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ON THE CHARACTERISTIC FUNCTION OF THE JOINT LIMIT DISTRIBUTION OF THE FIRST AND SECOND ORDER POWER SUMS¹⁾

Изучается предельная характеристическая функция совместного распределения нормированных сумм первых и вторых степеней последовательности независимых одинаково распределенных случайных величин, имеющих распределение, принадлежащее области притяжения α -устойчивого закона с $\alpha < 2$. Получена явная формула в терминах обобщенных гипергеометрических функций при $0 < \alpha < 2$, $\alpha \neq 1$, и в терминах S - и C -функций Френеля при $\alpha = 1$.

Ключевые слова и фразы: нормированные суммы степеней, область притяжения α -устойчивого закона, предельная характеристическая функция, обобщенные гипергеометрические функции, S - и C -функции Френеля, уравнения Шредингера.

Introduction

Let $X_1, X_2, \dots$ be a series of independent identically distributed (i.i.d.) random variables with the distribution function $F(x)$. Assume that $F(x)$ belongs to the domain of attraction of a stable distribution with exponent $0 < \alpha < 2$, i.e.,

$$\begin{aligned} F(-x) &= (C_- + o(1)) x^{-\alpha} L(x), \\ 1 - F(x) &= (C_+ + o(1)) x^{-\alpha} L(x), \quad x \rightarrow \infty, \end{aligned} \quad (1)$$

where

$$C_-, C_+ \geq 0, \quad C_- + C_+ > 0$$

and $L(x)$ is a slowly varying function at infinity.

Consider the sequence of random variables formed by the first and second order homogeneous power sums. Applying an appropriate linear normalization determined by the sequences $A_n = (A_{1n}, A_{2n})$, $B_n \neq 0$, $n = 1, 2, \dots$, the limit distribution

$$\left(B_n^{-1} \sum_{j=1}^n X_j - A_{1n}, B_n^{-2} \sum_{j=1}^n X_j^2 - A_{2n} \right) \xrightarrow{d} Y = (Y_1, Y_2) \quad \text{as } n \rightarrow \infty,$$

(where $\xrightarrow{d}$ denotes the convergence in distribution) exists (see [2] in a special case, and [7], [10] for the general case with higher order power sums, as well). The characteristic function of the random vector Y can be written in the following form:

$$\varphi_\alpha(x, y) = \mathbf{E} \exp\{ixY_1 + iyY_2\} = \exp\left(\psi_\alpha(x, y)\right), \quad x, y \in \mathbf{R},$$

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where

$$\psi_{\alpha}(x, y) = \alpha \int_{\mathbf{R} \setminus \{0\}} \kappa(u) \left[\exp\{ixu + iyu^2\} - 1 - iux \left(I\{\alpha = 1\} \frac{1}{1+u^2} + I\{1 < \alpha\} \right) \right] \frac{du}{|u|^{1+\alpha}}, \quad (2)$$

$\kappa(u) = I\{u < 0\} C_- + I\{u > 0\} C_+$, I denotes the indicator function.

Let us point out that the limit distributions of random symmetric polynomials under linear normalization can be written as distributions of a second order polynomial of Y (see [5] and [7]) and the characteristic function of the limit distribution can be described with the help of the characteristic function of Y (cf. [9] and [11]).

If instead of the characteristic function $\varphi_{\alpha}(x, y)$ we consider the mixed Fourier-Laplace transform $E \exp\{ixY_1 + yY_2\}$, $x \in \mathbf{R}$, $y > 0$, then this latter can be expressed using the parabolic cylinder function $D_{\nu}(x)$. These functions were used in [2] in connection to self-normalizing sums. The representation formula obtained in this paper for the characteristic function $\varphi_{\alpha}(x, y)$ in terms of the generalized hypergeometric functions ${}_pF_q$ and the Fresnel-functions S and C can be generalized to the linearly normed sums of higher order power sums. In the present case the analytic and asymptotic properties of the characteristic function can directly be obtained from these formulas.

The analysis of the joint characteristic function in the case when $\alpha \geq 2$ can easily be carried out. The limit distribution of the first and second order power sums under linear normalization exists. If $2 \leq \alpha < 4$, then Y_1 has normal limit distribution, which is independent from the random variable Y_2 , this latter having stable limit distribution with exponent $\alpha/2$. If $4 \leq \alpha < \infty$, then the joint limit distribution is normal.

There are examples in the literature of limit distributions for the use of generalized hypergeometric functions.

