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ON A SUBCLASS OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS DEFINED BY LINEAR OPERATOR

The present paper introduces and studies the subclass $A_n(m, \beta, p, q, \lambda)$ of univalent functions with negative coefficients defined by new linear operator J^λ in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. The main task is to investigate several properties such as coefficient estimates, distortion theorems, closure theorems. Neighborhood and radii of starlikeness, convexity and close-to-convexity of functions belonging to the class $A_n(m, \beta, p, q, \lambda)$ are studied.

Keywords: analytic univalent function, Hadamard product, Ruscheweyh derivative, distortion theorems, closure theorems.

§ 1. Introduction

Let $A(n)$ denote the class of functions normalized by

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (n \in N = 1, 2, 3, \dots), \quad (1.1)$$

which are analytic and univalent in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $A^-(n)$ be the subclass of $A(n)$, consisting of functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (n \in N, \quad a_k \geq 0). \quad (1.2)$$

For functions $f(z) \in A(n)$ given by (1.1) and $g(z) \in A(n)$ given by

$$g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k \quad (n \in N),$$

we define the Hadamard product (or convolution) of $f(z)$ and $g(z)$ by

$$f(z) * g(z) = z + \sum_{k=n+1}^{\infty} a_k b_k z^k \quad (z \in \mathcal{U}). \quad (1.3)$$

We define the function $\Phi(a, c; z)$ by

$$\Phi(a, c; z) = z + \sum_{k=n+1}^{\infty} \frac{(a)_{k-1}}{(b)_{k-1}} z^k \quad (a \in R, \quad c \in R - \{0, -1, -2, \dots\}),$$

where $(a)_k$ is the Pochhammer symbol defined by

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = \begin{cases} 1, & \text{if } k = 0, \\ a(a+1)(a+2) \cdots (a+k-1), & \text{if } k \in N. \end{cases}$$

We consider a functions $\Phi^-(a, c; z)$ defined by the convolution

$$\Phi(a, c; z) * \Phi^-(a, c; z) = \frac{z}{(1-z)^{\lambda+1}},$$

where $\lambda > -1, z \in \mathcal{U}$. This function yields the following family of linear operators

$$I^\lambda(a, c; z)f(z) = \Phi^-(a, c; z) * f(z), \quad z \in \mathcal{U},$$

where $a, c \in -\{0, -1, -2, \dots\}$. For a function $f \in A^-(n)$, it follows that for $\lambda > -1$

$$I^\lambda(a, c; z)f(z) = z - \sum_{k=n+1}^{\infty} \frac{(c)_{k-1}(\lambda+1)_{k-1}}{(1)_{k-1}(a)_{k-1}} a_k z^k$$

is the Cho–Kown–Srivastava integral operator [11]. Further, denote by $D^\lambda : A(n) \rightarrow A(n)$ the Ruscheweyh derivative of order λ defined by [8, 13, 14].

$$D^\lambda f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z) = \frac{z(z^{\lambda-1}f(z))^\lambda}{\lambda!} \quad (f(z) \in A(n)),$$

where $\lambda > -1$. Now, for $f \in A^-(n)$ we defined a new operator [4]

$$J^\lambda f(z) = I^\lambda(a, c; z)f(z) * D^\lambda f(z) = z - \sum_{k=n+1}^{\infty} \frac{(c)_{k-1}(\lambda+1)_{k-1}}{(1)_{k-1}(a)_{k-1}} B_k(\lambda) a_k z^k, \quad (1.4)$$

where $B_k(\lambda) = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda+1)(k-1)!}$, $\lambda > -1, z \in \mathcal{U}$. Next, we define the following

Definition 1. Let the function $f(z)$ be of the form (1.2). Then $f(z)$ is said to be in the class $A_n(m, \beta, p, q, \lambda)$ if it satisfies the following criterion

$$\left| \frac{\frac{z(J^\lambda f(z))'}{J^\lambda f(z)} - 1}{2p \left(\frac{z(J^\lambda f(z))'}{J^\lambda f(z)} - q \right) - m \left(\frac{z(J^\lambda f(z))'}{J^\lambda f(z)} - 1 \right)} \right| < \beta, \quad (1.5)$$

where $|z| < 1, 0 < \beta \leq 1, \frac{1}{2} \leq p \leq 1, 0 \leq q \leq \frac{1}{2}p$, and $\frac{1}{2} < m \leq 1$.

Normalized, univalent analytic functions have been extensively studied by (for example) Aghalary and Kulkarni [1], Aouf [2], Khairnar and Meena More [6], Juma and Kulkarni [5].

Remark 1. We note that by specializing the parameters m, β, p, q and λ we have the following subclasses.

