the first kind Γ_{ij}^k and $\Gamma'_{\alpha\beta}$, that for fixed N ,

$$\begin{aligned}\Gamma'_{\alpha\beta}(\theta, N) &= \sum_{i,j,k=1}^2 \Gamma_{ij}^k(\theta, N) \frac{\partial \psi_N^i(\mathbf{P}_\theta^N)}{\partial \psi_N'^\alpha(\mathbf{P}_\theta^N)} \frac{\partial \psi_N^j(\mathbf{P}_\theta^N)}{\partial \psi_N'^\beta(\mathbf{P}_\theta^N)} \frac{\partial \psi_N'^\gamma(\mathbf{P}_\theta^N)}{\partial \psi_N^k(\mathbf{P}_\theta^N)} \\ &\quad + \sum_{l=1}^2 \frac{\partial^2 \psi_N^l}{\partial \psi_N'^\alpha \partial \psi_N'^\beta}(\mathbf{P}_\theta^N) \frac{\partial \psi_N'^\gamma}{\partial \psi_N^l}(\mathbf{P}_\theta^N)\end{aligned}\quad (38)$$

(see [8; Vol. 1, 9-76]).

We have a symmetric connection in $S = \{\mathbf{P}_\theta, \theta \in \Theta\}$. For the time being, we omit θ in $\Gamma_{ij}^k(\theta, N)$.

Proposition 3.3. $\Gamma_{ij}^k(\infty)$ is a symmetric connection on a manifold S .

P r o o f. Since $\Gamma_{ij}^k(N)$ are the Christoffel symbols, $\Gamma_{ij}^k(N) = \Gamma_{ji}^k(N)$ for all N . Hence we have $\Gamma_{ij}^k(\infty) = \Gamma_{ji}^k(\infty)$. We denote the partial derivative of a function $f: \mathbf{R}^2 \rightarrow \mathbf{R}$ with respect to the α^{th} coordinate by $D_\alpha f$. Now by definition, for example for $\alpha = 1$,

$$\begin{aligned} \frac{\partial \psi_N^i}{\partial \psi_N'^\alpha}(\mathbf{P}_\theta^N) &= D_\alpha(\psi_N^i \circ (\psi_N')^{-1})(\psi_N'(\mathbf{P}_\theta^N)) \\ &= \lim_{h \rightarrow \infty} \frac{\psi_N^i \circ (\psi_N')^{-1}(\theta'^1 + h, \theta'^2) - \psi_N^i \circ (\psi_N')^{-1}(\theta'^1, \theta'^2)}{h}. \end{aligned}$$

There exists a unique $\mathbf{P}_{\theta_*}^N$ such that $\psi_N^i \circ (\psi_N')^{-1}(\theta'^1 + h, \theta'^2) = \psi_N^i(\mathbf{P}_{\theta_*}^N)$. It follows from the definition of ψ that

$$\psi_N^i(\mathbf{P}_{\theta_*}^N) = \psi^i(\mathbf{P}_{\theta_*}) = \psi^i \circ (\psi')^{-1}(\theta'^1 + h, \theta'^2).$$

Hence, for all N , $\psi_N^i \circ (\psi_N')^{-1}(\theta'^1 + h, \theta'^2) = \psi^i \circ (\psi')^{-1}(\theta'^1 + h, \theta'^2)$. Similarly, we have

$$\psi_N^i \circ (\psi_N')^{-1}(\theta'^1, \theta'^2) = \psi^i \circ (\psi')^{-1}(\theta'^1, \theta'^2).$$

Therefore, for all N ,

$$\frac{\partial \psi_N^i}{\partial \psi_N'^\alpha}(\mathbf{P}_\theta^N) = \frac{\partial \psi^i}{\partial \psi'^\alpha}(\mathbf{P}_\theta) \quad \text{and} \quad \frac{\partial \psi_N'^\alpha}{\partial \psi_N^i}(\mathbf{P}_\theta^N) = \frac{\partial \psi'^\alpha}{\partial \psi^i}(\mathbf{P}_\theta), \quad \text{for } i, \alpha = 1, 2.$$

Also we easily see that for all N ,

$$\frac{\partial^2 \psi_N^l}{\partial \psi_N'^\alpha \partial \psi_N'^\beta}(\mathbf{P}_\theta^N) = \frac{\partial^2 \psi^l}{\partial \psi'^\alpha \partial \psi'^\beta}(\mathbf{P}_\theta), \quad \text{for } \alpha, \beta, l = 1, 2.$$