As we have mentioned earlier the logarithm of the mixed Fourier-Laplace transform can be expressed by the parabolic cylinder functions. This latter can be written in terms of the generalized hypergeometric function ${}_1F_1$ (see [3, § 8.2.7]).

As an other example we can mention the studies of density functions of stable distributions ([1], [11]), where the density function is given using the so-called Meijer-functions. Let us point out that the Meijer-functions and the generalized hypergeometric functions can be expressed via each other ([3, §§ 8.3–8.4]).

Main results

In the sequel we consider only the symmetric case $C_- = C_+ = C$ making the formulation of the results and the proof easier, although the non-symmetric case ($C_- \neq C_+$) can be analysed similarly.

Denote by ${}_pF_q$ the generalized hypergeometric function, i.e.,

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!},$$

where a_i and b_j are arbitrary real numbers except for the negative integers and zero, and $(a)_k = a(a+1) \cdots (a+k-1)$, $k = 1, 2, \dots$, $(a)_0 = 1$.

Theorem. *If the distribution function $F(x)$ belongs to the domain of attraction of a symmetric stable distribution with exponent $0 < \alpha < 2$ (i.e., (1) holds with $C = C_- = C_+ > 0$), then for $y \neq 0$ the following representations are valid:*

A) If $0 < \alpha < 1$, then

$$\psi_{\alpha}(x, y) = -C \cos \frac{\alpha\pi}{2} \Gamma(1-\alpha) \Gamma\left(\frac{\alpha+1}{2}\right) \frac{1}{2\sqrt{\pi}} |y|^{\alpha/2}$$

$$\times \left\{ \exp \left(i \operatorname{sgn} y \frac{\alpha+1}{4} \pi \right) {}_2F_3 \left(\frac{\alpha+3}{4}, \frac{\alpha+1}{4}; \frac{1}{2}, \frac{1}{4}, \frac{3}{4}; -\frac{x^4}{64y^2} \right) + \frac{\alpha+3}{4} \exp \left(-i \operatorname{sgn} y \frac{\alpha+1}{4} \pi \right) \times \frac{x^2}{|y|} {}_2F_3 \left(\frac{\alpha+5}{4}, \frac{\alpha+3}{4}; \frac{3}{2}, \frac{3}{4}, \frac{5}{4}; -\frac{x^4}{64y^2} \right) \right\},$$

B) If $\alpha = 1$, then

$$\begin{aligned} \psi_1(x, y) = & -\pi |x| C \left(\frac{x^2}{|y|} \right) [1 + i \operatorname{sgn} y] - \pi |x| S \left(\frac{x^2}{|y|} \right) [1 - i \operatorname{sgn} y] \\ & + \sqrt{2\pi} \cos \left(\frac{x^2}{|y|} \right) \left[-\sqrt{|y|} + i|x| \operatorname{sgn} y \frac{\pi}{2} \right] \\ & + \sqrt{2\pi} \sin \left(\frac{x^2}{|y|} \right) \left[\sqrt{|y|} + i|x| \operatorname{sgn} y \frac{\pi}{2} \right], \end{aligned}$$

C) If $1 < \alpha < 2$, then

$$\psi_\alpha(x, y) = \int_0^x \psi_{\alpha-1}(u, y) du,$$

which in view of A) can also be expressed using the generalized hypergeometric functions ${}_2F_3$.

R e m a r k 1. It is easy to see that rotating the domain of integration on the complex plane with angle $\theta = (\pi/4) \operatorname{sgn} y$ (here Cauchy's theorem can be applied) we obtain such an integral representation of the function $\psi_\alpha(x, y)$ which is infinitely differentiable in both variables for any $y \neq 0$, but this representation does not provide any information about the behaviour of the function for small x and y values.

The first few terms of the power series of the generalized hypergeometric functions in part A) of the theorem are the following:

$$\begin{aligned} & {}_2F_3 \left(\frac{\alpha+3}{4}, \frac{\alpha+1}{4}; \frac{1}{2}, \frac{1}{4}, \frac{3}{4}; -\frac{x^4}{64y^2} \right) \\ & = 1 - \frac{(\alpha+1)(\alpha+3)}{32} \frac{x^4}{y^2} + \frac{(\alpha+1)(\alpha+3)(\alpha+5)(\alpha+7)}{6480} \frac{x^8}{y^4} \mp \dots \end{aligned} \quad (i)$$

and

$$\begin{aligned} & {}_2F_3 \left(\frac{\alpha+5}{4}, \frac{\alpha+3}{4}; \frac{3}{2}, \frac{3}{4}, \frac{5}{4}; -\frac{x^4}{64y^2} \right) = 1 - \frac{(\alpha+3)(\alpha+5)}{1120} \frac{x^4}{y^2} \\ & + \frac{(\alpha+1)(\alpha+3)(\alpha+5)(\alpha+7)}{226800} \frac{x^8}{y^4} \mp \dots \end{aligned} \quad (ii)$$

The error of the approximation can be estimated as follows.