- (i) If $f(z) \in A_n(m, \beta, p, q, \lambda)$ with $a = c = 1$ and $\lambda = 0$ in (1.4), then $J^\lambda f(z) = f(z)$.
- (ii) The class $A_n(1, 1, 1, 0, \lambda)$ is the class of starlike functions in $\mathcal{U}$.
- (iii) The class $A_n(1, 1, 1, q, \lambda)$ is the class of starlike functions of order q ($0 \leq q < 1$).
- (iv) The class $A_n(1, \beta, \frac{(\alpha+1)}{2}, 0, \lambda)$ is the class studied by Lakshminarasimhan [9].
- (v) The class $A_n(1, \beta, p, q, \lambda)$ is the class studied by S. R. Kulkarni [7].

In this paper we shall first deduce a necessary and sufficient condition for a function $f(z)$ to be in the class $A_n(m, \beta, p, q, \lambda)$. Then we obtain the distortion and growth theorems, closure theorems, neighborhood and radii of univalent starlikeness, convexity and close-to-convexity of order δ ($0 \leq \delta < 1$) for this functions.

§ 2. Coefficient inequality

Theorem 1. Let the function $f(z)$ be of the form (1.2). Then $f(z)$ is in the class $A_n(m, \beta, p, q, \lambda)$ if and only if

$$\sum_{k=n+1}^{\infty} [(k-1)(1-m\beta) - 2p\beta(k-q)] \frac{(c)_{k-1}(\lambda+1)_{k-1} B_k(\lambda)}{(1)_{k-1}(a)_{k-1}} a_k \leq 2p\beta(1-q), \quad (2.1)$$

where $0 < \beta \leq 1, \frac{1}{2} \leq p \leq 1, 0 \leq q \leq \frac{1}{2}p, \frac{1}{2} < m \leq 1, a, c \in \mathbb{R} - \{0, -1, -2, \dots\}, z \in \mathcal{U}$.

P r o o f. Suppose that (2.1) holds true. Then we find that

$$\begin{aligned} & \left| \frac{\frac{z(J^\lambda f(z))' - 1}{J^\lambda f(z)} - 1}{2p\left(\frac{z(J^\lambda f(z))' - 1}{J^\lambda f(z)} - q\right) - m\left(\frac{z(J^\lambda f(z))' - 1}{J^\lambda f(z)} - 1\right)} \right| = \\ &= \left| \frac{-\sum_{k=n+1}^{\infty} (k-1) \frac{(c)_{k-1}(\lambda+1)_{k-1} B_k(\lambda)}{(1)_{k-1}(a)_{k-1}} a_k z^k}{2p(1-q)z + \sum_{k=n+1}^{\infty} [m(k-1) - 2p(k-q)] \frac{(c)_{k-1}(\lambda+1)_{k-1} B_k(\lambda)}{(1)_{k-1}(a)_{k-1}} a_k z^k} \right| \leq \\ &\leq \frac{\sum_{k=n+1}^{\infty} (k-1) \frac{(c)_{k-1}(\lambda+1)_{k-1} B_k(\lambda)}{(1)_{k-1}(a)_{k-1}} a_k |z|^{k-1}}{2p(1-q) - \sum_{k=n+1}^{\infty} [m(k-1) - 2p(k-q)] \frac{(c)_{k-1}(\lambda+1)_{k-1} B_k(\lambda)}{(1)_{k-1}(a)_{k-1}} a_k |z|^{k-1}} < \beta. \end{aligned}$$

Choosing values of z on real axis and letting $z \rightarrow 1^-$, we have

$$[(k-1)(1-m\beta) - 2p\beta(k-q)] \frac{(c)_{k-1}(\lambda+1)_{k-1} B_k(\lambda)}{(1)_{k-1}(a)_{k-1}} a_k - 2p\beta(1-q) \leq 0.$$

By hypothesis, thus by maximum modulus theorem, we have

$$f(z) \in A_n(m, \beta, p, q, \lambda).$$

Conversely, assume that $f(z) \in A_n(m, \beta, p, q, \lambda)$, then from the definition of $f(z)$, we have

$$\left| \frac{\frac{z(J^\lambda f(z))' - 1}{J^\lambda f(z)} - 1}{2p\left(\frac{z(J^\lambda f(z))' - 1}{J^\lambda f(z)} - q\right) - m\left(\frac{z(J^\lambda f(z))' - 1}{J^\lambda f(z)} - 1\right)} \right| < \beta,$$

that is

$$\left| \frac{-\sum_{k=n+1}^{\infty} (k-1) \frac{(c)_{k-1}(\lambda+1)_{k-1} B_k(\lambda)}{(1)_{k-1}(a)_{k-1}} a_k z^k}{2p(1-q)z + \sum_{k=n+1}^{\infty} [m(k-1) - 2p(k-q)] \frac{(c)_{k-1}(\lambda+1)_{k-1} B_k(\lambda)}{(1)_{k-1}(a)_{k-1}} a_k z^k} \right| < \beta.$$