Since the eight quantities $\Gamma_{ij}^k(N)$ converge to finite values $\Gamma_{ij}^k(\infty)$, respectively, it follows from (38) that Γ_{ij}^k satisfy the following transformation law:

$$\Gamma_{\alpha\beta}^{\gamma}(\infty) = \sum_{i,j,k=1}^2 \Gamma_{ij}^k(\infty) \frac{\partial \theta^i}{\partial \theta'^\alpha} \frac{\partial \theta^j}{\partial \theta'^\beta} \frac{\partial \theta'^\gamma}{\partial \theta^k} + \sum_{l=1}^2 \frac{\partial^2 \theta^l}{\partial \theta'^\alpha \partial \theta'^\beta} \frac{\partial \theta'^\gamma}{\partial \theta^l}, \quad (39)$$

where $\theta = \psi(\mathbf{P}_\theta)$ and $\theta' = \psi'(\mathbf{P}_\theta)$ are used as coordinate systems in S .

Now we omit ∞ in all components appearing hereafter. Quantities satisfying transformation law (39) are called connections on a manifold S (see [8; Vol. 2, 5-17]). For given Γ_{ij}^k , we can determine a unique linear connection ∇ on S defined by

$$\nabla_{\partial/\partial \theta^i} \frac{\partial}{\partial \theta^j} = \sum_{k=1}^2 \Gamma_{ij}^k \frac{\partial}{\partial \theta^k}$$

(see [8; Vol. 2, 6-1, 6-2]). We will define the curvature tensor for a given connection ∇ later.

To determine whether a metric tensor on S , for which Γ_{ij}^k are Christoffel symbols of the second kind, exists, we have to find a symmetric positive definite matrix (a_{ij}) such that

$$\Gamma_{ij}^k = \sum_{l=1}^2 a^{lk} \frac{1}{2} \left\{ \frac{\partial a_{jl}}{\partial \theta^i} + \frac{\partial a_{il}}{\partial \theta^j} - \frac{\partial a_{ij}}{\partial \theta^l} \right\} \quad (40)$$

where $(a^{ij}) = (a_{ij})^{-1}$. If there exists a symmetric positive definite matrix (a_{ij}) satisfying (40), then S is a Riemannian manifold with a Riemannian metric a and there is a unique Riemannian connection ∇ on S , i.e., Γ_{ij}^k are Christoffel symbols of the second kind from Theorem 8.6 in [4]. With $\Gamma_{ij}^k(\theta)$ given in Proposition 3.2, we can write the PDE system from equation (40), but this PDE system is complicated and hence we need to find an equivalent system, which is simpler than the PDE system of (40).

Lemma 3.4. *Suppose that a symmetric connection Γ_{ij}^k is given and (a_{ij}) is a symmetric positive definite matrix. Then the equation*

$$\frac{\partial a_{jk}}{\partial \theta^i} = \sum_{l=1}^2 \Gamma_{ij}^l a_{lk} + \sum_{l=1}^2 \Gamma_{il}^l a_{jl} \quad (41)$$

is equivalent to the equation

$$\Gamma_{ij}^k = \sum_{l=1}^2 a^{lk} \frac{1}{2} \left\{ \frac{\partial a_{jl}}{\partial \theta^i} + \frac{\partial a_{il}}{\partial \theta^j} - \frac{\partial a_{ij}}{\partial \theta^l} \right\}. \quad (42)$$

Proof. Suppose that

$$\frac{\partial a_{jk}}{\partial \theta^i} = \sum_{l=1}^2 \Gamma_{ij}^l a_{lk} + \sum_{l=1}^2 \Gamma_{il}^l a_{jl}.$$

Permuting i, j, k , we then obtain

$$\begin{aligned} \frac{\partial a_{jk}}{\partial \theta^i} &= \sum_{r=1}^2 \Gamma_{ij}^r a_{rk} + \sum_{r=1}^2 \Gamma_{ik}^r a_{jr}, \\ \frac{\partial a_{ik}}{\partial \theta^j} &= \sum_{r=1}^2 \Gamma_{ji}^r a_{rk} + \sum_{r=1}^2 \Gamma_{jk}^r a_{ir}, \\ \frac{\partial a_{ij}}{\partial \theta^k} &= \sum_{r=1}^2 \Gamma_{ki}^r a_{rj} + \sum_{r=1}^2 \Gamma_{kj}^r a_{ir}. \end{aligned}$$