Let $z_0 > 0$, $n_0 > e\sqrt{z_0}$ be arbitrary numbers. Then an application of the Stirling formula provides an upper bound for the error in (i) and (ii) using the first n_0 terms:

$$\sum_{k=n_0}^{\infty} \frac{4}{3} \frac{1}{(k!)^2} z_0^k \leq \frac{4}{3} \sum_{k=n_0}^{\infty} \left[\sqrt{2\pi k} \left(\frac{k}{e} \right)^k \right]^{-2} z_0^k \leq \frac{2}{3\pi n_0} \left(\frac{e^2 z_0}{n_0^2} \right)^{n_0} \left(1 - \frac{e^2 z_0}{n_0^2} \right)^{-1}.$$

Proof. C) If $0 < \alpha < 1$, then the Fubini theorem can be applied in the following integral because the function is absolutely integrable leading to the expression

$$\begin{aligned} & i \int_0^z \int_{-\infty}^{\infty} [\exp\{isu + i \operatorname{sgn} y u^2\} - 1] \frac{du}{|u|^{1+\alpha}} ds \\ &= \int_{-\infty}^{\infty} \int_0^z [\exp\{isu + i \operatorname{sgn} y u^2\} - 1] \frac{du}{|u|^{1+\alpha}} ds \\ &= \int_{-\infty}^{\infty} [\exp\{izu + i \operatorname{sgn} y u^2\} - 1 - izu] \frac{du}{|u|^{2+\alpha}} ds = \psi_{1+\alpha}(z, \operatorname{sgn} y). \end{aligned}$$

In view of the connection between the generalized hypergeometric function and its derivatives ([3, § 5.2.2.]) to prove C) we have to prove A).

B) It is immediate that using partial integration the function $\psi_1(x, y)$ can be written as

$$\begin{aligned} \psi_1(x, y) &= 2 \int_0^{\infty} (\cos zu \cos yu^2 - 1) \frac{du}{u^2} + 2i \int_0^{\infty} (\cos zu \sin yu^2) \frac{du}{u^2} \\ &= 2\sqrt{|y|} \int_0^{\infty} (\cos |z|u \cos u^2 - 1) \frac{du}{u^2} + 2i\sqrt{|y|} \operatorname{sgn} y \int_0^{\infty} \cos |z|u \sin u^2 \frac{du}{u^2} \\ &= -2\sqrt{|y|} |z| \int_0^{\infty} \sin |z|u \cos u^2 \frac{du}{u} - 4\sqrt{|y|} \int_0^{\infty} \cos |z|u \sin u^2 du \\ &\quad + 2i\sqrt{|y|} \operatorname{sgn} y \int_0^{\infty} \cos |z|u \sin u^2 \frac{du}{u^2}, \end{aligned}$$

where $z = x/\sqrt{|y|}$. Using the expressions

$$\int_0^{\infty} \cos |z|u \sin u^2 \frac{du}{u^2} = |z| \frac{\pi}{2} \left[S\left(\frac{z^2}{4}\right) - C\left(\frac{z^2}{4}\right) + \sqrt{\frac{\pi}{2}} \left(\sin \frac{z^2}{4} + \cos \frac{z^2}{4} \right) \right],$$

(see [5, p. 431])

$$\begin{aligned} \int_0^{\infty} |z|u \cos u^2 \frac{du}{u} &= \frac{1}{2} \int_0^{\infty} \frac{1}{u} \left[\sin(u^2 + |z|u) - \sin(u^2 - |z|u) \right] du \\ &= \frac{\pi}{2} \left(C\left(\frac{z^2}{4}\right) + S\left(\frac{z^2}{4}\right) \right) \end{aligned}$$

(see [5, p. 430]),

$$\int_0^{\infty} \cos |z|u \sin u^2 du = \frac{1}{2} \sqrt{\frac{\pi}{2}} \left(\cos \frac{z^2}{4} + \sin \frac{z^2}{4} \right)$$

(see [5, p. 431]), we obtain the proof of B).

