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CERTAIN CLASS OF HARMONIC MULTIVALENT FUNCTIONS

Making use of the generalized derivative operator, we introduce a new subclass of harmonic multivalent functions. We obtain the coefficient bounds, distortion inequalities and inclusion relationships involving the neighborhoods of subclasses of harmonic multivalent functions.

Keywords: harmonic multivalent functions, derivative operator, neighborhood.

§ 1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply-connected complex domain D is said to be harmonic in D if both functions u and v are real harmonic in D . Such functions can be expressed as

$$f = h + \bar{g}, \quad (1)$$

where h and g are analytic in D . We call h the analytic part and g the co-analytic part of f . A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|\bar{h}(z)| > |\bar{g}(z)|$ for all z in D (see [1]). Many researcher introduced and studied certain classes of harmonic univalent functions (see [2–8]). For $p \geq 1$, $n \in \mathbb{N}$, denote by $SH(n, p)$ the class of functions of the form (1) that are harmonic multivalent and sense-preserving in the unit disk $U = \{z : |z| < 1\}$, where h and g are defined by

$$h(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k, \quad g(z) = \sum_{k=n+p-1}^{\infty} b_k z^k, \quad z \in U, \quad |b_n + p - 1| < 1, \quad (2)$$

which are analytic and multivalent functions in U .

Now we introduce a generalized derivative operator

$$D_{p,\lambda}^m f^{(q)} = D_{p,\lambda}^m h^{(q)} + (-1)^m \overline{D_{p,\lambda}^m g^{(q)}}.$$

The derivative operator $D_{p,\lambda}^m f^{(q)}$ of p -valent functions was introduced and studied by Eljamal and Darus in [9], where

$$D_{p,\lambda}^m h^{(q)} = \frac{p!}{(p-q)!} z^{p-q} + \sum_{k=n+p}^{\infty} \frac{k!}{(k-q)!} \left(\frac{k+\lambda-q}{p+\lambda-q} \right)^m a_k z^{k-q},$$

$$D_{p,\lambda}^m g^{(q)} = \sum_{k=n+p-1}^{\infty} \frac{k!}{(k-q)!} \left(\frac{k+\lambda-q}{p+\lambda-q} \right)^m b_k z^{k-q}, \quad m \in \mathbb{N}_0, \quad z \in U.$$

For $q = \lambda = 0$, $p = k = 1$, the differential operator $D^m f^{(q)}$ was introduced by Salagean [10] for the class A of analytic functions and modified for the class $SH(1, 1)$ by Jahangiri et. al [11].

Let

$$F(z) = (1 - \ell) D_{p,\lambda}^m f^{(q)}(z) + \ell D_{p,\lambda}^{m+1} f^{(q)}(z) = H(z) + \overline{G(z)}, \quad f(z) \in SH(n, p), \quad 0 \leq \ell \leq 1,$$

where H and G are of the form

$$H(z) = (1 - \ell + \ell(p + \lambda - q)) \frac{(p + \lambda - q)^m}{(p - q)!} p! z^{p-q} +$$

$$+ \sum_{k=n+p}^{\infty} (1 - \ell + \ell(k + \lambda - q)) \frac{(k + \lambda - q)^m}{(k - q)!} k! z^{k-q} a_k z^{k-q}, \quad (3)$$

$$G(z) = (-1)^m \sum_{k=n+p-1}^{\infty} (1 - \ell + \ell(k + \lambda - q)) \frac{(k + \lambda - q)^m}{(k - q)!} k! z^{k-q} b_k z^{k-q}. \quad (4)$$

Also, let $SH_{p,\lambda}^{m,n}(q, \ell, \alpha)$ denote the subclass of $SH(n, p)$ consisting of functions f defined in (1) that satisfy the following condition:

$$\operatorname{Re} \left\{ \frac{zH'(z) - \overline{zG'(z)}}{H(z) + \overline{G(z)}} \right\} > \alpha(p + \lambda - q) \quad (0 \leq \alpha < 1, p > q, p \in N, q \in N_0, z \in U), \quad (5)$$

where $H(z)$ and $G(z)$ are given by (3) and (4) respectively.

