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© *O. S. Kostromina***ON TWO-FREQUENCY QUASI-PERIODIC PERTURBATIONS OF SYSTEMS CLOSE TO TWO-DIMENSIONAL HAMILTONIAN ONES WITH A DOUBLE LIMIT CYCLE**

The problem of the effect of two-frequency quasi-periodic perturbations on systems close to arbitrary nonlinear two-dimensional Hamiltonian ones is studied in the case when the corresponding perturbed autonomous systems have a double limit cycle. Its solution is important both for the theory of synchronization of nonlinear oscillations and for the theory of bifurcations of dynamical systems. In the case of commensurability of the natural frequency of the unperturbed system with frequencies of quasi-periodic perturbation, resonance occurs. Averaged systems are derived that make it possible to ascertain the structure of the resonance zone, that is, to describe the behavior of solutions in the neighborhood of individual resonance levels. The study of these systems allows determining possible bifurcations arising when the resonance level deviates from the level of the unperturbed system, which generates a double limit cycle in a perturbed autonomous system. The theoretical results obtained are applied in the study of a two-frequency quasi-periodic perturbed pendulum-type equation and are illustrated by numerical computations.

Keywords: quasi-periodic perturbations, double limit cycle, resonances, averaged systems.

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Introduction

In recent years there has been considerable interest in the study of multi-frequency quasi-periodic oscillations [1–13]. Such phenomena are widespread in nature and technology, in almost all areas of physics (radiophysics, electronics, laser physics, astrophysics, and many others), as well as in biology, chemistry, and medicine (see the above papers as well as monographs [14–16]). Systems with a quasi-periodic dynamics are primarily non-autonomous system with two or more incommensurable frequencies. However, physicists have revealed the presence of quasi-periodic oscillations in autonomous systems (generators of quasi-periodic oscillations) [17–19].

Problems of the existence of quasi-periodic and almost periodic solutions, as well as complex dynamics in systems close to Hamiltonian ones, were studied in [6, 20–26]. The papers [20, 21] discussed the existence of quasi-periodic and almost periodic solutions for the Duffing equation. The questions of the existence of complex dynamics for the pendulum-type equation and the Duffing equation under quasi-periodic perturbations were considered in [6, 22–26]. More general results (for ordinary differential equations that contain a small parameter) on the existence of almost periodic solutions and integral manifolds are presented in [27]. In contrast to these papers, papers [3, 7, 8, 11] are devoted to the problem of the effect of small nonconservative quasi-periodic perturbations on systems close to arbitrary nonlinear two-dimensional Hamiltonian ones with limit cycles. Based on research methods for periodically perturbed systems [28, 29], the structure of non-degenerate resonance zones is revealed, conditions for the existence of quasi-periodic solutions in them are found, the problem of synchronization of quasi-periodic oscillations is solved, and the global behavior of solutions is analyzed. The theoretical results obtained in these studies are illustrated by the example of Duffing-type equations.

Let us consider the system

$$\begin{cases} \dot{x} = \frac{\partial H(x, y)}{\partial y} + \varepsilon g(x, y, \omega_1 t, \omega_2 t), \\ \dot{y} = -\frac{\partial H(x, y)}{\partial x} + \varepsilon f(x, y, \omega_1 t, \omega_2 t), \end{cases} \quad (0.1)$$

where ε is a small positive parameter, Hamiltonian function H and the functions g, f are sufficiently smooth (analytic) and uniformly bounded in x, y within some domain $D \subset \mathbb{R}^2$ (or $D \subset \mathbb{R}^1 \times \mathbb{S}^1$) along with partial derivatives up to the second order. In addition, functions g, f are continuous and quasi-periodic in t uniformly with respect to $(x, y) \in D$ with incommensurable frequencies ω_1, ω_2 .

Let us assume that the unperturbed system ($\varepsilon = 0$) is nonlinear and has a cell $D_0 \in D$ filled with closed phase curves $H(x, y) = h$, $h \in [h_{\min}, h_{\max}]$, and not containing any equilibria and separatrices. Let us suppose also that the perturbation is non-conservative, which is equivalent to the condition: $g'_x + f'_y \neq 0$.