A) Assume that $0 < \alpha < 1$. Let us introduce the notation

$$\begin{aligned} W(s, t) &= w(t) \exp \left\{ -is^2(4t)^{-1} \right\}, \quad s, t \in \mathbf{R}, \quad t \neq 0, \\ w(t) &= \frac{1}{2\sqrt{\pi|t|}} \exp \left\{ -i\frac{\pi}{2} \operatorname{sgn} t \right\}, \quad t \neq 0. \end{aligned}$$

Based on the well-known properties of the Fresnel integrals the function $\psi_{\alpha}(x, y)$ can be expressed in the following form:

$$\psi_{\alpha}(x, y) = \alpha C \int_{\mathbf{R} \setminus \{0\}} (\exp\{ixu + iyu^2\} - 1) \frac{du}{|u|^{1+\alpha}}$$

$$\begin{aligned}
&= \alpha C|y|^\alpha \int_{\mathbf{R} \setminus \{0\}} \left(\exp\{izu + iu^2\} - 1 \right) \frac{du}{|u|^{1+\alpha}} \\
&= \alpha C|y|^\alpha \int_{\mathbf{R} \setminus \{0\}} \int_{-\infty}^{\infty} \frac{1}{|u|^{1+\alpha}} \left(\exp\{izu + isu\} - 1 \right) W(s, \operatorname{sgn} y) ds du, \quad (3)
\end{aligned}$$

where $z = x/\sqrt{|y|}$, $y \neq 0$. First we prove that the order of integration in (3) can be changed. This would lead to the following representation:

$$\begin{aligned}
\psi_\alpha(x, y) &= \alpha C|y|^\alpha \int_{-\infty}^{\infty} \int_{\mathbf{R} \setminus \{0\}} \frac{1}{|u|^{1+\alpha}} (e^{isu} - 1) W(s - z, \operatorname{sgn} y) du ds \\
&= -C\Gamma(1 - \alpha) |y|^\alpha \int_{-\infty}^{\infty} |s|^\alpha \left(\exp\left\{i \frac{\pi}{2} \alpha\right\} I\{s < 0\} + \exp\left\{-i \frac{\pi}{2} \alpha\right\} I\{s > 0\} \right) \\
&\quad \times W(s - z, \operatorname{sgn} y) ds. \quad (4)
\end{aligned}$$

Denote

$$\Theta_1(z, \operatorname{sgn} y) = \int_0^\infty \int_{-\infty}^\infty \frac{1}{|u|^{1+\alpha}} (e^{isu} - 1) W(s - z, \operatorname{sgn} y) ds du, \quad (5)$$

$$\Theta_2(z, \operatorname{sgn} y) = \int_{-\infty}^0 \int_{-\infty}^\infty \frac{1}{|u|^{1+\alpha}} (e^{isu} - 1) W(s - z, \operatorname{sgn} y) ds du. \quad (6)$$

Since the integrals in (5) and (6) behave similarly it is enough to analyse (5). Let us introduce the notation

$$H_\delta(U, V_1, V_2) = \int_\delta^U \int_{-V_1}^{V_2} \frac{1}{|u|^{1+\alpha}} (e^{isu} - 1) W(s - z, \operatorname{sgn} y) ds du, \quad (7)$$

where U, V_1, V_2 and δ are arbitrary positive numbers.

The finite limits

$$H_\delta(U, -\infty, \infty) = \lim_{V_1, V_2 \rightarrow \infty} H_\delta(U, V_1, V_2) \quad (8)$$

and

$$H_\delta(\infty, V_1, V_2) = \lim_{U \rightarrow \infty} H_\delta(U, V_1, V_2) \quad (9)$$

obviously exist. We prove that the convergence in (8) is uniform in U as $V_1, V_2 \rightarrow \infty$, implying that

$$\lim_{V_1, V_2 \rightarrow \infty} \lim_{U \rightarrow \infty} H_\delta(U, V_1, V_2) = \lim_{U \rightarrow \infty} \lim_{V_1, V_2 \rightarrow \infty} H_\delta(U, V_1, V_2),$$

in other words

$$\begin{aligned}
H_\delta(\infty, -\infty, \infty) &= \int_\delta^\infty \int_{-\infty}^\infty \frac{1}{|u|^{1+\alpha}} (e^{isu} - 1) W(s - z, \operatorname{sgn} y) ds du \\
&= \int_{-\infty}^\infty \int_\delta^\infty \frac{1}{|u|^{1+\alpha}} (e^{isu} - 1) W(s - z, \operatorname{sgn} y) du ds. \quad (10)
\end{aligned}$$

