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ON PIECEWISE LINEAR APPROXIMATION FOR
NON-LINEAR STOCHASTIC EVOLUTION

Предлагается нелинейная модель эволюции случайных полей, основанная на аппроксимации кусочно-линейными дифференциальными уравнениями в частных производных со стохастическими граничными условиями.

Ключевые слова и фразы: обобщенные случайные поля, нелинейное эволюционное уравнение, стохастические граничные условия, соболевские суперпространства.

We have in mind a generalized random field

$$\xi(t) = \{(\varphi, \xi(t)), \varphi \in C_0^\infty(G)\}$$

in a region $G \subseteq \mathbf{R}^d$ such that, with time $t \in \mathbf{R}^1$ going, its evolution goes according to non-linear stochastic differential equation

$$d\xi(t) = A(t, \xi^t) dt + B(t, \xi^t) d\eta(t) \quad (1)$$

where drift and diffusion operators $A(t, \xi^t)$, $(B(t, \xi^t))$ depend on a whole «past» $\xi^t = \{\xi(s), s \leq t\}$, so

$$(\varphi, \xi(t)) = (\varphi, \xi(t_0)) + \int_{t_0}^t (\varphi, A(s, \xi^s)) ds + \int_{t_0}^t (\varphi, B(s, \xi^s) d\eta(s)), \quad t \geq t_0, \quad (2)$$

for all test functions $\varphi \in C_0^\infty(G)$. A problem is on *stochastic boundary conditions* for the corresponding random field $\xi = \{\xi(t), t \in I\}$ over a space-time cylinder $G \times I$, $I = (t_0, t_*)$, which determine $\xi \in \mathbf{W}(G \times I)$ as a unique solution from an appropriate functional class $\mathbf{W}(G \times I)$.

Suppose, the evolution goes in a such way that approximately we have

$$d\xi(t) = A\xi(t) dt + B d\eta(t) \quad (3)$$

on any sufficiently small time interval $I_{\Delta s}$: $s < t < s + \Delta s$, where A, B are appropriate linear operators, depending on the past ξ^s up to a «current» time moment s . Then, we can approach the boundary problem by means of a piecewise linear approximation for ξ , taking a partition $t_0 < t_1 < \dots < t_{n+1} = t_*$ of the interval $I = (t_0, t_*)$ and assuming that the evolution goes according to equation (3) on every subinterval (t_j, t_{j+1}) with the operators $A = A_j$, $B = B_j$ which depend on the past ξ^{t_j} up to the corresponding time moment t_j , $j = 0, \dots, n$. In this way, we consider just a case when for every time interval $I_{\Delta t_j} = (t_j, t_{j+1})$ we have $A = A_j$ as a non-degenerate elliptic operator $A = \sum_{|k| \leq 2p} a_k \partial^k$ and

$$\eta = \left\{ (\varphi, \eta) \stackrel{\text{def}}{=} \int_{I_{\Delta t_j}} (B^* \varphi(\cdot, t), d\eta(t)), \varphi \in C_0^\infty(G \times I_{\Delta t_j}) \right\} \quad (4)$$

as a generalized random field which is *meansquare continuous* over test functions $\varphi \in C_0^\infty(G \times I_{\Delta t_j})$ with respect to the L_2 -norm $\|\varphi\|_{L_2}$. In this case we look for $\xi \in \mathbf{W}(G \times I)$ from the corresponding *Sobolev super-space*

$$\mathbf{W}(G \times I) = \mathbf{W}_2^{(2p, 1)}(G \times I)$$

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with its representatives

$$\xi = \left\{ (\varphi, \eta) \stackrel{\text{def}}{=} \int_I (\varphi(\cdot, t), \xi(t)) dt, \varphi \in C_0^\infty(G \times I) \right\}$$

defined as generalized random fields which are *meansquare continuous* over test functions $\varphi \in C_0^\infty(G \times I)$ with respect to the $W_2^{-(2p,1)}$ -norm $\|\varphi\|_{-(2p,1)}$; recall, this norm can be determined by means of Fourier transform $\tilde{\varphi}$ as

$$\|\varphi\|_{-(2p,1)}^2 = \iint_{\mathbf{R}^d \times \mathbf{R}^1} |\tilde{\varphi}(\lambda, \mu)|^2 (1 + |\lambda|^{4p} + |\mu|^2)^{-1} d\lambda d\mu. \quad (5)$$