Denote by $\overline{SH}(n, p)$ the subclass of $SH(n, p)$ consisting of harmonic functions $f_m = h + \overline{g_m}$, where h and g_m are of the form

$$h(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k, \quad g_m(z) = (-1)^m \sum_{k=n+p-1}^{\infty} b_k z^k, \quad b_k \geq 0. \quad (6)$$

Define $\overline{SH}_{p,\lambda}^{m,n}(q, \ell, \alpha) = SH_{p,\lambda}^{m,n}(q, \ell, \alpha) \cap \overline{SH}(n, p)$.

The classes $SH_{p,\lambda}^{m,n}(q, \ell, \alpha)$ and $\overline{SH}_{p,\lambda}^{m,n}(q, \ell, \alpha)$ include well-known subclasses of $SH(n, p)$. For example:

(i) $SH_{1,0}^{0,1}(0, 0, 0) \equiv SH^*$ is the class of sense-preserving, harmonic univalent functions f which are starlike in U (see [12, 13]);

(ii) $\overline{SH}_{1,0}^{0,1}(0, 0, \alpha) \equiv SH^*(\alpha)$ is the class of sense-preserving, harmonic univalent functions f which are starlike of order α in U (see [14]);

(iii) $\overline{SH}_{1,0}^{1,1}(0, 0, \alpha) \equiv HK(\alpha)$ is the class of sense-preserving, harmonic univalent functions f which are convex of order α in U (see [12]);

(iv) $\overline{SH}_{1,p}^{0,1}(0, 0, 0) \equiv SH^*(p)$ is the class of sense-preserving, harmonic multivalent functions which are starlike in U (see [15]);

(v) $\overline{SH}_{p,0}^{m,n}(q, \ell, \alpha) \equiv \overline{SH}_p^{m,n}(q, \ell, \alpha)$ is the class of sense-preserving, harmonic multivalent functions in U (see [16]).

To prove our main results we need the following lemma.

Lemma 1. Let $f_m = h + \overline{g_m}$ be of the form (6). Then $f_m \in \overline{SH}(n, p)$ if and only if

$$\sum_{k=n+p}^{\infty} k a_k + \sum_{k=n+p-1}^{\infty} k b_k \leq p \quad (p \geq 1, n \in N). \quad (7)$$

§ 2. Main Result

We begin deriving a coefficient sufficient condition for the function f to belong to the class $SH_{p,\lambda}^{m,n}(q, \ell, \alpha)$. This result is contained in the following.

Theorem 1. Let $f = h + \bar{g}$ be given by (1). Furthermore, let

$$\sum_{k=n+p}^{\infty} \frac{(k + \lambda - q - \alpha(p + \lambda - q))(1 - \ell + \ell(k + \lambda - q))}{[(1 - \alpha)(p + \lambda - q) + 1] - |(1 - \alpha)(p + \lambda - q) - 1|}(1 - \ell + \ell(p + \lambda - q)) \frac{q_p^m}{q_k^m} |a_k| + \quad (8)$$

$$+ \sum_{k=n+p-1}^{\infty} \frac{(k + \lambda - q - \alpha(p + \lambda - q))(1 - \ell - \ell(k + \lambda - q))}{[(1 - \alpha)(p + \lambda - q) + 1] - |(1 - \alpha)(p + \lambda - q) - 1|}(1 - \ell + \ell(p + \lambda - q)) \frac{q_p^m}{q_k^m} |b_k| \leq \frac{1}{2},$$

where $q_p^m = \frac{(p+\lambda-q)^m}{(p-q)!} p!$, $q_k^m = \frac{(k+\lambda-q)^m}{(k-q)!} k!$, and $(0 \leq \alpha < 1, p > q, p \in N, q \in N_0, z \in U)$. Then f is sense-preserving, harmonic multivalent functions in U , and $f \in SH_{p,\lambda}^{m,n}(q, \ell, \alpha)$.