By the fact $|\operatorname{Re}(z)| \leq |z|$ for all z , we have

$$\operatorname{Re} \left\{ \frac{\sum_{k=n+1}^{\infty} (k-1) \frac{(c)_{k-1}(\lambda+1)_{k-1} B_k(\lambda)}{(1)_{k-1}(a)_{k-1}} a_k |z|^{k-1}}{2p(1-q)z + \sum_{k=n+1}^{\infty} [m(k-1) - 2p(k-q)] \frac{(c)_{k-1}(\lambda+1)_{k-1} B_k(\lambda)}{(1)_{k-1}(a)_{k-1}} a_k |z|^{k-1}} \right\} < \beta.$$

We choose the values of z on real axis such that $\frac{(J^\lambda f(z))'}{J^\lambda f(z)}$ is real and upon clearing, the denominator of the above expression and allowing $z \rightarrow 1^-$ through real values, we obtain

$$\sum_{k=n+1}^{\infty} [(k-1)(1-m\beta) - 2p\beta(k-q)] \frac{(c)_{k-1}(\lambda+1)_{k-1} B_k(\lambda)}{(1)_{k-1}(a)_{k-1}} a_k \leq 2p\beta(1-q). \quad \square$$

Corollary 1. Let the function $f(z)$ of the form (1.2) be in the class $A_n(m, \beta, p, q, \lambda)$. Then

$$a_k \leq \frac{2p\beta(1-q)(1)_{k-1}(a)_{k-1}}{[(k-1)(1-m\beta) - 2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1} B_k(\lambda)} \quad (k \geq n+1, n \in \mathbb{N}),$$

where $0 < \beta \leq 1$, $\frac{1}{2} \leq p \leq 1$, $0 \leq q \leq \frac{1}{2}p$, $\frac{1}{2} < m \leq 1$, $a, c \in \mathbb{R} - \{0, -1, -2, \dots\}$, $z \in \mathcal{U}$.

Remark 2. If $f(z) \in A_n(1, \beta, p, q, \lambda)$, then

$$a_k \leq \frac{2p\beta(1-q)(1)_{k-1}(a)_{k-1}}{[(k-1)(1-\beta) - 2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1} B_k(\lambda)} \quad (k \geq n+1, n \in \mathbb{N})$$

and equality holds for

$$f(z) = z - \frac{2p\beta(1-q)(1)_{k-1}(a)_{k-1}}{[(k-1)(1-\beta) - 2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1} B_k(\lambda)} z^k.$$

Remark 3. If $f(z) \in A_n(1, \beta, 1, q, \lambda)$, then

$$a_k \leq \frac{2\beta(1-q)(1)_{k-1}(a)_{k-1}}{[(k-1)(1-\beta) - 2\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)} \quad (k \geq n+1, n \in N)$$

and equality holds for

$$f(z) = z - \frac{2\beta(1-q)(1)_{k-1}(a)_{k-1}}{[(k-1)(1-\beta) - 2\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)} z^k.$$

Remark 4. If $f(z) \in A_n(1, 1, 1, q, \lambda)$, then

$$a_k \leq \frac{-(1-q)(1)_{k-1}(a)_{k-1}}{(k-q)(c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)} \quad (k \geq n+1, n \in N)$$

and equality holds for

$$f(z) = z + \frac{(1-q)(1)_{k-1}(a)_{k-1}}{(k-q)(c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)} z^k.$$

§ 3. Distortion theorems

Theorem 2. Let the function $f(z)$ of the form (1.2) be in the class $A_n(m, \beta, p, q, \lambda)$. Then for $|z| = r < 1$, we have

$$\begin{aligned} r - \frac{2p\beta(1-q)(1)_n(a)_n}{[n(1-m\beta) - 2p\beta(n+1-q)](c)_n(\lambda+1)_nB_{n+1}(\lambda)} r^{n+1} &\leq |f(z)| \leq \\ &\leq r + \frac{2p\beta(1-q)(1)_n(a)_n}{[n(1-m\beta) - 2p\beta(n+1-q)](c)_n(\lambda+1)_nB_{n+1}(\lambda)} r^{n+1}. \end{aligned}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z - \frac{2p\beta(1-q)(1)_n(a)_n}{[n(1-m\beta) - 2p\beta(n+1-q)](c)_n(\lambda+1)_nB_{n+1}(\lambda)} z^{n+1}.$$