Passing from variables x, y to the action I – angle θ variables in D_0 :

$$x = X(I, \theta), \quad y = Y(I, \theta), \quad (0.2)$$

where X, Y are functions periodic in θ with period 2π , we obtain the system of the form

$$\begin{cases} \dot{I} = \varepsilon F(I, \theta, \theta_1, \theta_2), \\ \dot{\theta} = \omega(I) + \varepsilon G(I, \theta, \theta_1, \theta_2), \\ \dot{\theta}_1 = \omega_1, \\ \dot{\theta}_2 = \omega_2, \end{cases} \quad (0.3)$$

where

$$\begin{aligned} F(I, \theta, \theta_1, \theta_2) &\equiv f(X, Y, \theta_1, \theta_2)X'_\theta - g(X, Y, \theta_1, \theta_2)Y'_\theta, \quad G(I, \theta, \theta_1, \theta_2) \equiv \\ &\equiv -f(X, Y, \theta_1, \theta_2)X'_I + g(X, Y, \theta_1, \theta_2)Y'_I. \end{aligned}$$

Here $\omega(I)$ is the frequency of motion in closed phase curves (natural frequency of the unperturbed system). We assume that the function $\omega(I)$ is monotonic and does not vanish on the interval $(I_{\min}, I_{\max}) \equiv (I(h_{\min}), I(h_{\max}))$. The functions F, G are smooth enough in $I, \theta, \theta_1, \theta_2$ within domain $[I_{\min}, I_{\max}] \times \mathbb{T}^3$, where $\mathbb{T}^3$ is a three-dimensional torus.

It is said that in system (0.3) there is a *resonance* if:

$$n\omega(I) = m_1\omega_1 + m_2\omega_2, \quad (0.4)$$

where n, m_1 and m_2 are coprime integer numbers. We denote the value of I , calculated from (0.4), by $I_{nm_1m_2}$. Accordingly, we will refer to the unperturbed levels $I = I_{nm_1m_2}$ (closed phase curve $H(x, y) = h_{nm_1m_2}$ of the unperturbed system) as the *resonance levels*.

Along with the system (0.1), let us consider the corresponding autonomous system

$$\begin{cases} \dot{x} = \frac{\partial H(x, y)}{\partial y} + \varepsilon g_0(x, y), \\ \dot{y} = -\frac{\partial H(x, y)}{\partial x} + \varepsilon f_0(x, y), \end{cases} \quad (0.5)$$

where

$$g_0(x, y) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} g(x, y, \theta_1, \theta_2) d\theta_1 d\theta_2, \quad f_0(x, y) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f(x, y, \theta_1, \theta_2) d\theta_1 d\theta_2.$$

Suppose that system (0.5) has a double limit cycle. This means that the Poincaré–Pontryagin generating function has a double root $h = h_*$ (for more details, see the next section).

The effect of quasi-periodic perturbation for a system of the form (0.5) with a structurally stable limit cycle (in this case, the Poincaré–Pontryagin generating function has a simple root $h = h_0$) was studied in [1, 3, 7, 8, 11]. The action of time-periodic perturbation for a system with a double limit cycle was investigated in [30]. However, this study excluded the case of parametric perturbations. A Duffing-type equation was considered as an illustrative example.

The problem of the effect of quasi-periodic perturbations on a perturbed autonomous system of the form (0.5), which has a double limit cycle, has not been previously studied and is considered for the first time in this paper. We study the rearrangements of phase portraits of the averaged system near the bifurcation case, when the resonance level $I = I_{nm_1m_2}$ coincides with the level $I = I_*$ in the neighborhood of which system (0.5) has a double limit cycle. The solution to this problem is essential for the theory of synchronization of oscillations, as well as the theory of bifurcations of dynamical systems. This study largely follows the paper [30], as well as the works [3, 7, 11].

As an illustrative example, we consider a pendulum-type equation of the form

$$\ddot{x} + \sin x = \varepsilon[(-1 + p_1 \cos 3x + p_2 x \alpha(t))\dot{x} + p_3 \alpha(t)], \quad (0.6)$$

where $p_1, p_2 > 0, p_3 > 0$ are parameters, ε is a small positive parameter, $\alpha(t) = \cos \omega_1 t \sin \omega_2 t$. We suppose that ω_1, ω_2 are incommensurable. The perturbation contains a nonlinear parametric term $p_2 x \dot{x} \alpha(t)$. The case when $p_2 = 0$ and the function $\alpha(t)$ is periodic was studied in [31].