P r o o f. We first show that if the inequality (8) holds for the coefficients of $f = h + \bar{g}$, then the required condition (7) is sense preserving and harmonic multivalent in U . In view of (5), we need to prove that $\operatorname{Re}\{w\} > 0$, where

$$w = \frac{zH'(z) - \overline{zG'(z)} - \alpha(p + \lambda - q)[H(z) + \overline{G(z)}]}{H(z) + \overline{G(z)}} = \frac{A(z)}{B(z)}.$$

By using the fact that $\operatorname{Re}\{w\} > 0 \Leftrightarrow |1 + w| > |1 - w|$, it suffices to show that $|A(z) + B(z)| - |A(z) - B(z)| > 0$, therefore we obtain $|A(z) + B(z)| - |A(z) - B(z)| \geq$

$$\begin{aligned} & \geq [((1 - \alpha)(p + \lambda - q) + 1) - |(1 - \alpha)(p + \lambda - q) - 1|](1 - \ell + \ell(p + \lambda - q))q_p^m |z|^{p-q} - \\ & - \sum_{k=n+p}^{\infty} 2(k + \lambda - q - \alpha(p + \lambda - q))(1 - \ell + \ell(k + \lambda - q))q_k^m |a_k| |z|^{k-q} - \\ & - \sum_{k=n+p}^{\infty} 2(k + \lambda - q - \alpha(p + \lambda - q))|(1 - \ell + \ell(k + \lambda - q))|q_k^m |b_k| |z|^{k-q} > \\ & > [((1 - \alpha)(p + \lambda - q) + 1) - |(1 - \alpha)(p + \lambda - q) - 1|](1 - \ell + \ell(p + \lambda - q))q_p^m |z|^{p-q} \times \\ & \times \left\{ 1 - \sum_{k=n+p}^{\infty} \frac{(k + \lambda - q - \alpha(p + \lambda - q))(1 - \ell + \ell(k + \lambda - q))}{[(1 - \alpha)(p + \lambda - q) + 1] - |(1 - \alpha)(p + \lambda - q) - 1|}(1 - \ell + \ell(p + \lambda - q)) \frac{q_k^m}{q_p^m} |a_k| - \right. \\ & \left. - \sum_{k=n+p}^{\infty} \frac{(k + \lambda - q - \alpha(p + \lambda - q))(1 - \ell + \ell(k + \lambda - q))}{[(1 - \alpha)(p + \lambda - q) + 1] - |(1 - \alpha)(p + \lambda - q) - 1|}(1 - \ell + \ell(p + \lambda - q)) \frac{q_k^m}{q_p^m} |b_k| \right\}. \quad \square \end{aligned}$$

Theorem 2. Let $f_m = h + \bar{g}_m$ be given by (6). Also, suppose that $\lambda < \frac{1}{n + p + \lambda - q}$ and $\alpha \geq 1 - \frac{1}{p + \lambda - q}$. Then $f_m \in \overline{SH}_{p,\lambda}^{m,n}(q, \ell, \alpha)$ if and only if

$$\begin{aligned} & \sum_{k=n+p}^{\infty} \frac{(k + \lambda - q - \alpha(p + \lambda - q))(1 - \ell + \ell(k + \lambda - q))}{(1 - \ell + \ell(p + \lambda - q))} \frac{q_k^m}{q_p^m} |a_k| + \\ & + \sum_{k=n+p-1}^{\infty} \frac{(k + \lambda - q - \alpha(p + \lambda - q))(1 - \ell - \ell(k + \lambda - q))}{(1 - \ell + \ell(p + \lambda - q))} \frac{q_k^m}{q_p^m} |b_k| \leq 1. \quad (9) \end{aligned}$$