Let us estimate the absolute value of the difference $D = H_\delta(U, V_1', V_2') - H_\delta(U, V_1'', V_2'')$ for arbitrary $V_1', V_2', V_1'', V_2'' > \Delta$, as $\Delta \rightarrow \infty$. Denote $a_1 = \max(V_1', V_1'')$, $a_2 = \min(V_1', V_1'')$, $b_1 = \min(V_2', V_2'')$ and $b_2 = \max(V_2', V_2'')$. Obviously $a_i, b_i > \Delta$, $i = 1, 2$. In this case

$$\begin{aligned}
|D| &\leq \left| \int_\delta^U \int_{-a_1}^{-a_2} \frac{1}{|u|^{1+\alpha}} (e^{isu} - 1) W(s - z, \operatorname{sgn} y) ds du \right| \\
&\quad + \left| \int_\delta^U \int_{b_1}^{b_2} \frac{1}{|u|^{1+\alpha}} (e^{isu} - 1) W(s - z, \operatorname{sgn} y) ds du \right| = D_1 + D_2. \quad (11)
\end{aligned}$$

Since D_1, D_2 can be handled similarly we analyse only one of them: for example D_2 . Obviously

$$D_2 \leq \left| \int_{\delta}^{\min(U, \Delta/4)} \int_{b_1}^{b_2} \frac{1}{|u|^{1+\alpha}} (e^{isu} - 1) W(s - z, \operatorname{sgn} y) ds du \right| + \left| \int_{\min(U, \Delta/4)}^U \int_{b_1}^{b_2} \frac{1}{|u|^{1+\alpha}} (e^{isu} - 1) W(s - z, \operatorname{sgn} y) ds du \right| = D_{21} + D_{22}.$$

The quantity $\int_{b_1}^{b_2} (e^{isu} - 1) W(s - z, \operatorname{sgn} y) ds$ is obviously uniformly bounded in b_1, b_2 , consequently

$$D_{22} \leq c \int_{\min(U, \Delta/4)}^U \frac{du}{u^{1+\alpha}} < \frac{c}{\alpha} \left(\frac{\Delta}{4} \right)^{-\alpha} \rightarrow 0, \quad \Delta \rightarrow \infty,$$

uniformly in U , where c is an appropriate constant. On the other hand

$$D_{21} \leq \left| \int_{\delta}^{\min(U, \Delta/4)} \int_{b_1}^{b_2} \frac{1}{|u|^{1+\alpha}} e^{isu} W(s - z, \operatorname{sgn} y) ds du \right| + \left| \int_{\delta}^{\min(U, \Delta/4)} \int_{b_1}^{b_2} \frac{1}{|u|^{1+\alpha}} W(s - z, \operatorname{sgn} y) ds du \right| = D_{31} + D_{32}. \quad (12)$$

The second term of (12) can be estimated as follows ($a_i, b_i > \Delta$):

$$D_{32} \leq \frac{1}{\alpha \delta^a} \left| \int_{b_1+z}^{b_2+z} W(s, \operatorname{sgn} y) ds \right| \rightarrow 0, \quad \Delta \rightarrow \infty,$$

the convergence to 0 is obviously uniform in U . As concerns the first term on the right-hand side of (12)

$$D_{31} = \left| \int_{\delta}^{\min(U, \Delta/4)} \frac{1}{|u|^{1+\alpha}} \exp\{iz^2 \operatorname{sgn} y + iuz\} \times \left(\int_{b_1+z-2u \operatorname{sgn} y}^{b_2+z-2u \operatorname{sgn} y} \exp\left\{-is^2 \operatorname{sgn} \frac{y}{4}\right\} w(\operatorname{sgn} y) ds \right) du \right|.$$

As for $\delta < u < \Delta/4$ the inequality $b_1 + z - 2u \operatorname{sgn} y > \Delta/4$ holds, if Δ large enough, consequently the existence of the Fresnel integral implies that there exists a constant c_{Δ} depending on Δ tending to 0 as $\Delta \rightarrow \infty$, estimating the absolute value of the inner integral from above. Thus

$$D_{31} \leq c_{\Delta} \frac{1}{\alpha \delta^a} \rightarrow 0 \quad \text{as } \Delta \rightarrow \infty,$$

independently of the value of U .

These imply that the convergence in (8) is uniform in U , so (10) holds.