P r o o f. Suppose that $f(z) \in A_n(m, \beta, p, q, \lambda)$. By the inequality (2.1), since

$$[n(1-m\beta) - 2p\beta(n+1-q)] \frac{(c)_n(\lambda+1)_n}{(1)_n(a)_n} B_{n+1}(\lambda)$$

is non decreasing and positive for $k \geq n+1$, we have

$$\begin{aligned} &[n(1-m\beta) - 2p\beta(n+1-q)] \frac{(c)_n(\lambda+1)_n}{(1)_n(a)_n} B_{n+1}(\lambda) \sum_{k=n+1}^{\infty} a_k \leq \\ &\leq \sum_{k=n+1}^{\infty} [(k-1)(1-m\beta) - 2p\beta(k-q)] \frac{(c)_{k-1}(\lambda+1)_{k-1}}{(1)_{k-1}(a)_{k-1}} B_k(\lambda) a_k \leq 2p\beta(1-q). \end{aligned}$$

This is equivalent to

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{2p\beta(1-q)(1)_n(a)_n}{[n(1-m\beta) - 2p\beta(n+1-q)](c)_n(\lambda+1)_nB_{n+1}(\lambda)}. \quad (3.1)$$

Using (1.2) and (3.1), we obtain

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k,$$

$$\begin{aligned}
|f(z)| &\geq |z| - \sum_{k=n+1}^{\infty} a_k |z|^k \geq r - \sum_{k=n+1}^{\infty} a_k r^k \geq r - r^{n+1} \sum_{k=n+1}^{\infty} a_k \geq \\
&\geq r - \frac{2p\beta(1-q)(1)_n(a)_n}{[n(1-m\beta) - 2p\beta(n+1-q)](c)_n(\lambda+1)_n B_{n+1}(\lambda)} r^{n+1}.
\end{aligned}$$

Similarly

$$|f(z)| \leq r + \frac{2p\beta(1-q)(1)_n(a)_n}{[n(1-m\beta) - 2p\beta(n+1-q)](c)_n(\lambda+1)_n B_{n+1}(\lambda)} r^{n+1}.$$

This completes the proof of Theorem 2. $\square$

Theorem 3. Let the function $f(z)$ of the form (1.2) be in the class $A_n(m, \beta, p, q, \lambda)$. Then for $|z| = r < 1$, we have

$$\begin{aligned}
1 - \frac{2p\beta(1+n)(1-q)(1)_n(a)_n}{[n(1-m\beta) - 2p\beta(n+1-q)](c)_n(\lambda+1)_n B_{n+1}(\lambda)} r^n &\leq |f'(z)| \leq \\
&\leq 1 + \frac{2p\beta(1+n)(1-q)(1)_n(a)_n}{[n(1-m\beta) - 2p\beta(n+1-q)](c)_n(\lambda+1)_n B_{n+1}(\lambda)} r^n.
\end{aligned}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z - \frac{2p\beta(1+n)(1-q)(1)_n(a)_n}{[n(1-m\beta) - 2p\beta(n+1-q)](c)_n(\lambda+1)_n B_{n+1}(\lambda)} z^{n+1}.$$

P r o o f. From (1.2) and (3.1) we have $f'(z) = 1 - \sum_{k=n+1}^{\infty} k a_k z^{k-1}$,

$$\begin{aligned}
|f'(z)| &\geq 1 - \sum_{k=n+1}^{\infty} k a_k |z|^{k-1} \geq 1 - \sum_{k=n+1}^{\infty} a_k r^{k-1} \geq 1 - r^n \sum_{k=n+1}^{\infty} k a_k \geq \\
&\geq 1 - \frac{2p\beta(1+n)(1-q)(1)_n(a)_n}{[n(1-m\beta) - 2p\beta(n+1-q)](c)_n(\lambda+1)_n B_{n+1}(\lambda)} r^n.
\end{aligned}$$

Similarly

$$|f(z)| \leq 1 + \frac{2p\beta(1+n)(1-q)(1)_n(a)_n}{[n(1-m\beta) - 2p\beta(n+1-q)](c)_n(\lambda+1)_n B_{n+1}(\lambda)} r^n.$$

This completes the proof of Theorem 3. $\square$

§ 4. Closure theorem

Theorem 4. Let the functions f_j ($j = 1, 2, \dots, s$) defined by

$$f_j(z) = z - \sum_{k=n+1}^{\infty} a_{k,j} z^k \quad (a_{k,j} \geq 0) \quad (4.1)$$

are in the class $A_n(m, \beta, p, q, \lambda)$. Then the function $h(z)$ defined by $h(z) = \sum_{j=1}^s \mu_j f_j(z)$ ($\mu_j \geq 0$) is

also in the class $A_n(m, \beta, p, q, \lambda)$ if $\sum_{j=1}^s \mu_j = 1$.

