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QUAISI INVARIANT CONHARMONIC TENSOR OF SPECIAL CLASSES OF LOCALLY CONFORMAL ALMOST COSYMPLECTIC MANIFOLD

The authors classified a locally conformal almost cosymplectic manifold ($\mathcal{LCAc}_S$ -manifold) according to the conharmonic curvature tensor. In particular, they have determined the necessary conditions for a conharmonic curvature tensor on the $\mathcal{LCAc}_S$ -manifold of classes $CT_i, i = 1, 2, 3$ to be Φ -quasi invariant. Moreover, it has been proved that any $\mathcal{LCAc}_S$ -manifold of the class CT_1 is conharmonically Φ -paracontact.

Keywords: locally conformal almost cosymplectic manifold, conharmonic curvature tensor, Φ -quasi invariant, conharmonically Φ -paracontact.

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Introduction

The first classification of an almost contact metric structure has been introduced by Chinea and Marrero [9]. Like an almost Hermitian structure, another principle of classification of an almost contact metric structure for differential-geometric invariants of the second order depends on the properties of the Riemannian curvature tensor R . Volkova introduced analogues of these classes in contact geometry CR_1, CR_2 and CR_3 [19].

Our study focuses on the conharmonic tensor of the locally conformal almost cosymplectic manifold of three spacial classes $CT_i, i = 1, 2, 3$ which are related with the classes CR_1, CR_2, CR_3 .

A harmonic function is a function whose Laplacian vanishes. It is known that the conformal transformation on the Riemannian manifold preserves the angle between two vectors. Generally, a harmonic function is not invariant. The condition to remain such a function invariant has been studied by Ishi [13]. Specifically, he introduced a conharmonic transformation that preserves the harmonicity of a certain function. On the other hand, Ghosh et al. [11] studied the $N(k)$ -contact metric manifolds satisfying curvature conditions on the conharmonic tensor. Dwivedi and Kim [10] obtained certain necessary and sufficient conditions for the K -contact and Sasakian manifolds to be quasi conharmonically flat, ξ -conharmonically flat and Φ -conharmonically flat. Furthermore, Asghari and Taleshian [4] studied the conharmonic curvature tensor on the Kenmotsu manifold. Chanyal and Uperti [8] proved that Φ -conharmonically flat (k, μ) -contact manifolds are η -Einstein manifolds. Taleshian et al. [18] considered LP -Sasakian manifolds admitting a conharmonic curvature tensor. Abood and Al-Hussaini [3] studied the geometrical properties of the conharmonic tensor of a locally conformal almost cosymplectic manifold. In particular, the authors established the necessary and sufficient conditions for the conharmonic tensor to be flat, the aforementioned manifold to be normal and an η -Einstein manifold.

§ 1. Preliminaries

The main purpose of this section is to construct an almost contact metric structure and a locally conformal almost cosymplectic manifold in the adjoined G -structure space.

Definition 1 (see [5]). Let M be $2n + 1$ dimensional smooth manifold, η be differential 1-form called a *contact form*, ξ be a vector field called a *characteristic*, Φ be an endomorphism of the module of the vector fields $X(M)$ called a *structure endomorphism*. The family of tensors $\{\eta, \xi, \Phi\}$ is called an *almost contact structure* if the following conditions hold

1. $\eta(\xi) = 1$;
2. $\Phi(\xi) = 0$;
3. $\eta \circ \Phi = 0$;
4. $\Phi^2 = -id + \eta \otimes \xi$.

In addition, if there is a Riemannian metric $g = \langle \cdot, \cdot \rangle$ on M such that

$$\langle \Phi X, \Phi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y), \quad X, Y \in X(M),$$

then the family of tensors $\{\eta, \xi, \Phi, g\}$ is called an almost contact metric structure. In this case, the mentioned manifold M endowed with this structure is called an almost contact metric manifold.