To prove (4) it is enough to check that the integrals

$$I_1 = \int_0^{\delta} \int_{-\infty}^{\infty} \frac{1}{|u|^{1+\alpha}} (e^{isu} - 1) W(s - z, \operatorname{sgn} y) ds du, \\ I_2 = \int_{-\infty}^{\infty} \int_0^{\delta} \frac{1}{|u|^{1+\alpha}} (e^{isu} - 1) W(s - z, \operatorname{sgn} y) du ds,$$

tend to zero as $\delta \rightarrow 0$. It is immediate that

$$|I_1| = \left| \int_0^{\delta} \frac{1}{|u|^{1+\alpha}} \left(\exp\{i(zu + u^2 \operatorname{sgn} y)\} - 1 \right) du \right|$$

$$\leq \int_0^\delta u^{-\alpha}(|z|+u) du \rightarrow 0, \quad \text{as } \delta \rightarrow 0.$$

We show that there exists a constant C_α depending only on the value α such that for any $\delta > 0$, which is small enough the inequality

$$|I_2| \leq C_\alpha \delta^{(1-\alpha)/2} \quad (13)$$

holds.

Let $A > |z|$ be an arbitrary number. Since $W(s, \operatorname{sgn} y) = 2^{-1} \pi^{-1/2} e^{iy_0 s^2}$, $y_0 = -4^{-1} \operatorname{sgn} y$ ($|y_0| = \frac{1}{4}$), consequently

$$\begin{aligned} & \left| \int_{-A}^A \left(\int_0^\delta \frac{1}{|u|^{1+\alpha}} (e^{i(s+z)u} - 1) du \right) W(s, \operatorname{sgn} y) ds \right| \\ & \leq \int_{-A}^A \int_0^\delta \frac{|s+z|}{u^\alpha} \frac{1}{2\sqrt{\pi}} du ds \leq \frac{1}{2\sqrt{\pi}(1-\alpha)} (A^2 + 2A|z|) \delta^{1-\alpha} \\ & \leq \frac{3}{\sqrt{\pi}(1-\alpha)} A^2 \delta^{1-\alpha}. \end{aligned} \quad (14)$$

On the other hand using partial integration with respect to the variable s (first on a finite interval, then tending to infinity with the upper limit) we obtain

$$\begin{aligned} & \int_A^\infty \left(\frac{1}{2iy_0 s} \int_0^\delta (e^{i(s+z)u} - 1) \frac{du}{u^{1+\alpha}} \right) (2iy_0 s e^{iy_0 s^2}) ds \\ & = -\frac{1}{2iy_0} \frac{1}{A} e^{iy_0 A^2} \int_0^\delta (e^{i(A+z)u} - 1) \frac{du}{u^{1+\alpha}} \\ & \quad - \frac{1}{2iy_0} \int_A^\infty \left(-\frac{1}{s^2} \int_0^\delta (e^{i(s+z)u} - 1) \frac{du}{u^{1+\alpha}} \right) e^{iy_0 s^2} ds \\ & \quad - \frac{1}{2iy_0} \int_A^\infty \left(\frac{1}{s} \int_0^\delta i e^{i(s+z)u} \frac{du}{u^\alpha} \right) e^{iy_0 s^2} ds = E_1 + E_2 + E_3. \end{aligned}$$

The integrals E_1 and E_2 can easily be estimated. Since

$$\int_0^\infty \frac{|e^{iu} - 1|}{u^{1+\alpha}} du \leq \int_0^1 \frac{1}{u^\alpha} du + \int_1^\infty \frac{2}{u^{1+\alpha}} du = c_\alpha = \frac{1}{1-\alpha} + \frac{2}{\alpha},$$

so

$$|E_1| \leq \frac{1}{2|y_0|} \frac{1}{A} \int_0^{\delta(A+z)} \frac{|A+z|^\alpha |e^{iu} - 1|}{u^{1+\alpha}} du \leq 4c_\alpha A^{-(1-\alpha)}, \quad (15)$$

$$\begin{aligned} |E_2| & \leq \frac{1}{2|y_0|} \int_A^\infty \frac{1}{s^2} \left(\int_0^\delta \frac{|e^{i(s+z)u} - 1|}{u^{1+\alpha}} du \right) ds \\ & \leq 2 \int_A^\infty \frac{|s+z|^\alpha}{s^2} \left(\int_0^\infty \frac{|e^{iu} - 1|}{u^{1+\alpha}} du \right) ds \leq 4c_\alpha^2 A^{-(1-\alpha)}. \end{aligned} \quad (16)$$