According to its meansquare continuity, we can treat $\xi \in \mathbf{W}_2^{(2p,1)}(G \times I)$ as a generalized random field

$$\xi = \left\{ (x, \xi), x \in X(G \times I) \right\} \quad (6)$$

over all test functions $x = \{(x, u), u \in C_0^\infty(\mathbf{R}^d \times \mathbf{R}^1)\}$ which are Schwartz distributions on $\mathbf{R}^d \times \mathbf{R}^1$ from the corresponding Sobolev space

$$X(G \times I) = \mathbf{W}_2^{-(2p,1)}(G \times I) = [C_0^\infty(G \times I)]$$

appearing as a closure $[C_0^\infty(G \times I)]$ of Schwartz test functions $\varphi \in C_0^\infty(G \times I)$ with respect to norm (5).

Considering evolution equation (3) on any time interval $I_{\Delta t_j} = (t_j, t_{j+1})$, $j = 0, \dots, n$, we have $\xi \in \mathbf{W}_2^{(2p,1)}(G \times I_{\Delta t_j})$ with a variety of generalized *boundary values*; in particular, on bottom and top $(G \times \{t_j\}) \cup (G \times \{t_{j+1}\})$ of the cylinder $G \times I_{\Delta t_j}$, we can get

$$(\varphi, \xi(t)) \stackrel{\text{def}}{=} (x, \xi), \quad x = \varphi \times \delta_t, \quad \varphi \in C_0^\infty(G), \quad (7)$$

by *boundary test functions* $x = \varphi \times \delta_t$ with delta-functions δ_t on $\mathbf{R}^1$ at the end points $t = t_j, t_{j+1}$ of $I_{\Delta t_j}$, acting as

$$(x, u) = (\varphi, u(\cdot, t)) \quad (8)$$

on $u \in C_0^\infty(\mathbf{R}^d \times \mathbf{R}^1)$, and on side boundary $\partial G \times I_{\Delta t_j}$ of $G \times I_{\Delta t_j}$, we can get boundary values

$$(\varphi, \xi^{(k)}) \stackrel{\text{def}}{=} (x, \xi), \quad \varphi \in C_0(\partial G \times I_{\Delta t_j}), \quad (9)$$

by *boundary test functions* $x = (-1)^k (\varphi \times \delta^{(k)})$ with non-tangent derivatives $\delta^{(k)}$, $k = 0, \dots, 2p-1$, of delta-functions on $\mathbf{R}^d$ at points $r \in \partial G$ of a boundary ∂G , acting as

$$(x, u) = \iint_{\partial G \times I_{\Delta t_j}} \varphi(r, t) u^{(k)}(r, t) dr dt \quad (10)$$

on $u \in C_0^\infty(\mathbf{R}^d \times \mathbf{R}^1)$. Note, that (8), (10) represent a complete system of *boundary test functions* $x \in X(G \times I_{\Delta t_j})$, $\text{supp } x \in \Gamma_j$, on the cylinder $G \times I_{\Delta t_j}$ boundary

$$\Gamma_j = (G \times \{t_j\}) \cup (\partial G \times I_{\Delta t_j}) \cup (G \times \{t_{j+1}\}).$$