Definition 2 (see [16]). At each point $p \in M^{2n+1}$, there is a frame in a complexification of a tangent space $T_p^c(M)$ of the form $(p, \varepsilon_0, \varepsilon_1, \dots, \varepsilon_n, \varepsilon_{\hat{1}}, \dots, \varepsilon_{\hat{n}})$, where $\varepsilon_a = \sqrt{2}\pi(e_a)$, $\varepsilon_{\hat{a}} = \sqrt{2}\bar{\pi}(e_a)$, $\hat{a} = a + n$, $\varepsilon_0 = \xi_p$, $\pi = -\frac{1}{2}(\Phi^2 + \sqrt{-1}\Phi)$ and $\bar{\pi} = \frac{1}{2}(-\Phi^2 + \sqrt{-1}\Phi)$. The frame $(p, \varepsilon_0, \varepsilon_1, \dots, \varepsilon_n, \varepsilon_{\hat{1}}, \dots, \varepsilon_{\hat{n}})$ is called an A -frame.

A set of such frames defines G -structure on M with a structure group $1 \times U(n)$. This structure is called a G -adjoined structure.

Lemma 1 (see [15]). The component matrices of the tensors Φ_p and g_p in A -frame have the following forms respectively:

$$(\Phi_j^i) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{-1}I_n & 0 \\ 0 & 0 & -\sqrt{-1}I_n \end{pmatrix}, \quad (g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix},$$

where I_n is the identity matrix of order n .

Definition 3 (see [5]). A skew-symmetric tensor $\Omega(X, Y) = g(X, \Phi Y)$ is called a fundamental form of the $\mathcal{AC}$ -structure.

Definition 4 (see [12]). An almost contact metric structure $S = (\eta, \xi, \Phi, g)$ is called an almost cosymplectic structure ($\mathcal{AC}_f$ -structure) if

1. $d\eta = 0$;
2. $d\Omega = 0$.

Definition 5 (see [17]). A conformal transformation of an $\mathcal{AC}$ -structure $S = (\eta, \xi, \Phi, g)$ on a manifold M is a mapping from S to an $\mathcal{AC}$ -structure $\tilde{S} = (\tilde{\eta}, \tilde{\xi}, \tilde{\Phi}, \tilde{g})$ such that

$$\tilde{\eta} = e^{-\sigma}\eta, \quad \tilde{\xi} = e^{\sigma}\xi, \quad \tilde{\Phi} = \Phi, \quad \tilde{g} = e^{-2\sigma}g,$$

where σ is the determining function of the conformal transformation. If $\sigma = \text{const}$, then the conformal transformation is said to be trivial.

Definition 6 (see [17]). An $\mathcal{AC}$ -structure S on a manifold M is said to be a locally conformal almost cosymplectic ($\mathcal{LCA}_S$ -structure) if the restriction of S on a some neighborhood U of an arbitrary point $p \in M$, admits a conformal transformation of an almost cosymplectic structure. This transformation is called a locally conformal. A manifold M equipped with a $\mathcal{LCA}_S$ -structure is called a $\mathcal{LCA}_S$ -manifold.

Lemma 2 (see [14]). *In the adjoined G -structure space, the structure equations of $\mathcal{LCAC}_S$ -manifold have the following forms:*

1. $d\omega^a = -\omega_b^a \wedge \omega^b + B_c^{ab}\omega^c \wedge \omega_b + B^{abc}\omega_b \wedge \omega_c + B_b^a\omega \wedge \omega^b + B^{ab}\omega \wedge \omega_b;$
2. $d\omega_a = \omega_a^b \wedge \omega_b + B_{ab}^c\omega_c \wedge \omega^b + B_{abc}\omega^b \wedge \omega^c + B_a^b\omega \wedge \omega_b + B_{ab}\omega \wedge \omega^b;$
3. $d\omega = C_b\omega \wedge \omega^b + C^b\omega \wedge \omega_b;$

where $A_{[bd]}^{ac} = -2\delta_{[b}^{[c}\sigma_{d]}^a] + 2\sigma^{[a}\delta_b^{e]}\sigma_{[c}\delta_{d]}^e] + \frac{1}{2}B^{dca}B_{ebd}$, Here B^{abc} , B_{abc} ; B^{ab} , B_{ab} ; B_a^a , B_a^b ; C^{ab} , C_{ab} ; C^b , C_b ; A_b^{acd} , A_{acd}^b ; A_{bd}^{ac} ; A_b^{ac0} , A_{ac0}^b ; B^{abci} , B_{abci} ; D^{abi} , D_{abi} and σ_{ij} are smooth functions in the adjoined G -structure space. The tensors B^{abc} ; B^{ab} are called the second and third structure tensors respectively.