Using $a_{ij} = a_{ji}$ and $\Gamma_{ij}^k = \Gamma_{ji}^k$, we have

$$\sum_{r=1}^2 \Gamma_{ij}^r a_{rl} = \frac{1}{2} \left\{ \frac{\partial a_{jl}}{\partial \theta^i} + \frac{\partial a_{il}}{\partial \theta^j} - \frac{\partial a_{ij}}{\partial \theta^l} \right\}$$

which implies that

$$\Gamma_{ij}^k = \sum_{l=1}^2 a^{lk} \frac{1}{2} \left\{ \frac{\partial a_{jl}}{\partial \theta^i} + \frac{\partial a_{il}}{\partial \theta^j} - \frac{\partial a_{ij}}{\partial \theta^l} \right\}.$$

Conversely, suppose that for given Γ_{ij}^k and (a_{ij}) , the equation (42) is satisfied. Then it is clear that

$$\frac{\partial a_{jk}}{\partial \theta^i} = \frac{1}{2} \left\{ \frac{\partial a_{ik}}{\partial \theta^j} + \frac{\partial a_{jk}}{\partial \theta^i} - \frac{\partial a_{ji}}{\partial \theta^k} \right\} + \frac{1}{2} \left\{ \frac{\partial a_{ij}}{\partial \theta^k} + \frac{\partial a_{kj}}{\partial \theta^i} - \frac{\partial a_{ki}}{\partial \theta^j} \right\}.$$

Since $(a^{ij}) = (a_{ij})^{-1}$, we have

$$\begin{aligned} \frac{1}{2} \left\{ \frac{\partial a_{ik}}{\partial \theta^j} + \frac{\partial a_{jk}}{\partial \theta^i} - \frac{\partial a_{ji}}{\partial \theta^k} \right\} &= \sum_{r=1}^2 \Gamma_{ji}^r a_{rk}, \\ \frac{1}{2} \left\{ \frac{\partial a_{ij}}{\partial \theta^k} + \frac{\partial a_{kj}}{\partial \theta^i} - \frac{\partial a_{ki}}{\partial \theta^j} \right\} &= \sum_{r=1}^2 \Gamma_{ik}^r a_{rj}. \end{aligned}$$

Hence it follows from the symmetry of a_{ij} and Γ_{ij}^k that the equation (42) implies (41).

Before proving the main theorem in this paper, we consider the curvature tensor R of a linear connection ∇ on S , because the components R_{jkl}^i of the curvature tensor R play an important role in the proof of the theorem. If X, Y, Z are vector fields, we consider the following vector field

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X, Y]}Z \quad (43)$$

where $[X, Y]$ is the Lie bracket of X and Y (see [8; Vol. 2, 6-18]). The map R in the equation (43) is called the curvature tensor of ∇ . In local coordinates we have

$$R\left(\frac{\partial}{\partial \theta^k}, \frac{\partial}{\partial \theta^l}\right) \frac{\partial}{\partial \theta^j} = \sum_{i=1}^2 R_{jkl}^i \frac{\partial}{\partial \theta^i}$$

where $R_{j,k,l}^i$ is defined by the equation

$$R_{jkl}^i(\theta) = \frac{\partial \Gamma_{lj}^i(\theta)}{\partial \theta^k} - \frac{\partial \Gamma_{kj}^i(\theta)}{\partial \theta^l} + \sum_{p=1}^2 (\Gamma_{lj}^p(\theta) \Gamma_{kp}^i(\theta) - \Gamma_{kj}^p(\theta) \Gamma_{lp}^i(\theta)).$$

From Proposition 3.2, we can obtain the limit

$$R_{jkl}^i(\theta) := \lim_{N \rightarrow \infty} R_{jkl}^i(\theta, N)$$

of the Riemannian curvature tensor components $R_{jkl}^i(N)$ corresponding to $\Gamma_{ij}^k(N)$ with respect to a coordinate (S, ψ) after a tedious calculation.