P r o o f. We can write

$$h(z) = \sum_{j=1}^s \mu_j \left(z - \sum_{k=n+1}^{\infty} a_{k,j} z^k \right) = \sum_{j=1}^s \mu_j z - \sum_{j=1}^s \sum_{k=n+1}^{\infty} \mu_j a_{k,j} z^k = z - \sum_{k=n+1}^{\infty} \sum_{j=1}^s \mu_j a_{k,j} z^k.$$

Furthermore, since the functions $f_j(z)$ ($j = 1, 2, \dots, s$) are in the class $A_n(m, \beta, p, q, \lambda)$, then from Theorem 1 we have

$$\sum_{k=n+1}^{\infty} [(k-1)(1-m\beta) - 2p\beta(k-q)] \frac{(c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{(1)_{k-1}(a)_{k-1}} a_{k,j} \leq 2p\beta(1-q).$$

Thus it is enough to prove that

$$\begin{aligned} & \sum_{k=n+1}^{\infty} [(k-1)(1-m\beta) - 2p\beta(k-q)] \frac{(c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{(1)_{k-1}(a)_{k-1}} \left(\sum_{j=1}^s \mu_j a_{k,j} \right) = \\ &= \sum_{j=1}^s \mu_j \sum_{k=n+1}^{\infty} [(k-1)(1-m\beta) - 2p\beta(k-q)] \frac{(c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{(1)_{k-1}(a)_{k-1}} a_{k,j} \leq \\ &\leq \sum_{j=1}^s \mu_j 2p\beta(1-q) = 2p\beta(1-q). \end{aligned}$$

Hence the proof is complete. $\square$

Corollary 2. Let the functions f_j ($j = 1, 2$) defined by (4.1) are in the class $A_n(m, \beta, p, q, \lambda)$. Then the function $h(z)$ defined by $h(z) = (1-\zeta)f_1(z) + \zeta f_2(z)$, $0 \leq \zeta \leq 1$, is also in the class $A_n(m, \beta, p, q, \lambda)$.

§ 5. Extreme points

Theorem 5. Let $f_1(z) = z$ and

$$f_k(z) = z - \frac{2p\beta(1-q)(1)_{k-1}(a)_{k-1}}{[(k-1)(1-m\beta) - 2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)} z^k \quad (k \geq n+1, n \in N).$$

Then the function $f(z)$ of the form (1.2) is in the class $A_n(m, \beta, p, q, \lambda)$ if and only if it can be expressed in the form

$$f(z) = \mu_1 f_1(z) + \sum_{k=n+1}^{\infty} \mu_k f_k(z),$$

where $\mu_1 \geq 0$, $\mu_k \geq 0$, $k \geq n+1$, and $\mu_1 + \sum_{k=n+1}^{\infty} \mu_k = 1$.

P r o o f. Assume that $f(z)$ can be expressed in the form $f(z) = \mu_1 f_1(z) + \sum_{k=n+1}^{\infty} \mu_k f_k(z)$,

$$f(z) = z - \sum_{k=n+1}^{\infty} \frac{2p\beta(1-q)(1)_{k-1}(a)_{k-1}}{[(k-1)(1-m\beta) - 2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)} \mu_k z^k.$$

Thus

$$\begin{aligned} & \sum_{k=n+1}^{\infty} \frac{[(k-1)(1-m\beta) - 2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{2p\beta(1-q)(1)_{k-1}(a)_{k-1}} \cdot \\ & \cdot \sum_{k=n+1}^{\infty} \frac{2p\beta(1-q)(1)_{k-1}(a)_{k-1}}{[(k-1)(1-m\beta) - 2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)} \mu_k = \sum_{k=n+1}^{\infty} \mu_k = 1 - \mu_1 \leq 1. \end{aligned}$$

Hence $f(z) \in A_n(m, \beta, p, q, \lambda)$.