There are various types of *stochastic boundary conditions*

$$\sum_k (\varphi_k, \xi^{(k)}) = \sum_k (\varphi_k, \xi_+^{(k)}), \quad (11)$$

which involve certain combinations of generalized non-tangent derivatives $\xi^{(k)}$, $k = 0, \dots, 2p-1$, on the side boundary $\partial G \times I_{\Delta t_j}$ by means of certain collections of

$$\left\{ \varphi_k \in C_0^\infty(\partial G \times I_{\Delta t_j}), k = 0, \dots, 2p-1 \right\},$$

such that, with these conditions given arbitrarily — with an arbitrary stochastic boundary sample $\xi_+ \in W_2^{(2p,1)}(G \times I_{\Delta t_j})$ — we have a unique solution $\xi \in W_2^{(2p,1)}(G \times I_{\Delta t_j})$ of equation (3) on the time interval $I_{\Delta t_j} = (t_j, t_{j+1})$ with the arbitrary «initial values» $\xi(t_j) = \{(\varphi, \xi(t_j)), \varphi \in C_0^\infty(G)\}$ fixed. For example, suppose we deal with a symmetric elliptic operator $A \leq 0$ and consider one of its self-adjoint extensions with a domain $D_A \subseteq W_2^{2p}(G)$ determined by boundary conditions of a type $\sum_{k=0}^{2p-1} (f_k, u^{(k)}) = 0$, $u \in D_A$, which involve some combinations of normal derivatives $u^{(k)}$ by means of certain collections $\{f_k \in C_0(\partial G), k = 0, \dots, 2p-1\}$; then we can set arbitrarily stochastic boundary conditions (11) by means of the corresponding collections

$$\{\varphi_k(r, t) = f_k(r) \times g_k(t), g_k \in C_0(I_{\Delta t_j}), k = 0, \dots, 2p-1\}$$

— to illustrate, we refer just on *Dirichlet type stochastic boundary conditions* which can be given in the whole as $\xi_+^{(k)} = \xi_+^{(k)}$, $k = 0, \dots, p-1$, by means of the generalized normal derivatives — see [1], [3].

Thus, having in mind the certain stochastic boundary conditions of type (11), we can set them for every time interval $I_{\Delta t_j} = (t_j, t_{j+1})$ depending on the past $\xi^{t_j} = \{\xi(s), s \leq t_j\}$, and correspondingly, with the initial $\xi(t_j)$, get a unique solution $\xi = \xi_j \in W_2^{(2p,1)}(G \times I_{\Delta t_j})$ of equation (3) on $I_{\Delta t_j} = (t_j, t_{j+1})$.

For $j = 0, \dots, n$ these very equations mean that

$$(\varphi, \xi(t)) - (\varphi, \xi(t_j)) = \int_{t_j}^t (A^* \varphi, \xi(s)) ds + \int_{t_j}^t (B^* \varphi, d\eta(s)),$$

$$\varphi \in C_0^\infty(G), \quad t_j \leq t \leq t_{j+1};$$

obviously, we can stick them one to another and get the corresponding equation (2) with

$$\left. \begin{aligned} (\varphi, A(t, \xi^t)) &= (A_j^* \varphi, \xi(t)) \\ (\varphi, B(t, \xi^t) d\eta(t)) &= (B_j^* \varphi, d\eta(t)) \end{aligned} \right\} \quad t_j \leq t < t_{j+1}, \quad j = 0, \dots, n, \quad (12)$$

for all test functions $\varphi \in C_0^\infty(G)$.

Theorem. A piecewise formula

$$\xi(t) = \xi_j(t), \quad t_j \leq t < t_{j+1}, \quad j = 0, \dots, n, \quad (13)$$

determines a unique solution $\xi \in W_2^{(2p,1)}(G \times I)$ of equation (1) with the drift and diffusion operators of type (12).