Proposition 3.4. *The components $R_{jkl}^i(\theta)$ are given by*

$$\begin{aligned} R_{111}^1(\theta) &= R_{211}^1(\theta) = R_{122}^1(\theta) = R_{222}^1(\theta) = 0, \\ R_{121}^1(\theta) &= R_{112}^1(\theta) = R_{221}^1(\theta) = R_{212}^1(\theta) = 0, \quad \text{and} \\ R_{111}^2(\theta) &= R_{211}^2(\theta) = R_{122}^2(\theta) = R_{222}^2(\theta) = R_{221}^2(\theta) = R_{212}^2(\theta) = 0, \\ R_{121}^2(\theta) &= -R_{112}^2(\theta) = \frac{(\partial_1 g_{22}(\theta))^2}{4(g_{22}(\theta))^2} + \frac{1}{4\theta^1} \left(\frac{\partial_1 g_{22}(\theta)}{g_{22}(\theta)} \right) - \frac{\partial_2 g_{22}(\theta)}{4(g_{22}(\theta))^2} \\ &\quad \times \left\{ \frac{T}{2} \left(\frac{\theta^2}{\theta^1} \right) \sum_{k=1}^{\infty} \frac{\lambda_k a_k^2}{(\theta^1 \lambda_k - \theta^2 a_k)^2} - \frac{1}{4} \sum_{k=1}^{\infty} \frac{\lambda_k^2 a_k}{(\theta^1 \lambda_k - \theta^2 a_k)^3} \right. \\ &\quad - \frac{1}{4} \left(\frac{\theta^2}{\theta^1} \right) \sum_{k=1}^{\infty} \frac{\lambda_k a_k^2}{(\theta^1 \lambda_k - \theta^2 a_k)^3} \\ &\quad + \frac{T}{2} \sum_{k=1}^{\infty} \frac{\lambda_k^2 a_k \exp\{-2(\theta^1 \lambda_k - \theta^2 a_k)T\}}{(\theta^1 \lambda_k - \theta^2 a_k)^2} \\ &\quad + \frac{1}{4} \sum_{k=1}^{\infty} \frac{\lambda_k^2 a_k \exp\{-2(\theta^1 \lambda_k - \theta^2 a_k)T\}}{(\theta^1 \lambda_k - \theta^2 a_k)^3} \\ &\quad \left. + \frac{1}{4} \left(\frac{\theta^2}{\theta^1} \right) \sum_{k=1}^{\infty} \frac{\lambda_k a_k^2 \exp\{-2(\theta^1 \lambda_k - \theta^2 a_k)T\}}{(\theta^1 \lambda_k - \theta^2 a_k)^3} \right\}. \end{aligned}$$

Remark 3.4. Direct computation also shows that the limit values R_{jkl}^i coincide with the curvature tensor corresponding to Γ_{ij}^k given in Proposition 3.2. But R_{jkl}^i are not necessarily components of a Riemannian curvature tensor because we don't know that Γ_{ij}^k define a Riemannian connection.

3.3. The main theorem. Now we prove the main theorem in this paper.

Theorem 3.1. *Either there is no Riemannian metric for which Γ_{ij}^k are the Christoffel symbols of the second kind in which case we have a curved manifold or every Riemannian metric compatible with Γ_{ij}^k defines a flat manifold.*

Proof. Suppose that there exists a Riemannian metric (a_{ij}) , for which Γ_{ij}^k are the Christoffel symbols of the second kind. Then (a_{ij}) satisfy the equation (41) with $\Gamma_{ij}^k(\theta)$ given in Proposition 3.2. Hence we have the

following 1st order linear PDE system: for all $\theta \in \Theta$,

$$\frac{\partial a_{11}(\theta)}{\partial \theta^1} = 2(\Gamma_{11}^1(\theta) a_{11}(\theta) + \Gamma_{11}^2(\theta) a_{21}(\theta)), \quad (44)$$

$$\frac{\partial a_{12}(\theta)}{\partial \theta^1} = \Gamma_{11}^1(\theta) a_{12}(\theta) + \Gamma_{11}^2(\theta) a_{22}(\theta) + \Gamma_{12}^2(\theta) a_{12}(\theta), \quad (45)$$

$$\frac{\partial a_{22}(\theta)}{\partial \theta^1} = 2\Gamma_{12}^2(\theta) a_{22}(\theta), \quad (46)$$

$$\frac{\partial a_{11}(\theta)}{\partial \theta^2} = 2\Gamma_{21}^2(\theta) a_{12}(\theta), \quad (47)$$

$$\frac{\partial a_{12}(\theta)}{\partial \theta^2} = \Gamma_{21}^2(\theta) a_{22}(\theta) + \Gamma_{22}^2(\theta) a_{12}(\theta), \quad (48)$$

$$\frac{\partial a_{22}(\theta)}{\partial \theta^2} = 2\Gamma_{22}^2(\theta) a_{22}(\theta). \quad (49)$$