To estimate E_3 we have to perform an extra partial integration:

$$\begin{aligned} E_3 & = \frac{1}{2iy_0} \int_0^A \left(-\frac{1}{2iy_0 s^2} \int_0^\delta i e^{i(s+z)u} \frac{du}{u^\alpha} \right) (2iy_0 s e^{iy_0 s^2}) ds \\ & = \frac{1}{4y_0^2} \frac{1}{A^2} e^{iy_0 A^2} \int_0^\delta i e^{i(A+z)u} \frac{du}{u^\alpha} \\ & \quad - \frac{1}{4y_0^2} \int_A^\infty \left(\frac{-2}{s^3} \int_0^\delta e^{i(s+z)u} \frac{du}{u^\alpha} + \frac{1}{s^2} \int_0^\delta e^{i(s+z)u} u^{1-\alpha} du \right) e^{iy_0 s^2} ds, \end{aligned}$$

implying that

$$|E_3| \leq 4c_\alpha \left(\frac{1}{A^2} + \frac{1}{A^2} + \frac{\delta}{A} \right) \delta^{1-\alpha} = 4c_\alpha \left(\frac{2}{A^2} + \frac{\delta}{A} \right) \delta^{1-\alpha}. \quad (17)$$

Let us remark that analogous estimations are obtained in the case when the limits of the integrals are $(-\infty, -A)$ instead of (A, ∞) .

Defining $A = A_\delta = \delta^{-(1-\alpha)/4}$ we obtain the inequality (13), consequently $I_2 \rightarrow 0$ as $\delta \rightarrow 0$ proving (4).

Separating the real and imaginary parts of the integral in (4) and applying the well-known integral-formulae (see [5, p. 430-431])

$$\begin{aligned} \int_0^\infty x^\alpha \left\{ \frac{\sin \alpha x^2}{\cos \alpha x^2} \right\} \cos bx \, dx &= \frac{1}{2} a^{-(\alpha+1)/2} \Gamma\left(\frac{\alpha+1}{2}\right) \left\{ \frac{\sin\left(\frac{(\alpha+1)\pi}{4}\right)}{\cos\left(\frac{(\alpha+1)\pi}{4}\right)} \right\} \\ &\quad \times {}_2F_3\left(\frac{\alpha+3}{4}, \frac{\alpha+1}{4}; \frac{1}{2}, \frac{1}{4}, \frac{3}{4}; \frac{b^4}{64a^2}\right) \\ &\quad \mp \frac{b^2}{2} a^{-(\alpha+3)/2} \Gamma\left(\frac{\alpha+1}{2}\right) \left\{ \frac{\cos\left(\frac{(\alpha+1)\pi}{4}\right)}{\sin\left(\frac{(\alpha+1)\pi}{4}\right)} \right\} \\ &\quad \times {}_2F_3\left(\frac{\alpha+5}{4}, \frac{\alpha+3}{4}; \frac{3}{2}, \frac{3}{4}, \frac{5}{4}; -\frac{b^4}{64a^2}\right) \end{aligned}$$

and

$$\begin{aligned} \int_0^\infty x^\alpha \left\{ \frac{\sin \alpha x^2}{\cos \alpha x^2} \right\} \sin bx \, dx &= \frac{b}{2} a^{-(\alpha+2)/2} \Gamma\left(\frac{\alpha+2}{2}\right) \left\{ \frac{\cos\left(\frac{\alpha\pi}{4}\right)}{-\sin\left(\frac{\alpha\pi}{4}\right)} \right\} \\ &\quad \times {}_2F_3\left(\frac{\alpha+4}{4}, \frac{\alpha+2}{4}; \frac{1}{2}, \frac{3}{4}, \frac{5}{4}; -\frac{b^4}{64a^2}\right) \\ &\quad \mp \frac{b^3}{12} a^{-(\alpha+4)/2} \Gamma\left(\frac{\alpha+4}{2}\right) \left\{ \frac{\cos\left(\frac{(\alpha+2)\pi}{4}\right)}{\sin\left(\frac{(\alpha+2)\pi}{4}\right)} \right\} \\ &\quad \times {}_2F_3\left(\frac{\alpha+6}{4}, \frac{\alpha+4}{4}; \frac{3}{2}, \frac{5}{4}, \frac{7}{4}; -\frac{b^4}{64a^2}\right), \end{aligned}$$

where $a, b > 0$, we obtain the proof of the case A).

Remark 2. Rotating the domain of integration in the formula defining the function $\psi_\alpha(x, y)$ with the angle $\beta = \operatorname{sgn} y \pi/4$ (using the Cauchy's theorem) we get the following form

$$\begin{aligned} \psi_\alpha(x, y) &= e^{i\alpha\beta} \int_{\mathbf{R} \setminus \{0\}} \left[\exp \left\{ ix(re^{i\beta}) + iy(r^2 e^{i2\beta}) \right\} - 1 \right] \frac{dr}{|r|^{1+\alpha}} \\ &= \exp \left\{ i\alpha\beta - i\frac{\alpha\pi}{4} \operatorname{sgn} y \right\} \\ &\quad \times 2^{-\alpha/2} \int_{\mathbf{R} \setminus \{0\}} \left[\exp \left\{ ixr - xr \operatorname{sgn} y - 2|y|r^2 \right\} - 1 \right] \frac{dr}{|r|^{1+\alpha}}. \end{aligned}$$