Definition 7 (see [7]). A Ricci tensor is a tensor of type $(2, 0)$ which is a contracting of the Riemannian curvature tensor and defined by

$$r_{ij} = -R_{ijk}^k.$$

Lemma 3 (see [1]). *In the adjoined G -structure space, the components of the Ricci tensor of $\mathcal{LCAC}_S$ -manifold are given by the following forms:*

1. $r_{ab} = 2(-2A_{(ab)c}^c - 4(\sigma^{[c}\delta_{[b}^{h]}B_{c]ha} + \sigma^{[c}\delta_{[a}^{h]}B_{c]hb}) + \sigma_0 B_{a[c}\delta_{b]}^c + \sigma_0 B_{b[c}\delta_{a]}^c + 2\sigma_0 B_{ab} - D_{ab0} - \sigma_{ab} - \sigma_a\sigma_b + 2B_{bah}\sigma^h;$
2. $r_{\hat{a}b} = -4(\delta_{[b}^{[a}\sigma_{c]}^c] - \sigma_{[c}\delta_{h]}^b\sigma^{[h}\delta_c^a] - \frac{1}{2}\sigma^{[a}\delta_b^{h]}\sigma_h + B^{hca}B_{hcb} + B^{bch}B_{cha}) + (B^{cb}B_{ac} - B_{hb}B^{ah}) + A_{ac}^{cb} - \delta_b^a\sigma_{00} - 2n\sigma_0^2 - \sigma_b^a - \sigma^a\sigma_b;$
3. $r_{a0} = -A_{ac0}^c - \sigma^c B_{ac} + n\sigma_0\sigma_a + 2(\sigma_0[c\delta_a^c] + B^{cb}B_{bca} - 2\sigma^{[c}\delta_{[c}^{h]}B_{a]h});$
4. $r_{oo} = -2n(\sigma_{00} + \sigma_0^2) - 2B_{hc}B^{ch} - 2(\sigma_c^c + \sigma^c\sigma_c) + 4\sigma^{[c}\delta_c^{h]}\sigma_h.$

The remaining components are conjugate to the above-mentioned components.

Definition 8 (see [2]). A $\mathcal{LCAC}_S$ -manifold is said to have Φ -invariant Ricci tensor, if $\Phi \circ r = r \circ \Phi$.

Lemma 4 (see [2]). *A $\mathcal{LCAC}_S$ -manifold has Φ -invariant Ricci tensor if and only if, in the adjoined G -structure space, the following condition*

$$r_{\hat{b}}^{\hat{a}} = r_{ab} = r_0^{\hat{a}} = r_{a0} = 0$$

holds.

Definition 9 (see [6]). A pseudo-Riemannian manifold M is known as an η -Einstein of type (α, β) if its Ricci tensor satisfies the following condition:

$$r = \alpha g + \beta \eta \otimes \eta,$$

where α and β are suitable smooth functions. If $\beta = 0$, then M is referred to as an Einstein manifold.

Definition 10 (see [12]). Let M be an $\mathcal{AC}$ -manifold of dimension $2n + 1$. A tensor T of type $(4, 0)$ which is invariant under conharmonic transformation and defined by the form

$$T_{ijkl} = R_{ijkl} - \frac{1}{2n-1}(r_{il}g_{jk} - r_{jl}g_{ik} + r_{jk}g_{il} - r_{ik}g_{jl})$$

is called a conharmonic curvature tensor.

In the work [1], we have calculated all possible components of the mentioned tensor which are listed in the next lemma.