P r o o f. Since $\overline{SH}_{p,\lambda}^{m,n}(q, \ell, \alpha) \subset SH_{p,\lambda}^{m,n}(q, \ell, \alpha)$, we only need to prove the necessary part of the theorem. Assume that $f_m \in \overline{SH}_{p,\lambda}^{m,n}(q, \ell, \alpha)$, then by virtue of (8) and (7), we obtain

$$\operatorname{Re} \left\{ \frac{(1-\alpha)(p+\lambda-q)z^{p-q} - \sum_{k=n+p}^{\infty} (k+\lambda-q-\alpha(p+\lambda-q)) \frac{(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(p+\lambda-q))} \frac{q_k^m}{q_p^m} a_k z^{k-q}}{z^{p-q} - \sum_{k=n+p}^{\infty} \frac{(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(p+\lambda-q))} \frac{q_k^m}{q_p^m} a_k z^{k-q} + (-1)^{2m} \sum_{k=n+p-1}^{\infty} \frac{(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(p+\lambda-q))} \frac{q_k^m}{q_p^m} b_k \overline{z^{k-q}}} \times \right. \\ \left. \times \frac{-(-1)^{2m} \sum_{k=n+p-1}^{\infty} (k+\lambda-q+\alpha(p+\lambda-q)) \frac{(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(p+\lambda-q))} \frac{q_k^{m+1}}{q_p^{m+1}} b_k \overline{z^{k-q}}}{z^{p-q} - \sum_{k=n+p}^{\infty} \frac{(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(p+\lambda-q))} \frac{q_k^m}{q_p^m} a_k z^{k-q} + (-1)^{2m} \sum_{k=n+p-1}^{\infty} \frac{(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(p+\lambda-q))} \frac{q_k^m}{q_p^m} b_k \overline{z^{k-q}}} \right\} \geq 0. \quad (10)$$

The above condition must hold for all values of z , $|z| = r < 1$. Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$ we have

$$\frac{(1-\alpha)(p+\lambda-q) - \sum_{k=n+p}^{\infty} (k+\lambda-q-\alpha(p+\lambda-q)) \frac{(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(p+\lambda-q))} \frac{q_k^m}{q_p^m} a_k r^{k-q}}{1 - \sum_{k=n+p}^{\infty} \frac{(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(p+\lambda-q))} \frac{q_k^m}{q_p^m} a_k r^{k-p} + \sum_{k=n+p-1}^{\infty} \frac{(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(p+\lambda-q))} \frac{q_k^m}{q_p^m} b_k r^{k-p}} \times \\ \times \frac{-\sum_{k=n+p-1}^{\infty} (k+\lambda-q+\alpha(p+\lambda-q)) \frac{(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(p+\lambda-q))} \frac{q_k^{m+1}}{q_p^{m+1}} b_k r^{k-q}}{1 - \sum_{k=n+p}^{\infty} \frac{(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(p+\lambda-q))} \frac{q_k^m}{q_p^m} a_k r^{k-q} + \sum_{k=n+p-1}^{\infty} \frac{(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(p+\lambda-q))} \frac{q_k^m}{q_p^m} b_k r^{k-q}} \geq 0. \quad (11)$$

If (9) does not hold, then the numerator in (11) is negative for r sufficiently close to 1. Therefore, there exists a point $z_0 = r_0$ in $(0, 1)$ for which the quotient in (11) is negative. This contradicts our assumption that $f_m \in \overline{SH}_{p,\lambda}^{m,n}(q, \ell, \alpha)$. We thus conclude that it is both necessary and sufficient that the coefficient bound inequality (9) holds true when $f_m \in \overline{SH}_{p,\lambda}^{m,n}(q, \ell, \alpha)$. $\square$

Theorem 3. The class $\overline{SH}_{p,\lambda}^{m,n}(q, \ell, \alpha)$ is closed under convex combinations.

