A. Sh. Shukurov, About one type of sequences that are not a Schauder basis in Hilbert spaces, *Vestn. Udmurtsk. Univ. Mat. Mekh. Komp. Nauki*, 2015, Volume 25, Issue 2, 244–247

Use of the all-Russian mathematical portal Math-Net.Ru implies that you have read and agreed to these terms of use
<http://www.mathnet.ru/eng/agreement>

Download details:

IP: 52.23.180.231

April 26, 2023, 14:52:03

MSC: 46A35, 46B15, 46C05

© A. Sh. Shukurov

ABOUT ONE TYPE OF SEQUENCES THAT ARE NOT A SCHAUDER BASIS IN HILBERT SPACES

Let H be a Hilbert space and a (not necessarily bounded) sequence of its elements $\{e_n\}_{n=1}^\infty$ has a bounded subsequence $\{e_{n_k}\}_{k=1}^\infty$ such that $|(e_{n_k}, e_{n_m})| \geq \alpha > 0$ for all sufficiently large $k, m \in N, k \neq m$. It is proved that such a sequence $\{e_n\}_{n=1}^\infty$ is not a basic sequence and thus is not a Schauder basis in H . Note that the results of this paper generalize and offer a short and more simple proof of some recent results obtained in this direction.

Keywords: Schauder basis, basic sequence, Hilbert space, orthonormal sequence and orthonormal basis, weakly convergent sequences.

Introduction

We begin by recalling some notions.

Definition 1 (see [1, 2, 3]). A sequence of vectors $\{x_n\}_{n=1}^\infty$ in an infinite-dimensional Banach space X is said to be a *Schauder basis* for X if to each vector x in the space there corresponds a unique sequence of scalars $\{\alpha_n\}_{n=1}^\infty$ such that

$$x = \alpha_1 \cdot x_1 + \dots + \alpha_n \cdot x_n + \dots$$

Definition 2 (see [3]). A sequence of vectors $\{x_n\}_{n=1}^\infty$ in a Banach space X is said to be a *basic sequence* if it is a Schauder basis for the closure of its linear span.

It is well known that every separable Hilbert space possesses an orthonormal Schauder bases, i. e. a Schauder basis $\{e_n\}_{n=1}^\infty$ for which $\|e_n\| = 1$ and $(e_n, e_m) = 0$ for every $n, m \in N, n \neq m$. Besides it, it can easily be shown that every sequence $\{e_n\}_{n=1}^\infty$ of elements in any Hilbert space with the properties $\|e_n\| = 1$ and $(e_n, e_m) = 0$ for every $n, m \in N, n \neq m$, is a basic sequence in this space. It is easy to see that the number 1 in this formulation can easily be replaced by the other number by retaining the mentioned property. Therefore, it is natural to ask what would happen if the number 0 is replaced by the another number. First result in this direction (that is known to us) is obtained in [4].

Theorem 1 (see [4]). Let H be a Hilbert space and $\{x_n\}_{n=1}^\infty$ be a sequence of elements in H with the following properties:

- 1) $\|x_n\| = 1$ for all $n \in N$; 2) $(x_n, x_m) = a, 0 < |a| < 1; n, m \in N, n \neq m$.

Then $\{x_n\}_{n=1}^\infty$ is not a basic sequence in H .

The next result in this direction is the following.

Theorem 2 (see [5]). Let H be a Hilbert space and $\{x_n\}_{n=1}^\infty$ be a complete sequence of elements in H with the following properties:

- 1) $\|x_n\| = 1$ for all $n \in N$; 2) $(x_n, x_m) \geq a > 0, n, m \in N, n \neq m$.

Then $\{x_n\}_{n=1}^\infty$ is not a Schauder basis in H .

Some time later Sadybekov and Sarsenby [6], investigating the unconditional basicity of sequences, obtained the following result, where they require from the sequence to be almost normalized instead of being normalized.

Theorem 3 (see [6]). *Let H be a Hilbert space and $\{x_n\}_{n=1}^\infty$ be a complete, minimal and an almost normalized sequence of elements in H with the property $|(x_n, x_m)| \geq a > 0$ for all sufficiently large numbers n and m . Then $\{x_n\}_{n=1}^\infty$ is not an unconditional basis in H .*

In this note we state a result that generalizes all the mentioned results and, besides it, offers a short and simpler proof of these results.