Conversely, assume that $f(z) \in A_n(m, \beta, p, q, \lambda)$. Setting

$$\mu_k = \frac{[(k-1)(1-m\beta) - 2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{2p\beta(1-q)(1)_{k-1}(a)_{k-1}} a_{k,j},$$

since $\mu_1 = 1 - \sum_{k=n+1}^{\infty} \mu_k$. Thus $f(z) = \mu_1 f_1(z) + \sum_{k=n+1}^{\infty} \mu_k f_k(z)$. Hence the proof is complete. $\square$

Corollary 3. *The extreme points of the class $A_n(m, \beta, p, q, \lambda)$ are the functions $f_1(z) = z$ and*

$$f_k(z) = z - \frac{2p\beta(1-q)(1)_{k-1}(a)_{k-1}}{[(k-1)(1-m\beta) - 2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)} z^k \quad (k \geq n+1, n \in N).$$

§ 6. Neighborhood property

The concept of neighborhood of analytic function was first introduced by Goodman [3]. Ruscheweyh [14], Raina and Srivastava [12], Orhan and Kamali [10], Silverman [15] investigated this concept for the elements of several famous subclass of analytic functions. We define the (n, δ) -neighborhood of a function $f(z) \in A^-(n)$ by

$$N_{n,\delta}(f) = \{g \in A^-(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \text{ and } \sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \delta\}. \quad (6.1)$$

In particular, for $e(z) = z$

$$N_{n,\delta}(e) = \{g \in A^-(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \text{ and } \sum_{k=n+1}^{\infty} k|b_k| \leq \delta\}. \quad (6.2)$$

Furthermore, a function of the form (1.2) is said to be in the class $A_n^\sigma(m, \beta, p, q, \lambda)$ if there exists a function $h(z) \in A_n(m, \beta, p, q, \lambda)$ such that

$$\left| \frac{f(z)}{h(z)} - 1 \right| < 1 - \sigma \quad (z \in \mathcal{U}, 0 \leq \sigma < 1). \quad (6.3)$$

Theorem 6. *If*

$$\frac{(c)_{k-1}(\lambda+1)_{k-1}}{(1)_{k-1}(a)_{k-1}} B_k(\lambda) \geq \frac{(c)_n(\lambda+1)_n}{(1)_n(a)_n} B_{n+1}(\lambda) \quad (k \geq n+1, n \in N) \quad (6.4)$$

and

$$\delta = \frac{2p\beta(1-q)(1)_n(a)_n}{[(n)(1-m\beta) - 2p\beta(n+1-q)](c)_n(\lambda+1)_n B_{n+1}(\lambda)},$$

then $A_n(m, \beta, p, q, \lambda) \in N_{n,\delta}(e)$.

P r o o f. Let $f(z) \in A_n(m, \beta, p, q, \lambda)$. Then in view of assertion (2.1) of Theorem 1 and the condition (6.4), we get

$$\begin{aligned} & [n(1-m\beta) - 2p\beta(n+1-q)] \frac{(c)_n(\lambda+1)_n}{(1)_n(a)_n} B_{n+1}(\lambda) \sum_{k=n+1}^{\infty} a_k \leq \\ & \leq \sum_{k=n+1}^{\infty} [(k-1)(1-m\beta) - 2p\beta(k-q)] \frac{(c)_{k-1}(\lambda+1)_{k-1}}{(1)_{k-1}(a)_{k-1}} B_k(\lambda) a_k \leq 2p\beta(1-q), \end{aligned}$$

which implies

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{2p\beta(1-q)(1)_n(a)_n}{[n(1-m\beta) - 2p\beta(n+1-q)](c)_n(\lambda+1)_n B_{n+1}(\lambda)}. \quad (6.5)$$

Applying assertion (2.1) of Theorem 1 in conjunction with (6.5), we obtain

$$[n(1-m\beta) - 2p\beta(n+1-q)] \frac{(c)_n(\lambda+1)_n}{(1)_n(a)_n} B_{n+1}(\lambda) \sum_{k=n+1}^{\infty} a_k \leq 2p\beta(1-q),$$

$$(n+1)[n(1-m\beta) - 2p\beta(n+1-q)] \frac{(c)_n(\lambda+1)_n}{(1)_n(a)_n} B_{n+1}(\lambda) \sum_{k=n+1}^{\infty} a_k \leq 2p\beta(n+1)(1-q),$$

$$\sum_{k=n+1}^{\infty} ka_k \leq \frac{2p\beta(n+1)(1-q)(1)_n(a)_n}{[(n)(1-m\beta) - 2p\beta(n+1-q)](c)_n(\lambda+1)_n B_{n+1}(\lambda)} = \delta.$$

By virtue of (6.1), we have $f(z) \in N_{n,\delta}(e)$. This completes the proof of Theorem 6. $\square$