Lemma 5. *In the adjoined G -structure space, the components of the conharmonic curvature tensor of $\mathcal{LCA}_S$ -manifold are given by the following forms:*

1. $T_{abcd} = 2(2B_{[c|ab|d]} - 2\sigma_{[a}B_{b]cd} + B_{a[c}B_{d]b});$
2. $T_{\hat{a}bcd} = 2(A_{bcd}^a + 4\sigma^{[a}\delta_{[c}^{h]}B_{d]hb} - \sigma_0 B_{b[d}\delta_{c]}^a) - \frac{1}{2n-1}(r_{bc}\delta_d^a - r_{bd}\delta_c^a);$
3. $T_{\hat{a}bcd} = A_{bc}^{ad} + 4\sigma^{[a}\delta_c^{h]} \sigma_{[h}\delta_{b]}^d - 4B^{dah}B_{chb} + B^{ad}B_{bc} - \delta_c^a\delta_b^d\sigma_0^2 + \frac{1}{2n-1}(r_b^d\delta_c^a + r_c^a\delta_b^d);$
4. $T_{\hat{a}\hat{b}cd} = 2(2\delta_{[c}^{[b}\sigma_{d]}^{a]} + 2B^{hab}B_{hdc} - \delta_{[c}^a\delta_{d]}^b\sigma_0^2) - \frac{4}{2n-1}(r_{[a}^{[d}\delta_{b]}^{c]});$
5. $T_{\hat{a}0cd} = 2(\sigma_{0[c}\delta_{d]}^a + B^{ab}B_{bcd} - 2\sigma^{[a}\delta_{[c}^{h]}B_{d]h}) + \frac{1}{2n-1}(r_{0d}\delta_c^a - r_{0c}\delta_d^a);$
6. $T_{\hat{a}\hat{b}c0} = A_b^{ac0} + \sigma_b B^{ac} - \delta_b^c\sigma_0\sigma^a - \frac{1}{2n-1}(r_0^a\delta_b^c);$
7. $T_{abc0} = 2B_{cab0} + 2B_{cab}\sigma_0;$
8. $T_{\hat{a}0b0} = -\delta_b^a\sigma_{00} - \delta_b^a\sigma_0^2 - B_{cb}B^{ac} - \sigma_b^a - \sigma^a\sigma_b + 2\sigma^{[a}\delta_b^{c]}\sigma_c + \frac{1}{2n-1}(r_{00}\delta_b^a + r_b^a);$
9. $T_{\hat{a}0\hat{b}0} = 2\sigma_0 B^{ab} - D^{ab0} - \sigma^{ab} - \sigma^a\sigma^b + 2B^{bac}\sigma_c + \frac{1}{2n-1}(r_{\hat{a}\hat{b}}).$

The remaining components are conjugate to the above-mentioned components or can be obtained by the property of symmetry for T or identically equal to zero.

The next lemma gives analogues to the Gray's identities in the adjoined G -structure space.

Lemma 6 (see [19]). *An $\mathcal{AC}$ -manifold is called a manifold of class*

1. CR_1 if and only if, $R_{abcd} = R_{\hat{a}bcd} = R_{\hat{a}\hat{b}cd} = 0;$
2. CR_2 if and only if, $R_{abcd} = R_{\hat{a}bcd} = 0;$
3. CR_3 if and only if, $R_{\hat{a}bcd} = 0.$

§ 2. The main results

In this section, we introduce analogues to the Gray's identities for the conharmonic curvature tensor of a $\mathcal{LCA}_S$ -manifold.