P r o o f. Let $\ell < \frac{1}{p+\lambda-q}$, $\alpha \geq 1 - \frac{1}{p+\lambda-q}$ and $f_{mi} \in \overline{SH}_{p,\lambda}^{m,n}(q, \ell, \alpha)$ for $i = 1, 2, \dots$, where f_{mi} is given by

$$f_{mi}(z) = z^p - \sum_{k=n+p}^{\infty} a_{ki} z^k + (-1)^m \sum_{k=n+p-1}^{\infty} b_{ki} \overline{z^k}.$$

Then by (9),

$$\sum_{k=n+p}^{\infty} \frac{(k+\lambda-q-\alpha(p+\lambda-q))(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(k+\lambda-q))} \frac{q_k^m}{q_p^m} a_{ki} + \\ + \sum_{k=n+p-1}^{\infty} \frac{(k+\lambda-q-\alpha(p+\lambda-q))(1-\ell-\ell(k+\lambda-q))}{(1-\ell+\ell(k+\lambda-q))} \frac{q_k^m}{q_p^m} b_{ki} \leq (1-\alpha(p+\lambda-q)). \quad (12)$$

For $\sum_{i=1}^{\infty} t_i$, $0 \leq t_i \leq 1$, the convex combination of f_{mi} may be written as

$$\sum_{i=1}^{\infty} t_i f_{mi}(z) = z^p - \sum_{k=n+p}^{\infty} \left(\sum_{i=1}^{\infty} t_i a_{ki} \right) z^k + (-1)^m \sum_{k=n+p-1}^{\infty} \left(\sum_{i=1}^{\infty} t_i b_{ki} \right) \overline{z^k}.$$

Then by (12),

$$\sum_{k=n+p}^{\infty} \frac{(k+\lambda-q-\alpha(p+\lambda-q))(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(k+\lambda-q))} \frac{q_k^m}{q_p^m} \left(\sum_{i=1}^{\infty} t_i a_{ki} \right) + \\ + \sum_{k=n+p-1}^{\infty} \frac{(k+\lambda-q-\alpha(p+\lambda-q))(1-\ell-\ell(k+\lambda-q))}{(1-\ell+\ell(k+\lambda-q))} \frac{q_k^m}{q_p^m} \left(\sum_{i=1}^{\infty} t_i b_{ki} \right) =$$

$$\begin{aligned}
&= \sum_{i=1}^{\infty} t_i \left\{ \sum_{k=n+p}^{\infty} \frac{(k+\lambda-q-\alpha(p+\lambda-q))(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(k+\lambda-q))} \frac{q_k^m}{q_p^m} a_{ki} + \right. \\
&\quad \left. + \sum_{k=n+p-1}^{\infty} \frac{(k+\lambda-q-\alpha(p+\lambda-q))(1-\ell-\ell(k+\lambda-q))}{(1-\ell+\ell(k+\lambda-q))} \frac{q_k^m}{q_p^m} b_{ki} \right\} \leq \\
&\leq (1-\alpha)(p+\lambda-q) \sum_{i=1}^{\infty} t_i = (1-\alpha)(p+\lambda-q).
\end{aligned}$$

This is the condition required by (10) so $\sum_{i=1}^{\infty} t_i f_{mi}(z) \in \overline{SH}_{p,\lambda}^{m,n}(q, \ell, \alpha)$. $\square$

Theorem 4. Let $\ell < \frac{1}{p+\lambda-q}$ and $\alpha_1 \geq 1 - \frac{1}{p+\lambda-q}$. For $\alpha_1 < \alpha_2$,

$$\overline{SH}_{p,\lambda}^{m,n}(q, \ell, \alpha_2) \subset \overline{SH}_{p,\lambda}^{m,n}(q, \ell, \alpha_1).$$

P r o o f. Let $\ell < \frac{1}{p+\lambda-q}$, $\alpha_1 \geq 1 - \frac{1}{p+\lambda-q}$, and $f_m(z) \in \overline{SH}_{p,\lambda}^{m,n}(q, \ell, \alpha_2)$.