§ 1. Main result and its proof

Theorem 4. *Let H be a Hilbert space and a bounded sequence of its elements $\{e_n\}_{n=1}^\infty$ satisfies $|(e_n, e_m)| \geq \alpha > 0$ for $n, m \in N, n \neq m$. Then $\{e_n\}_{n=1}^\infty$ is not a basic sequence (and thus a Schauder basis) in H .*

P r o o f. Assume the contrary: the sequence $\{e_n\}_{n=1}^\infty$ satisfies the conditions of the theorem and is a basic sequence.

Since $\{e_n\}_{n=1}^\infty$ is assumed to be bounded, it has a weakly convergent subsequence $\{e_{n_k}\}_{k=1}^\infty$ (see, for example, [7, p. 81]); let x_0 be its weak limit. It is known that every subsequence of a basic sequence is also a basic sequence (it follows, for example, from [3, Theorem 1.1']). Therefore, the subsequence $\{e_{n_k}\}_{k=1}^\infty$ is also a basic sequence. Hence, it has a biorthogonal system, i. e. a sequence of elements $\{b_k\}_{k=1}^\infty$ such that

$$(b_k, e_{n_m}) = \delta_{km}, \quad (1)$$

where δ_{km} is a Kronecker symbol. Here, passing to the limit as $m \rightarrow \infty$ and taking into account that x_0 is a weak limit of $\{e_{n_m}\}_{m=1}^\infty$, we obtain that

$$(b_k, x_0) = 0 \quad \forall k \in N. \quad (2)$$

By the condition of the theorem we have $|(e_{n_k}, e_{n_m})| \geq a > 0$ for all $k, m \in N, n \neq m$. Therefore, taking into account that x_0 is a weak limit of $\{e_{n_m}\}_{m=1}^\infty$, by passing to the limit at first as $k \rightarrow \infty$ and then passing to the limit as $m \rightarrow \infty$, we obtain that $|(x_0, x_0)| > 0$. This relation implies that $x_0 \neq \theta$.

Now, since the weak limit of a sequence of elements lies in the closure of its linear span (see, for example, [7, p. 81] or [1, p. 216]), x_0 must have a representation

$$x_0 = \alpha_1 \cdot e_{n_1} + \alpha_2 \cdot e_{n_2} + \dots + \alpha_k \cdot e_{n_k} + \dots \quad (3)$$

We find from here that

$$(b_k, x_0) = \alpha_1 \cdot (b_k, e_{n_1}) + \alpha_2 \cdot (b_k, e_{n_2}) + \dots + \alpha_k \cdot (b_k, e_{n_k}) + \dots$$

for all $k \in N$. From here, by using (1) and (2), we obtain that

$$\alpha_k = 0 \quad \forall k \in N.$$

These relations and (3) imply that $x_0 = \theta$. But this contradicts to the fact that $x_0 \neq \theta$. The obtained contradiction shows that our assumption is false. The theorem is proved. $\square$

§ 2. Concluding remarks

As was already mentioned, if a sequence is a basic sequence then every subsequence of this original sequence is also a basic sequence (it follows, for example, from [3, Theorem 1.1']). This observation and the proof of the theorem from the previous section imply that the following more general result holds true.

Theorem 5. *Let H be a Hilbert space and a (not necessarily bounded) sequence of its elements $\{e_n\}_{n=1}^\infty$ has a bounded subsequence $\{e_{n_k}\}_{k=1}^\infty$ such that $|(e_{n_k}, e_{n_m})| \geq \alpha > 0$ for all sufficiently large $k, m \in N, k \neq m$. Then $\{e_n\}_{n=1}^\infty$ is not a basic sequence (and thus a Schauder basis) in H .*