Definition 11. In the adjoined G -structure space, a $\mathcal{LCA}_S$ -manifold is called a manifold of class

1. CT_1 if and only if, $T_{abcd} = T_{\hat{a}bcd} = T_{\hat{a}\hat{b}cd} = 0;$
2. CT_2 if and only if, $T_{abcd} = T_{\hat{a}bcd} = 0;$

3. CT_3 if and only if, $T_{\bar{a}bcd} = 0$.

Definition 12. The conharmonic tensor is called Φ -quasi invariant, if

$$\begin{aligned} \langle T(\Phi X, \Phi Y)\Phi X, \Phi Y \rangle - \langle T(\Phi^2 X, \Phi^2 Y)\Phi^2 X, \Phi^2 Y \rangle &= \langle T(\Phi^2 X, \Phi Y)\Phi^2 X, \Phi Y \rangle \\ &- \langle T(\Phi X, \Phi^2 Y)\Phi X, \Phi^2 Y \rangle \end{aligned}$$

holds for each $X, Y, Z \in X(M)$.

Definition 13. A $\mathcal{LCA}_S$ -manifold is called a conharmonically Φ -paracontact, if its conharmonic tensor satisfies the identity

$$\begin{aligned} \langle T(\Phi^2 X, \Phi^2 Y)\Phi^2 Z, \Phi^2 W \rangle &= \langle T(\Phi^2 X, \Phi^2 Y)\Phi Z, \Phi W \rangle - \langle T(\Phi X, \Phi^2 Y)\Phi Z, \Phi^2 W \rangle \\ &- \langle T(\Phi X, \Phi^2 Y)\Phi^2 Z, \Phi W \rangle, \end{aligned}$$

where $X, Y \in X(M)$.

Theorem 1. A conharmonic tensor on $\mathcal{LCA}_S$ -manifold of class CT_i , $i = 1, 2, 3$, is Φ -quasi invariant.

P r o o f. Consider the component $T_{\bar{a}bab} = 0$. Write this equation in A -frame, we have

$$\langle T(\varepsilon_{\bar{a}}, \varepsilon_b)\varepsilon_a, \varepsilon_b \rangle = 0.$$

Then we get

$$\langle T(\bar{\sigma}X, \sigma Y)\sigma X, \sigma Y \rangle = 0, \quad X, Y \in X(M)$$

We compensate for the value of the projections σ and $\bar{\sigma}$ and, using the linear property of the tensor, we can rewrite the previous expression in the form

$$\begin{aligned} &\{ \langle T(X, Y)X, Y \rangle + \langle T(\Phi X, \Phi Y)X, Y \rangle + \langle T(\Phi X, Y)\Phi X, Y \rangle + \langle T(\Phi X, Y)X, \Phi Y \rangle - \\ &\langle T(X, \Phi Y)\Phi X, Y \rangle - \langle T(X, Y)\Phi X, \Phi Y \rangle - \langle T(X, \Phi Y)X, \Phi Y \rangle - \langle T(\Phi X, \Phi Y)\Phi X, \Phi Y \rangle \} + \\ &i \{ \langle T(X, \Phi Y)\Phi X, \Phi Y \rangle - \langle T(X, \Phi Y)X, Y \rangle - \langle T(X, Y)\Phi X, Y \rangle - \langle T(X, Y)X, \Phi Y \rangle + \\ &\langle T(\Phi X, Y)X, Y \rangle - \langle T(\Phi X, \Phi Y)\Phi X, Y \rangle - \langle T(\Phi X, \Phi Y)X, \Phi Y \rangle - \langle T(\Phi X, Y)\Phi X, \Phi Y \rangle \} = 0. \end{aligned}$$

Note that the real part equals to zero, and regarding the symmetrical properties of the coharmonic curvature tensor, we have

$$\langle T(X, Y)X, Y \rangle + \langle T(\Phi X, Y)\Phi X, Y \rangle - \langle T(X, \Phi Y)X, \Phi Y \rangle - \langle T(\Phi X, \Phi Y)\Phi X, \Phi Y \rangle = 0. \quad (1)$$

Then by substituting $X \rightarrow -\Phi^2 X; Y \rightarrow -\Phi^2 Y$ in equation (1) and using the fact $\Phi^3(X) = -\Phi(X)$, we obtain the assertion of the theorem. $\square$

Theorem 2. A conharmonic curvature tensor on $\mathcal{LCA}_S$ -manifold of class CR_3 with Φ -invariant Ricci tensor is Φ -quasi invariant.