Theorem 7. If $h(z) \in A_n(m, \beta, p, q, \lambda)$ and

$$\sigma = 1 - \frac{\delta}{(n+1)} \frac{[n(1-m\beta) - 2p\beta(n+1-q)](c)_n(\lambda+1)_n B_{n+1}(\lambda)}{[n(1-m\beta) - 2p\beta(n+1-q)](c)_n(\lambda+1)_n B_{n+1}(\lambda) - 2p\beta(1-q)(1)_n(a)_n}, \quad (6.6)$$

then

$$N_{n,\delta}(h) \subset A_n^\sigma(m, \beta, p, q, \lambda).$$

P r o o f. Suppose that $f \in N_{n,\delta}(h)$, we then find from (1.2) that

$$\sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \delta,$$

which readily implies the following coefficient inequality

$$\sum_{k=n+1}^{\infty} |a_k - b_k| \leq \frac{\delta}{n+1} \quad (n \in N). \quad (6.7)$$

Next, since $h \in A_n(m, \beta, p, q, \lambda)$ in the view of (6.5), we have

$$\sum_{k=n+1}^{\infty} b_k \leq \frac{2p\beta(1-q)(1)_n(a)_n}{[(n)(1-m\beta) - 2p\beta(n+1-q)](c)_n(\lambda+1)_n B_{n+1}(\lambda)}. \quad (6.8)$$

Using (6.7) and (6.8), we get

$$\begin{aligned} \left| \frac{f(z)}{h(z)} - 1 \right| &\leq \frac{\sum_{k=n+1}^{\infty} |a_k - b_k|}{1 - \sum_{k=n+1}^{\infty} b_k} \leq \frac{\delta}{(n+1) \left(1 - \frac{2p\beta(1-q)(1)_n(a)_n}{[(n)(1-m\beta) - 2p\beta(n+1-q)](c)_n(\lambda+1)_n B_{n+1}(\lambda)} \right)} \leq \\ &\leq \frac{\delta}{(n+1)} \left(\frac{[n(1-m\beta) - 2p\beta(n+1-q)](c)_n(\lambda+1)_n B_{n+1}(\lambda)}{[n(1-m\beta) - 2p\beta(n+1-q)](c)_n(\lambda+1)_n B_{n+1}(\lambda) - 2p\beta(1-q)(1)_n(a)_n} \right) = 1 - \sigma, \end{aligned}$$

provided that σ is given by (6.6), thus by condition (6.3), $f \in A^\sigma(m, \beta, p, q, \lambda)$. $\square$

§ 7. Radii of starlikeness, convexity and close-to-convexity

Theorem 8. Let the function $f(z)$ of the form (1.2) be in the class $A_n(m, \beta, p, q, \lambda)$. Then f is univalent starlike of order δ ($0 \leq \delta < 1$) in $|z| < r_1$, where

$$r_1 = \inf_k \left\{ \frac{(1-\delta)[(k-1)(1-m\beta) - 2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1} B_k(\lambda)}{2p\beta(1-q)(k-\delta)(1)_{k-1}(a)_{k-1}} \right\}^{\frac{1}{k-1}}.$$

The result is sharp for the function $f(z)$ given by

$$f_k(z) = z - \frac{2p\beta(1-q)(1)_{k-1}(a)_{k-1}}{[(k-1)(1-m\beta) - 2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1} B_k(\lambda)} \quad (k \geq n+1, n \in N).$$

P r o o f. It suffices to show that $\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \delta$, $|z| < r_1$. We have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \frac{-\sum_{k=n+1}^{\infty} (k-1)a_k z^{k-1}}{1 - \sum_{k=n+1}^{\infty} a_k z^{k-1}} \right| \leq \frac{\sum_{k=n+1}^{\infty} (k-1)a_k |z|^{k-1}}{1 - \sum_{k=n+1}^{\infty} a_k |z|^{k-1}}.$$

To prove the theorem, we must show that

$$\frac{\sum_{k=n+1}^{\infty} (k-1)a_k |z|^{k-1}}{1 - \sum_{k=n+1}^{\infty} a_k |z|^{k-1}} \leq 1 - \delta.$$

It is equivalent to

$$\sum_{k=n+1}^{\infty} (k-\delta)a_k |z|^{k-1} \leq 1 - \delta.$$

Using Theorem 1, we obtain

$$|z| \leq \left\{ \frac{(1-\delta)[(k-1)(1-m\beta) - 2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{2p\beta(1-q)(k-\delta)(1)_{k-1}(a)_{k-1}} \right\}^{\frac{1}{k-1}}.$$

Hence the proof is complete. $\square$

Theorem 9. Let the function $f(z)$ of the form (1.2) be in the class $A_n(m, \beta, p, q, \lambda)$. Then f is univalent convex of order δ ($0 \leq \delta < 1$) in $|z| < r_2$, where

$$r_2 = \inf_k \left\{ \frac{(1-\delta)[(k-1)(1-m\beta) - 2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{2p\beta k(1-q)(k-\delta)(1)_{k-1}(a)_{k-1}} \right\}^{\frac{1}{k-1}}.$$