REFERENCES

1. Lyusternik L.A., Sobolev V.I. *Elementy funktsional'nogo analiza* (Elements of functional analysis), Moscow: Nauka, 1965.
2. Young R.M. *An introduction to nonharmonic Fourier series*, New York: Academic Press, 1980.
3. Mil'man V.D. Geometric theory of Banach spaces. Part I. The theory of basis and minimal systems, *Russian Mathematical Surveys*, 1970, vol. 25, issue 3, pp. 111–170.
4. Khmyleva T.E., Bukhtina I.P. On some sequence of Hilbert space elements, which is not basis, *Vestn. Tomsk. Gos. Univ. Mat. Mekh.*, 2007, no. 1, pp. 58–62 (in Russian).
5. Khmyleva T.E., Ivanova O.G. On some systems of a Hilbert space which are not bases, *Vestn. Tomsk. Gos. Univ. Mat. Mekh.*, 2010, no. 3 (11), pp. 53–60 (in Russian).
6. Sadybekov M.A., Sarsenbi A.M. On a necessary condition for a system of normalized elements to be a basis in a Hilbert space, *Vestn. Tomsk. Gos. Univ. Mat. Mekh.*, 2011, no. 1 (13), pp. 44–46 (in Russian).
7. Dunford N., Schwartz J. *Lineinye operatory. Obshchaya teoriya* (Linear operators. General theory), Moscow: Izd. Inostr. Lit., 1962, 895 p.

Received 01.04.2015

Shukurov Aydin Shukur, PhD in Mathematics, Research Fellow, Department of Non-harmonic Analysis, Institute of Mathematics and Mechanics, Azerbaijan National Academy of Sciences, ul. B. Vahabzade, 9, Baku, AZ1141, Azerbaijan.
E-mail: ashshukurov@gmail.com

А. Ш. Шукүров**Об одном классе последовательностей, не являющихся базисом Шаудера в гильбертовом пространстве**

Ключевые слова: базис Шаудера, базисная последовательность, гильбертово пространство, ортонормированная последовательность и ортонормированный базис, слабо сходящиеся последовательности.

УДК 517.982

Пусть H — гильбертово пространство и (необязательно ограниченная) последовательность $\{e_n\}_{n=1}^{\infty}$ его элементов содержит ограниченную подпоследовательность $\{e_{n_k}\}_{k=1}^{\infty}$ такую, что $|(e_{n_k}, e_{n_m})| \geq \alpha > 0$ для любых достаточно больших $k, m \in N, k \neq m$. Доказано, что такая последовательность $\{e_n\}_{n=1}^{\infty}$ не является базисной последовательностью и, следовательно, базисом Шаудера в пространстве H . Полученные результаты обобщают и предлагают короткое и более простое доказательство некоторых недавних результатов, полученных в этом направлении.

СПИСОК ЛИТЕРАТУРЫ

1. Люстерник Л.А., Соболев В.И. Элементы функционального анализа. М.: Наука, 1965.
2. Young R.M. *An introduction to nonharmonic Fourier series*. New York: Academic Press, 1980.
3. Мильман В.Д. Геометрическая теория пространств Банаха. Часть. I. Теория базисных и минимальных систем // Успехи математических наук. 1970. Т. 25. Вып. 3 (153). С. 113–174.
4. Хмылева Т.Е., Бухтина И.П. О некоторой последовательности элементов в гильбертовом пространстве, не являющейся базисом // Вестник Томского государственного университета. Математика и механика. 2007. № 1. С. 58–62.
5. Хмылева Т.Е., Иванова О.Г. О некоторых системах в гильбертовом пространстве, не являющихся базисом // Вестник Томского государственного университета. Математика и механика. 2010. № 3. С. 53–60.
6. Садыбеков М.А., Сарсенби А.М. Об одном необходимом условии базисности системы нормированных элементов в гильбертовом пространстве // Вестник Томского государственного университета. Математика и механика. 2011. № 1 (13). С. 44–46.
7. Данфорд Н., Шварц Дж. Линейные операторы: общая теория. Москва: Изд-во иностр. лит., 1962. 896 с.

Поступила в редакцию 01.04.2015

Шукюров Айдын Шукюр оглы, доктор философии по математике, научный сотрудник, отдела негармонического анализа, Институт математики и механики НАН Азербайджана, AZ1141, Азербайджан, г. Баку, ул. Б. Вагабзаде, 9.

E-mail: ashshukurov@gmail.com