$$\begin{aligned}
&\sum_{k=n+p}^{\infty} \frac{(k+\lambda-q-\alpha_1(p+\lambda-q))(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(k+\lambda-q))(1-\alpha_1)} \frac{q_k^m}{q_p^{m+1}} |a_k| + \\
&+ \sum_{k=n+p-1}^{\infty} \frac{(k+\lambda-q-\alpha_1(p+\lambda-q))(1-\ell-\ell(k+\lambda-q))}{(1-\ell+\ell(k+\lambda-q))(1-\alpha_1)} \frac{q_k^m}{q_p^{m+1}} |b_k| \leq \\
&\leq \sum_{k=n+p}^{\infty} \frac{(k+\lambda-q-\alpha_2(p+\lambda-q))(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(k+\lambda-q))(1-\alpha_2)} \frac{q_k^m}{q_p^{m+1}} |a_k| + \\
&+ \sum_{k=n+p-1}^{\infty} \frac{(k+\lambda-q-\alpha_2(p+\lambda-q))(1-\ell-\ell(k+\lambda-q))}{(1-\ell+\ell(k+\lambda-q))(1-\alpha_2)} \frac{q_k^m}{q_p^{m+1}} |b_k| \leq 1,
\end{aligned}$$

then $f_m(z) \in \overline{SH}_{p,\lambda}^{m,n}(q, \ell, \alpha_1)$. $\square$

Following Goodman [17] and Ruscheweyh [18] we refer to the neighborhood of $f = h + \bar{g} \in \overline{SH}(n, p)$,

$$\begin{aligned}
&N_{n,p,\lambda}^{\delta}(f_m^{(q)}, g_m^{(q)}) = \{g_m \in \overline{SH}(n, p) : \\
&g_m(z) = z^p - \sum_{k=n+p}^{\infty} A_k z^k + (-1)^m \sum_{k=n+p-1}^{\infty} B_k \bar{z}^k, A_k, B_k \geq 0, B_{n+p-1} < 1, \text{ and} \quad (13) \\
&\sum_{k=n+p}^{\infty} \frac{k!}{(k+\lambda-q)!} k(|a_k - A_k| + |b_k - B_k|) + \frac{(n+p-1)!(n+p-1)}{(n+p-1+\lambda-q)!} |b_{n+p-1} - B_{n+p-1}| \leq \delta, \delta > 0\}.
\end{aligned}$$

In particular, for the function $h(z) = z^p$ we have

$$\begin{aligned}
&N_{n,p,\lambda}^{\delta}(h^{(q)}, g_m^{(q)}) = \{g_m \in \overline{SH}(n, p) : \\
&g_m(z) = z^p - \sum_{k=n+p}^{\infty} A_k z^k + (-1)^m \sum_{k=n+p-1}^{\infty} B_k \bar{z}^k, A_k, B_k \geq 0, B_{n+p-1} < 1, \text{ and} \quad (14) \\
&\sum_{k=n+p}^{\infty} \frac{k!}{(k-q)!} k(A_k + B_k) + \frac{(n+p-1)!(n+p-1)}{(n+p-1-q)!} |B_{n+p-1}| \leq \delta, \delta > 0\}.
\end{aligned}$$

Theorem 5. Let $\ell < \frac{1}{p+\lambda-q}$ and $\alpha \geq 1 - \frac{1}{p+\lambda-q}$. If $g_m \in \overline{SH}_{p,\lambda}^{m,n}(q, \ell, \alpha)$, then

$$\overline{SH}_{p,\lambda}^{m,n}(q, \ell, \alpha) \subset N_{n,p,\lambda}^\delta(h^{(q)}, g_m^{(q)}),$$

where $h(z)$ and $g_m(z)$ are given by (14),

$$\begin{aligned} \delta = & \frac{(n+p)(1-\alpha)(p+\lambda-q)}{(n+(1-\alpha)(p+\lambda-q))\psi} - \\ & - \left(\frac{(n+p)(n-1+(1+\alpha)(p+\lambda-q))(1-\ell-\ell(n+p+\lambda-1-q))}{(n+(1-\alpha)(p+\lambda-q))(1-\ell-\ell(n+p+\lambda-1-q))(n+p+\lambda-q)^m} - \right. \\ & \left. - \frac{(n+p-1)!(n+p-1)}{(n+p-1-q)!} \right) B_{n+p-1}, \end{aligned}$$

and

$$\psi = \left(\frac{(1-\ell-\ell(n+p+\lambda-q))(n+p+\lambda-q)^m}{((1-\ell)+\ell(p+\lambda-q))q_{n+p-1}^m} \right),$$

where $q_{n+p-1}^m = \frac{(p+\lambda-q)^m}{(n+p-1-q)!}(n+p-1)!$.