To prove it, we consider ξ as a linear continuous operator $\xi: \varphi \in W_2^{-(2p,1)}(G \times I) \rightarrow H \ni (\varphi, \xi)$ to a Hilbert space H of random variables h , $E|h|^2 < \infty$, with a standard mean-square norm $\|h\| = (E|h|^2)^{1/2}$. Then, its continuity boundedness is equivalent to continuity boundedness of scalar linear functionals

$$u = \{(\varphi, u) \stackrel{\text{def}}{=} E(\varphi, \xi) \bar{h}, \varphi \in W_2^{-(2p,1)}(G \times I)\}$$

for all $h \in H$. In its turn, the latter one is equivalent to a condition $u \in W_2^{(2p,1)}(G \times I)$ if the boundary ∂G is regular in a way that any function $u \in W_2^{(2p,1)}(G \times I)$ on the cylinder $G \times I$ can be extended up to a function $u \in W_2^{(2p,1)}(\mathbf{R}^d \times \mathbf{R}^1)$ from the same type Sobolev space on the whole $\mathbf{R}^d \times \mathbf{R}^1$. As is known, the very Sobolev space $W_2^{(2p,1)}(G \times I)$ can be characterized by the following property of its representatives: as functions of any space coordinate in $G \subseteq \mathbf{R}^d$ (for almost all other coordinates in $G \times I$) they have absolutely continuous derivatives $\partial^k u$, $k = 0, \dots, 2p-1$, with $\partial^{2p} u \in L_2(G \times I)$, and as functions of $t \in I$ (for almost all other coordinates in G) they are absolutely continuous with

$\partial u / \partial t \in L_2(G \times I)$. Obviously, this characterization holds true for functions defined as $u(\cdot, t) = u_j(\cdot, t)$, $t_j \leq t < t_{j+1}$, $j = 0, \dots, n$, with $u_j \in W_2^{(2p,1)}(G \times I_{\Delta t_j})$ such that $u_j(\cdot, t_{j+1}) = u_{j+1}(\cdot, t_{j+1})$, $j = 0, \dots, n-1$.

Hence, in (13) we have $\xi \in W_2^{(2p,1)}(G \times I)$, since $\xi_j \in W_2^{(2p,1)}(G \times I_{\Delta t_j})$ with $\xi_j(t_{j+1}) = \xi_{j+1}(t_{j+1})$, $j = 0, \dots, n-1$. And in (13) we have the unique solution of equation (1) from the Sobolev superspace $W_2^{(2p,1)}(G \times I)$, since its parts $\xi_j(t) = \xi(t)$, $t_j \leq t \leq t_{j+1}$, represent unique solutions from the corresponding $W_2^{(2p,1)}(G \times I_{\Delta t_j})$, $j = 0, \dots, n$. Thus, the proof is over.

Having in mind a sequence of approximations (13) for a certain non-linear evolution model, we evaluate a difference of one from another. For this, we consider just a piece of these approximations on an arbitrary time interval $I_{\Delta s} = (s, s + \Delta s)$, where they obey equation (3) with the corresponding elliptic operators $A = \sum_{|k| \leq 2p} a_k \partial^k \leq 0$ with constant (real) coefficients such that

$$0 < e \leq \frac{|\sum_{|k| \leq 2p} a_k (i\lambda)^k|^2}{|\lambda|^{4p}} \leq C < \infty, \quad |\lambda| \geq 1 \quad (i = \sqrt{-1}). \quad (14)$$

Considering stochastic boundary conditions (11), we assume that its type is fixed for the whole time interval $I = (t_0, t_*)$, i.e., the corresponding system of boundary test functions $x = \sum_k (-1)^k (\varphi_k \times \delta^{(k)})$ is fixed. The very type of this system can be characterized in the following way: a deterministic boundary problem

$$\begin{cases} \frac{\partial u}{\partial t} = Au + v, \\ \sum_k (\varphi_k, u^{(k)}) = 0 \end{cases} \quad (15)$$

has a unique solution $u \in W_2^{(2p,1)}(G \times I)$, $u(t_0) = 0$, for any $v \in L_2(G \times I)$. Equivalently, the considered system of boundary test functions $x = \sum_k (-1)^k \times (\varphi_k \times \delta^{(k)})$ on the side boundary $\partial G \times I$ is such that, altogether with the boundary test functions $x = \varphi \times \delta_{t_0}$, $\varphi \in C_0^\infty(G)$, on the bottom boundary $G \times \{t_0\}$ of the cylinder $G \times I$, it generates a direct complement X_+ to a subspace $\mathcal{L}^* L_2(G \times I)$ in the test functions space $X(G \times I) = W_2^{-(2p,1)}(G \times I)$ — see Fig. 1,

Fig. 1

where $\mathcal{L}^* = -\partial/\partial t - A$ is formally adjoint operator to the differential operator $\mathcal{L} = \partial/\partial t - A$ in (15).