P r o o f. Since the $\mathcal{LCA}_S$ -manifold M has Φ -invariant Ricci tensor, then CT_3 and CR_3 are coincide. Regarding the Theorem 1, we get that the conharmonic curvature tensor is Φ -quasi invariant. $\square$

Theorem 3. Any $\mathcal{LCA}_S$ -manifold M of class CT_1 is conharmonically Φ -paracontact.

P r o o f. On the G -adjoined structure space, we have $T_{\hat{a}\hat{b}cd} = 0$. In A -frame, we get

$$\langle T(\varepsilon_{\hat{a}}, \varepsilon_{\hat{b}})\varepsilon_c, \varepsilon_d \rangle = 0.$$

Consequently, we have

$$\langle T(\bar{\sigma}X, \sigma Y)\sigma X, \sigma Y \rangle = 0, \quad X, Y, Z, W \in X(M).$$

We make up for the value of the projections σ and $\bar{\sigma}$ and, using the linear property of the tensor, we can rewrite this expression in the form

$$\langle T(X, Y)Z, W \rangle - \langle T(\Phi X, \Phi Y)Z, W \rangle + \langle T(X, \Phi Y)\Phi Z, W \rangle + \langle T(X, \Phi Y)Z, \Phi W \rangle - \langle T(X, Y)\Phi Z, \Phi W \rangle + \langle T(\Phi X, Y)\Phi Z, W \rangle + \langle T(\Phi X, Y)Z, \Phi W \rangle + \langle T(\Phi X, \Phi Y)\Phi Z, \Phi W \rangle = 0. \quad (2)$$

According to the Definition 11, we have $T_{abcd} = 0$. Using the same technique, consequently we obtain

$$\langle T(X, Y)Z, W \rangle + \langle T(\Phi X, \Phi Y)Z, W \rangle - \langle T(X, \Phi Y)\Phi Z, W \rangle - \langle T(X, \Phi Y)Z, \Phi W \rangle - \langle T(X, Y)\Phi Z, \Phi W \rangle + \langle T(\Phi X, Y)\Phi Z, W \rangle + \langle T(\Phi X, Y)Z, \Phi W \rangle - \langle T(\Phi X, \Phi Y)\Phi Z, \Phi W \rangle = 0. \quad (3)$$

From the combination of the equations (2) and (3), it follows that

$$\langle T(X, Y)Z, W \rangle = \langle T(X, Y)\Phi Z, \Phi W \rangle - \langle T(\Phi X, Y)\Phi Z, W \rangle - \langle T(\Phi X, Y)Z, \Phi W \rangle. \quad (4)$$

Then by substituting $X \rightarrow -\Phi^2 X; Y \rightarrow -\Phi^2 Y; Z \rightarrow -\Phi^2 Z, W \rightarrow -\Phi^2 W$ in the equation (4) and using the fact $\Phi^3(X) = -\Phi(X)$, we get the result. $\square$

Theorem 4. *Let M be $\mathcal{LCA}_S$ -manifold of class CR_2 , then the first structure tensor is parallel in the first canonical connection.*

P r o o f. Suppose that M is $\mathcal{LCA}_S$ -manifold of the class CR_2 . So we have

$$2((B_{cabd} - B_{dabc}) - 2\sigma_{[a}B_{b]cd} + B_{a[c}B_{d]b}) = 0. \quad (5)$$

Symmetrizing (5) by the indices (c, d) and then by the indices (c, b) , we get

$$B_{cabd} = 0.$$

Regarding the fundamental theorem of tensor analysis, we have

$$\nabla B_{abc} = dB_{abc} + B_{dbc}\omega_a^d + B_{adc}\omega_b^d + B_{abd}\omega_c^d = B_{abcd}\omega^d.$$

So, we get

$$\nabla B_{abc} = B_{abcd}\omega^d.$$

It follows that

$$\nabla B_{abc} = 0.$$

Therefore, the tensor B_{abc} is parallel. $\square$

Theorem 5. *Let M be $\mathcal{LCA}_S$ -manifold of class CR_1 , then its second and third structure tensors identically vanish.*