Let us now check the compatibility condition. First consider (44) and (47).

$$\begin{aligned} & \frac{\partial^2 a_{11}(\theta)}{\partial \theta^2 \partial \theta^1} - \frac{\partial^2 a_{11}(\theta)}{\partial \theta^1 \partial \theta^2} \\ &= 2 \left(\frac{\partial \Gamma_{11}^2(\theta)}{\partial \theta^2} - \frac{\partial \Gamma_{21}^2(\theta)}{\partial \theta^1} + \Gamma_{11}^1(\theta) \Gamma_{21}^2(\theta) \right. \\ & \quad \left. + \Gamma_{11}^2(\theta) \Gamma_{22}^2(\theta) - \Gamma_{12}^2(\theta) \Gamma_{22}^2(\theta) \right) a_{12}(\theta) = 2R_{121}^2(\theta) a_{12}(\theta). \end{aligned} \quad (50)$$

Now consider (45) and (48).

$$\begin{aligned} & \frac{\partial^2 a_{12}(\theta)}{\partial \theta^2 \partial \theta^1} - \frac{\partial^2 a_{12}(\theta)}{\partial \theta^1 \partial \theta^2} \\ &= \left(\frac{\partial \Gamma_{11}^2(\theta)}{\partial \theta^2} - \frac{\partial \Gamma_{21}^2(\theta)}{\partial \theta^1} + \Gamma_{11}^1(\theta) \Gamma_{21}^2(\theta) \right. \\ & \quad \left. + \Gamma_{11}^2(\theta) \Gamma_{12}^2(\theta) - \Gamma_{12}^2(\theta) \Gamma_{22}^2(\theta) \right) a_{22}(\theta) = R_{121}^2(\theta) a_{22}(\theta). \end{aligned} \quad (51)$$

By definition, $a(\theta) \equiv (a_{ij}(\theta))$ is positive definite for all $\theta \in \Theta$. Hence, in particular, $a_{11}(\theta) > 0$, $a_{22}(\theta) > 0$ for all $\theta \in \Theta$. For satisfying the compatibility, from (51), we should then have $R_{121}^2(\theta) \equiv 0$. Therefore R vanishes in some coordinate system and being a tensor, $R \equiv 0$ in all coordinate systems, i.e., the space is flat, if R_{121}^2 is not identically 0 in some coordinate system. Let $R_{121}^2(\theta') \neq 0$ say, then from (50) and (51), we have $a_{12}(\theta') = a_{22}(\theta') = 0$. Hence $a_{ij}(\theta)$ cannot be a Riemannian metric.

E x a m p l e. Consider the following SPDE: $0 \leq t \leq T$, $0 < x < \pi$,

$$du(t, x) = (\theta^1 \Delta u(t, x) + \theta^2 u(t, x)) dt + (I - \Delta)^{-1/2} dW(t, x), \quad (52)$$

with initial and boundary conditions

$$u(0, x) = 0, \quad f \in L^2(0, \pi) \quad \text{and} \quad u(t, 0) = u(t, \pi) = 0, \quad 0 \leq t \leq T, \quad (53)$$

where $W(t)$ is a cylindrical Brownian motion on $H = L^2(0, \pi)$, $\Delta = \partial^2 / \partial x^2$ and $\theta = (\theta^1, \theta^2) \in \Theta$. The SPDE (52) is a special case of the Hilbert space valued SDE (19), i.e.,

$$L = -\Delta, \quad A = I, \quad B = (I - \Delta)^{-1/2} \quad \text{and} \quad X_0 = 0$$

where I is an identity operator. In this case $a_k = 1, b_k = 1/\sqrt{k^2 + 1}$ and $\varphi_k(x) = \sin kx$, and the parameter space is given by

$$\Theta = \{(\theta^1, \theta^2) \mid \theta^1 > 0 \text{ and } \theta^1 > \theta^2\}.$$

Then it is easy to check that the sequences $\{a_k\}$ and $\{b_k\}$ satisfy the conditions (C2) and (C3) mentioned above.

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