P r o o f. Let $f_m \in \overline{SH}_{p,\lambda}^{m,n}(q, \ell, \alpha)$. We need to show that $g_m(z) \in N_{n,p,\lambda}^\delta(h^{(q)}, g_m^{(q)})$. It suffices to show that g_m satisfies the condition (14). In view of Theorem 2, we have

$$\begin{aligned} \psi \left[\sum_{k=n+p}^{\infty} (k+\lambda-q-\alpha(p+\lambda-q)) \frac{k!}{(k-q)!} A_k + \sum_{k=n+p}^{\infty} (k+\lambda-q-\alpha(p+\lambda-q)) \frac{k!}{(k-q)!} B_k \right] & \leq \\ & \leq (1-\alpha)(p+\lambda-q) - \frac{(n-1+(1+\alpha)(p+\lambda-q))(1-\ell-\ell(n+p+\lambda-1-q))q_{n+p-1}^m}{((1-\ell)+\ell(p+\lambda-q))q_p^m} B_{n+p-1}. \end{aligned}$$

Then

$$\begin{aligned} \sum_{k=n+p}^{\infty} \left(\frac{k!}{(k-q)!} k \right) (A_k + B_k) & \leq \frac{(1-\alpha)(p+\lambda-q)}{\psi} - \\ & - \frac{(n-1+(1+\alpha)(p+\lambda-q))(1-\ell-\ell(n+p+\lambda-1-q))q_{n+p-1}^m}{\psi((1-\ell)+\ell(p+\lambda-q))q_p^m} B_{n+p-1} + \\ & + (q+\alpha(p+\lambda-q)) \sum_{k=n+p}^{\infty} \frac{k!}{(k-q)!} k (A_k + B_k) \leq \frac{(1-\alpha)(p+\lambda-q)}{\psi} - \\ & - \frac{(n-1+(1+\alpha)(p+\lambda-q))(1-\ell-\ell(n+p+\lambda-1-q))q_{n+p-1}^m}{\psi((1-\ell)-\ell(p+\lambda-q))(n+p+\lambda-q)^m} B_{n+p-1} + \\ & + \frac{(q+\alpha(p+\lambda-q))}{n+p} \sum_{k=n+p}^{\infty} \frac{k!}{(k-q)!} k (A_k + B_k), \end{aligned}$$

so that,

$$\begin{aligned} \sum_{k=n+p}^{\infty} \frac{k!}{(k-q)!} k (A_k + B_k) & \leq \frac{(n+p)(1-\alpha)(p+\lambda-q)}{\psi(n+((1-\alpha)(p+\lambda-q)))} - \\ & - \frac{(n+p)(n-1+(1+\alpha)(p+\lambda-q))(1-\ell-\ell(n+p+\lambda-1-q))q_{n+p-1}^m}{(n+(1-\alpha)(p+\lambda-q))(1-\ell-\ell(n+p+\lambda-1-q))(n+p+\lambda-q)^m} B_{n+p-1} = \\ & = \delta - \frac{(n+p-1)!(n+p-1)}{(n+p-1-q)!} B_{n+p-1}, \end{aligned}$$

which, in view of definition (14), completes the proof. $\square$

Theorem 6. Let $\ell < \frac{1}{p+\lambda-q}$ and $\alpha \geq 1 - \frac{1}{p+\lambda-q}$. If $g_m \in \overline{SH}_{p,\lambda}^{m,n}(q, \ell, \alpha)$, then

$$N_{n,p,\lambda}^\delta(h^{(q)}, g_m^{(q)}) \subset SH^*(n, p),$$

where $h(z)$ and $g_m(z)$ are given by (14),

$$\begin{aligned} \delta \leq & \frac{(n+p-1)!}{(n+p-1-q)!} p - \chi(1-\alpha)(p-q) + \\ & + \left(\frac{(n+p-q)!(n-1+(1+\alpha)(p+\lambda-q))(1-\ell-\ell)(n+p+\lambda-1-q)q_{n+p-1}^m}{(n+(1-\alpha)(p+\lambda-q))(1-\ell-\ell(n+p+\lambda-q))(n+p+\lambda-q)^m(n+p-1)!} - \right. \\ & \left. - (n+p-1) \right) b_{n+p-1}, \end{aligned}$$

and

$$\chi = \frac{(1-\ell+\ell(n+p+\lambda-q))(n+p-q)!q_p^m}{((1-\ell)-\ell(p+\lambda-q))(n+p+\lambda-q)^m(n+p-1)!}.$$