From this it is immediate that $\psi_\alpha(x, y)$ has derivatives of arbitrary order with respect to

x and y if $y \neq 0$. Furthermore $\psi_\alpha(x, y)$ satisfies the following partial differential equation

$$\frac{\partial}{\partial y} \psi_\alpha(x, -y) = i \frac{\partial^2}{\partial y^2} \psi_\alpha(x, -y), \quad (18)$$

$$\psi_\alpha(x, 0) = -2 \frac{\Gamma(1-\alpha)}{\alpha} \cos \frac{\alpha}{2} \pi |x|^\alpha \quad (19)$$

on the regions $(x, y) \in \mathbf{R} \times \mathbf{R}_+$, and $(x, y) \in \mathbf{R} \times \mathbf{R}_-$. Since the analytic properties of the boundary function $\psi_\alpha(x, 0)$ in (19) are not good enough the well-known theorems for the solutions of the Schrödinger equation (18) cannot be applied. Note that the form of the function $\psi_\alpha(x, y)$ given in (4) is analogous to that of the solution of the Cauchy problem, when for the boundary condition the assumptions

$$\begin{aligned} \psi_\alpha(x, 0) &\in C^2(\mathbf{R}^1), & |x|^4 \psi_\alpha(x, y) &\leq M, \\ |x|^4 \frac{\partial}{\partial x} \psi_\alpha(x, 0) &\leq M, & |x|^2 \frac{\partial^2}{\partial x^2} \psi_\alpha(x, 0) &\leq M, & x \in \mathbf{R}^1, \end{aligned}$$

hold (cf. [12, 14.5]).

R e m a r k 3. If instead of the i.i.d. random variables $X_1, X_2, \dots$ we consider the characteristic function of the joint limit distribution of the first and the second order sums of independent series of infinitesimal random variables $\{X_{n_1}, \dots, X_{n_k}\}$, $n = 1, 2, \dots$ ($k_n \rightarrow 0$, $n \rightarrow \infty$) (see [4]), then a form analogous to (2) can be obtained with more general Lévy's spectral function. As a consequence it may hold that the function $\psi_\alpha(x, y)$ satisfies the Schrödinger equation for a more general class of limit distributions, with possibly more complicated boundary functions.

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ON A CHARACTERIZATION OF STOCHASTIC PROCESSES BY THE ABSOLUTE MOMENTS OF STOCHASTIC INTEGRALS

Условие идентичности двух непрерывных по вероятности стохастических процессов с независимыми стационарными симметричными приращениями дано в терминах абсолютных моментов стохастических интегралов.

Ключевые слова и фразы: характеристизация, стохастический интеграл, стохастический процесс, абсолютный момент.

1. Introduction. Braverman [1] gave a condition for the equality of the distributions of two independent symmetric random variables in terms of some absolute moments. Here we extend the result to characterize stochastic processes with independent stationary and symmetric increments. We first state a few definitions. Inter alia, we extend the result of Braverman [1] to random vectors.

Definition 1. A random variable X is said to be *symmetric* if X and $-X$ are identically distributed.

Definition 2. A k -dimensional random vector $\mathbf{X}$ is said to be *symmetric* if $\lambda^t \mathbf{X}$ is symmetric for every $\lambda \in \mathbf{R}^k$.

Suppose $\mathbf{X} \equiv \{X(t), 0 \leq t \leq T\}$ is a stochastic process continuous in probability and with stationary independent symmetric increments. It can be checked that, for any continuous function $\gamma(t)$ defined on $[0, T]$, the stochastic integral

$$S_{\mathbf{X}} \equiv \int_0^T \gamma(t) dX(t) \quad (1)$$

exists in probability and the characteristic function $\psi(\cdot)$ of $S_{\mathbf{X}}$ is given by the relation

$$\log \psi(u) = \int_0^T \varphi(u\gamma(t)) dt,$$

where $\varphi(\cdot)$ is the log of the characteristic function of $X(t+1) - X(t)$. Note that the characteristic function of $X(t+1) - X(t)$ is a real-valued function by the symmetry of $X(t+1) - X(t)$, non-zero by the infinite divisibility of $X(t+1) - X(t)$ and does not depend on t by the stationarity of the increments of the process $\mathbf{X}$. In particular $\psi(\cdot)$ is

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