P r o o f. Let M be $\mathcal{LCA}_S$ -manifold of the class CR_1 . According to the component of Riemannian curvature tensor, we have

$$2(2B[c|ab|d] - 2\sigma_{[a}B_{b]cd}B_{a[c}B_{d]b}) = 0. \quad (6)$$

Symmetrizing and then antisymmetrizing (6) by the indices (a, b) , we deduce

$$B_{ac}B_{db} - B_{ad}B_{cb} = 0. \quad (7)$$

Antisymmetrizing (7) by the indices (a, d) , we get

$$B_{ac}B_{db} = 0. \quad (8)$$

Contracting (8) by the indices (a, d) and then by (c, b) , we get $B_{ac}^2 = 0$, then

$$B_{ac} = 0.$$

From the condition $R_{\hat{a}bcd} = 0$, we obtain

$$B^{abc} = 0.$$

□

In the rest of this section, we establish a relation between the classes CT_1 and CR_1 .

Theorem 6. Suppose that M is $\mathcal{LCA}_S$ -manifold of class CT_1 with Φ -invariant Ricci tensor; then M is a manifold of class CR_1 if and only if M has flat Ricci tensor.

P r o o f. Suppose that M has flat Ricci tensor. Since M is a $\mathcal{LCA}_S$ -manifold of class CT_1 , then according to the Definition 11, we have

$$T_{abcd} = T_{\hat{a}bcd} = T_{\hat{a}bcd} = 0.$$

The condition $T_{abcd} = 0$, implies that $R_{abcd} = 0$; while the condition $T_{\hat{a}bcd} = 0$ with Φ -invariant Ricci tensor, implies that $R_{\hat{a}bcd} = 0$. In addition, $T_{\hat{a}bcd} = 0$ with flat Ricci tensor, indicates $R_{\hat{a}bcd} = 0$. Then M is a manifold of class CR_1 . Conversely, according to the condition $T_{\hat{a}bcd} = 0$, we have

$$2(2\delta_{[c}^{[b}\sigma_{d]}^a] + 2B^{hab}B_{hdc} - \delta_{[c}^a\delta_{d]}^b\sigma_0^2) - \frac{4}{2n-1}(r_{[c}^{[a}\delta_{d]}^b]) = 0.$$

Since M is a manifold of class CR_1 , we get

$$-\frac{4}{2n-1}(r_{[c}^{[a}\delta_{d]}^b]) = 0. \quad (9)$$

Taking the contraction operation for (9) by the indices (b, d) , we deduce

$$r_c^a\delta_b^b - r_b^a\delta_c^b - r_c^b\delta_b^a + r_b^b\delta_c^a = 0.$$

Consequently, we obtain

$$(n-2)r_c^a + r_b^b\delta_c^a = 0. \quad (10)$$

Taking the symmetrization and antisymmetrization operations for (10) by the indices (b, a) , we conclude

$$r_c^a = 0. \quad (11)$$

Making use of the Theorems 3 and 5, we have

$$-4(\delta_{[b}^{[a} \sigma_{c]}^c] - \sigma_{[c} \delta_{h]}^c \sigma^{[h} \delta_c^a] - \frac{1}{2} \sigma^{[a} \delta_b^{h]} \sigma_h) + A_{ac}^{cb} - \delta_b^a \sigma_{00} - 2n \delta_b^a \sigma_0^2 - \sigma_b^a - \sigma^a \sigma_b = 0. \quad (12)$$

Symmetrizing (12) by the indices (c, h) , we obtain

$$-\sigma_{[c} \delta_{h]}^c \sigma^{[h} \delta_c^a] = 0. \quad (13)$$

Contracting (13) by the indices (a, c) , we get

$$\sigma_a \delta_h^a \sigma^h \delta_a^a - \sigma_a \delta_h^a \sigma^a \delta_a^h - \sigma_h \delta_a^a \sigma^h \delta_a^a + \sigma_h \delta_a^a \sigma^a \delta_a^h = 0.$$