The result is sharp for the function $f(z)$ given by

$$f_k(z) = z - \frac{2p\beta(1-q)(1)_{k-1}(a)_{k-1}}{[(k-1)(1-m\beta) - 2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)} \quad (k \geq n+1, n \in \mathbb{N}). \quad (7.1)$$

P r o o f. It suffices to show that $\left| \frac{zf''(z)}{f'(z)} \right| \leq 1 - \delta$, $|z| < r_2$. We have

$$\left| \frac{zf''(z)}{f'(z)} \right| = \left| \frac{-\sum_{k=n+1}^{\infty} k(k-1)a_k z^{k-1}}{1 - \sum_{k=n+1}^{\infty} k a_k z^{k-1}} \right| \leq \frac{\sum_{k=n+1}^{\infty} k(k-1)a_k |z|^{k-1}}{1 - \sum_{k=n+1}^{\infty} k a_k |z|^{k-1}}.$$

To prove the theorem, we must show that

$$\frac{\sum_{k=n+1}^{\infty} k(k-1)a_k |z|^{k-1}}{1 - \sum_{k=n+1}^{\infty} k a_k |z|^{k-1}} \leq 1 - \delta.$$

It is equivalent to

$$\sum_{k=n+1}^{\infty} k(k-\delta)a_k |z|^{k-1} \leq 1 - \delta.$$

Using Theorem 1, we obtain

$$|z|^{k-1} \leq \left\{ \frac{(1-\delta)[(k-1)(1-m\beta) - 2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{2p\beta k(1-q)(k-\delta)(1)_{k-1}(a)_{k-1}} \right\},$$

$$|z| \leq \left\{ \frac{(1-\delta)[(k-1)(1-m\beta) - 2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{2p\beta k(1-q)(k-\delta)(1)_{k-1}(a)_{k-1}} \right\}^{\frac{1}{k-1}}.$$

Hence the proof is complete. $\square$

Theorem 10. Let the function $f(z)$ of the form (1.2) be in the class $A_n(m, \beta, p, q, \lambda)$. Then f is univalent close-to-convex of order δ ($0 \leq \delta < 1$) in $|z| < r_3$, where

$$r_3 = \inf_k \left\{ \frac{(1-\delta)[(k-1)(1-m\beta) - 2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{2p\beta k(1-q)(1)_{k-1}(a)_{k-1}} \right\}^{\frac{1}{k-1}}.$$

The result is sharp for the function $f(z)$ given by (7.1).

P r o o f. It suffices to show that $|f'(z) - 1| \leq 1 - \delta$, $|z| < r_3$. We have

$$|f'(z) - 1| = \left| - \sum_{k=n+1}^{\infty} k a_k z^{k-1} \right| \leq \sum_{k=n+1}^{\infty} k a_k |z|^{k-1}.$$

Thus $|f'(z) - 1| \leq 1 - \delta$, if

$$\sum_{k=n+1}^{\infty} \frac{k a_k}{1-\delta} |z|^{k-1} \leq 1.$$

Using Theorem 1, the above inequality holds true if

$$|z|^{k-1} \leq \left\{ \frac{(1-\delta)[(k-1)(1-m\beta) - 2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{2p\beta k(1-q)(1)_{k-1}(a)_{k-1}} \right\}$$

or

$$|z| \leq \left\{ \frac{(1-\delta)[(k-1)(1-m\beta) - 2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{2p\beta k(1-q)(1)_{k-1}(a)_{k-1}} \right\}^{\frac{1}{k-1}}.$$

Hence the proof is complete. $\square$

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А. Р. С. Джума, М. Ш. Абдул-Хуссейн, М. Ф. Хани

Об одном подклассе однолистных функций с отрицательными коэффициентами, заданном линейным оператором

Ключевые слова: аналитические однолистные функции, произведение Адамара, производная Рушевея, теоремы искажения, теоремы о замыкании.

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В работе вводится и исследуется подкласс $A_n(m, \beta, p, q, \lambda)$ однолистных функций с отрицательными коэффициентами, определяемый новым линейным оператором J^λ в открытом единичном круге $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. Основной задачей является изучение следующих свойств и характеристик: оценки коэффициентов, теоремы искажения, теоремы о замыкании, окрестность функции, радиусы звездообразности, выпуклости и почти выпуклости функций, принадлежащих классу $A_n(m, \beta, p, q, \lambda)$.

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