P r o o f. Suppose that $f_m(z) \in \overline{SH}_{p,\lambda}^{m,n}(q, \ell, \alpha)$ and $g_m(z) \in N_{n,p,\lambda}^\delta(h^{(q)}, g_m^{(q)})$. We need to show that g_m satisfies the condition (7). We have

$$\begin{aligned} \sum_{k=n+p}^{\infty} k(A_k+B_k)+(n+p-1)B_{n+p-1} & \leq \sum_{k=n+p}^{\infty} k[|a_k-A_k|+|b_k-B_k|]+(n+p-1)|b_{n+p-1}-B_{n+p-1}|+ \\ & + \sum_{k=n+p}^{\infty} k(a_k+b_k)+(n+p-1)b_{n+p-1} \leq \\ & \leq \frac{(n+p-1-q)!}{(n+p-1)!} \left[\sum_{k=n+p}^{\infty} \frac{k!}{(k-q)!} k[|a_k-A_k|+|b_k-B_k|] + \right. \\ & \left. + \frac{(n+p-1)!(n+p-1)}{(n+p-1-q)!|b_{n+p-1}-B_{n+p-1}|} \right] + (n+p-1)b_{n+p-1} \\ & \chi \left(\sum_{k=n+p}^{\infty} \left(\frac{(k+\lambda-q-\alpha(p+\lambda-q))(1-\ell+\ell(k+\lambda-q))\frac{q_k^m}{q_p^m}a_k}{(1-\ell+\ell(p+\lambda-q))} + \right. \right. \\ & \left. \left. + \frac{(k+\lambda-q-\alpha(p+\lambda-q))(1-\ell-\ell(k+\lambda-q))\frac{q_k^m}{q_p^m}b_k}{(1-\ell+\ell(p+\lambda-q))} \right) \right) \leq \\ & \leq \frac{(n+p-1-q)!}{(n+p-1)!} \delta + (n+p-1)b_{n+p-1} + \\ & + \chi \left((1-\alpha)(p+\lambda-q) - \frac{(n-1+(1+\alpha)(p+\lambda-q))(1-\ell-\ell(n+p-1+\lambda-q))}{(1-\ell+\ell(p+\lambda-q))q_p^m} \frac{q_{n+p-1}^m}{q_p^m} b_{n+p-1} \right). \end{aligned}$$

This expression is never greater than p provided that

$$\begin{aligned} \delta \leq & \frac{(n+p-1)!}{(n+p-1-q)!} \left[p - \chi(1-\alpha)(p-q) + \right. \\ & + \frac{(n+p-q)!(n-1+(1+\alpha)(p+\lambda-q))(1-\ell-\ell(n+p-1+\lambda-q))q_{n+p-1}^m}{(n+(1-\alpha)(p+\lambda-q))(1-\ell-\ell(n+p+\lambda-q))(n+p+\lambda-q)^m(n+p-1)!} - \\ & \left. - (n+p-1) \right] b_{n+p-1}. \end{aligned}$$

The proof of the other case is similar and so is omitted. $\square$

Remarks. Different type of results involving the harmonic functions can be read in [2–8].

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Об одном классе гармонических многолистных функций

Ключевые слова: гармонические многолистные функции, оператор производной, окрестность.

УДК 517.53

С помощью обобщенного оператора производной вводится новый подкласс в классе гармонических многолистных функций. Установлены оценки на коэффициенты, неравенства искажения и включения с окрестностями для различных подклассов в классе гармонических многолистных функций.

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