Hence

$$-n(n-1)(\sigma_h \sigma^h) = 0.$$

Or equivalently,

$$\sigma_h \bar{\sigma}_h = 0 \Leftrightarrow \sum_h |\sigma_h|^2 = 0 \Leftrightarrow \sigma_h = 0. \quad (14)$$

Once again, by using the equation (12), we have

$$A_{ac}^{cb} - \delta_b^a \sigma_{00} - 2n \delta_b^a \sigma_0^2 = 0. \quad (15)$$

By virtue of the Theorems 2, 5 and equation (14), the tensors $A_{[bd]}^{ac} = A_{bd}^{[ac]} = 0$ become symmetric regarding the lower and upper indices. Antisymmetrizing (15) by the indices (c, b) , we obtain

$$-\delta_b^a \sigma_{00} - 2n \delta_b^a \sigma_0^2 = 0. \quad (16)$$

Contracting (16) by the indices (a, b) , consequently, we get

$$r_{00} = 0. \quad (17)$$

According to the equations (11), (17) and Φ -invariant Ricci property, we get that M has flat Ricci tensor. $\square$

Finally, we established an application by identifying the necessary and sufficient condition for the locally conformal almost contact manifold to be η -Einstein manifold

Theorem 7. Suppose that M is $\mathcal{LCA}_S$ -manifold of class CT_1 with Φ -invariant Ricci tensor. Then the necessary and sufficient condition for M to be η -Einstein manifold is $\sigma_d^a = E \delta_d^a$, where $E = \frac{2(1-n)\sigma_0^2}{n-2} - \frac{\alpha}{2n-1}$.

P r o o f. Suppose that M is $\mathcal{LCA}_S$ -manifold of class CT_1 . According to the Definition 11 and Lemma 5, we have

$$2(2\delta_{[c}^{[b} \sigma_{d]}^a] - \delta_{[c}^a \delta_{d]}^b \sigma_0^2) - \frac{4}{2n-1} (r_{[d}^{[a} \delta_{c]}^{b]}) = 0. \quad (18)$$

Contracting (18) by indices (b, c) , we get

$$(n-2)\sigma_d^a + \delta_d^a \sigma_b^b - 2(1-n)\sigma_0^2 \delta_d^a - \frac{1}{2n-1} ((n-2)r_d^a + \delta_d^a r_b^b) = 0.$$

Then, we deduce

$$\sigma_d^a = \frac{2(1-n)\sigma_0^2\delta_d^a}{n-2} - \frac{r_d^a}{2n-1}. \quad (19)$$

Let M be η -Einstein manifold. We use the Definition 9, so the equation (19) becomes

$$\sigma_d^a = E\delta_d^a. \quad (20)$$

Conversely, from the equations (19) and (20), we have

$$r_d^a = \alpha\delta_d^a$$

where α is the cosmological constant. Also

$$r_{00} = \alpha + \beta$$

where $\alpha = -2n(\sigma_{00} + \sigma_0^2) - 2B_{hc}B^{ch} - 2(\sigma_c^c + \sigma^c\sigma_c) + 4\sigma^{[c}\delta_c^{h]}\sigma_h - \alpha$

According to Φ -invariant of Ricci tensor, we obtain that M is η -Einstein manifold. $\square$

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Квазиинвариантный конгармонический тензор специальных классов локально конформного почти косимплектического многообразия

Ключевые слова: локально конформное почти косимплектическое многообразие, тензор конгармонической кривизны, Φ -квазиинвариант, конгармонический Φ -параконтакт.

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В работе описывается классификация локально конформного почти косимплектического многообразия ($\mathcal{LCA}_S$ -многообразия) в соответствии с тензором конгармонической кривизны. В частности, были получены необходимые условия Φ инвариантности тензора конгармонической кривизны на $\mathcal{LCA}_S$ -многообразии классов CT_i , $i = 1, 2, 3$. Кроме того, доказано, что любое $\mathcal{LCA}_S$ -многообразие класса CT_1 оказывается конгармоническим и Φ -параконтактным.

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