With condition (14), we have a strictly positive angle $\alpha \geq \alpha_0 > 0$ for any operator $\mathcal{L}^* = -\partial/\partial t - A$ which can appear — see Fig. 1; recall,

$$\cos \alpha = \frac{\sup \langle \mathcal{L}f, x \rangle}{\|\mathcal{L}^* f\|_{-(2p,1)}, \|x\|_{-(2p,1)}}$$

is defined by means of a scalar product $\langle \cdot, \cdot \rangle$ in $X = W_2^{-(2p,1)}(G \times I)$; also, we have strictly bounded norms $\|\mathcal{L}^*\|_{L_2}, \|\mathcal{L}^{*-1}\|_{W_2^{-(2p,1)}} \leq C_0$ for operators $\mathcal{L}^*: f \in L_2(G \times I) \rightarrow X = W_2^{-(2p,1)}(G \times I)$ — see [2], [3].

Obviously, the very similar picture holds for any sub-cylinder $G \times I_{\Delta s}$, $I_{\Delta s} \subseteq I$ (at least, with the same constants α_0, C_0).

Note that, according to this picture, for any sub-cylinder $G \times I_{\Delta s}$ we can obtain a solution $\xi \in W_2^{(2p,1)}(G \times I_{\Delta s})$ of the corresponding stochastic boundary problem as

$$(\varphi, \xi) = (f, \eta) + (x, \xi_+), \quad \varphi \in C_0^\infty(G \times I_{\Delta s}), \quad (16)$$

by means of a decomposition $\varphi = \mathcal{L}f + x$ with $f \in L_2$, $x \in X_+$, — see Fig. 1.

For the space-time sub-cylinder $G \times I_{\Delta s}$, let

$$\partial A = \sum_{|k| \leq 2p} (\partial a_k) \partial^k, \quad \delta \eta, \delta \xi_+ \quad (17)$$

be differences of the drift operators in (3), stochastic sources (4), stochastic boundary samples in boundary conditions (11). Let us introduce

$$\begin{aligned} \|\delta A\| &= \max_k |\delta a_k|, \quad \|\delta \eta\|_{L_2} = \sup_{f \in L_2: \|f\|_{L_2} \leq 1} (\mathbb{E}|(f, \eta)|^2)^{1/2}, \\ \|\delta \xi_+\|_{W_2^{(2p,1)}} &= \sup_{x \in X_+: \|x\|_{-(2p,1)} \leq 1} (\mathbb{E}|(x, \delta \xi_+)|^2)^{1/2}. \end{aligned}$$

Then, as a particular case of general estimate of perturbations in solutions of stochastic partial differential equations — see [2], we can evaluate a quantity

$$\|\delta \xi\|_{W_2^{(2p,1)}} = \sup_{\varphi \in W_2^{-(2p,1)}(G \times I_{\Delta s}): \|\varphi\|_{-(2p,1)} \leq 1} (\mathbb{E}|(\varphi, \delta \xi)|^2)^{1/2}$$

by means of (17) as follows.

Lemma. *The estimate*

$$\|\delta \xi\|_{W_2^{(2p,1)}} \leq C \left(\|\delta A\| \cdot \|\eta\|_{L_2} + \|\delta \eta\|_{L_2} + \|\delta \xi_+\|_{W_2^{(2p,1)}} \right) \quad (18)$$

holds with a constant C which does not depend on the interval $I_{\Delta s} \subseteq I$.

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