§ 1. Poincaré–Pontryagin generating function and double cycles in the perturbed autonomous system

In cell D_0 , pass to the action I – angle θ variables in system (0.5), writing this transformation in the form (0.2), and averaging the resulting system over θ , we obtain the system

$$\dot{u} = \varepsilon B_0(u), \quad (1.1)$$

where

$$B_0(u) = \frac{1}{2\pi} \int_0^{2\pi} [f_0(X, Y) X'_\theta - g_0(X, Y) Y'_\theta] d\theta.$$

Here, $u = I + O(\varepsilon)$. The function $B_0(u)$ is called the Poincaré–Pontryagin generating function.

It is known [28, 29], if $u = u_0$ is a simple equilibrium state ($B_0(u_0) = 0, B'_0(u_0) \neq 0$) of the averaged system (1.1), then in system (0.5) for sufficiently small ε this equilibrium state corresponds to a rough limit cycle. Moreover, the limit cycle will be stable if the equilibrium state is stable, that is, $B'(u_0) < 0$, and unstable otherwise. Let us denote

$$B_1(u_0) \equiv B'_0(u_0) = \frac{1}{2\pi} \int_0^{2\pi} (g'_{0x} + f'_{0y}) \Big|_{\substack{x=X(u_0, \theta) \\ y=Y(u_0, \theta)}} d\theta.$$

The double root $u = u_*$ of the equation $B_0(u) = 0$ determines the unperturbed level $I = I_*$ (closed phase curve $h = h_*$ of the unperturbed system), from which a double limit cycle is generated under the action of the perturbation. In this case, $B_1(u_*) = 0$, $B'_1(u_*) \neq 0$.

When non-autonomous disturbances affect an autonomous system, a limit cycle may coincide with the resonance level.

§ 2. Averaging system in the neighborhood of the individual resonance level

It is known [3, 7], in the neighborhood

$$U_\mu = \{(I, \theta) : I_{nm_1m_2} - C\mu < I < I_{nm_1m_2} + C\mu, 0 \leq \theta < 2\pi, C = \text{const} > 0\},$$

$\mu = \sqrt{\varepsilon}$ of the individual resonance level $I = I_{nm_1m_2}$ (it will be called the *resonance zone*) the system (0.3) after replacing

$$I = I_{nm_1m_2} + \mu u, \quad \theta = v + \frac{m_1\theta_1 + m_2\theta_2}{n}$$

reduces to an averaged system of the form

$$\begin{cases} \dot{u} = \mu A(v; I_{nm_1m_2}) + \mu^2 P_0(v; I_{nm_1m_2})u + O(\mu^3), \\ \dot{v} = \mu b_1 u + \mu^2 (b_2 u^2 + Q_0(v; I_{nm_1m_2})) + O(\mu^3), \end{cases} \quad (2.1)$$

where

$$A(v; I_{nm_1m_2}) = \frac{1}{4\pi^2 n^2} \int_0^{2\pi n} \int_0^{2\pi n} F \left(I_{nm_1m_2}, v + \frac{m_1\theta_1 + m_2\theta_2}{n}, \theta_1, \theta_2 \right) d\theta_1 d\theta_2,$$

$$P_0(v; I_{nm_1m_2}) = \frac{1}{4\pi^2 n^2} \int_0^{2\pi n} \int_0^{2\pi n} F'_I \left(I_{nm_1m_2}, v + \frac{m_1\theta_1 + m_2\theta_2}{n}, \theta_1, \theta_2 \right) d\theta_1 d\theta_2,$$

$$Q_0(v; I_{nm_1m_2}) = \frac{1}{4\pi^2 n^2} \int_0^{2\pi n} \int_0^{2\pi n} G \left(I_{nm_1m_2}, v + \frac{m_1\theta_1 + m_2\theta_2}{n}, \theta_1, \theta_2 \right) d\theta_1 d\theta_2,$$

$$b_1 = \omega'(I_{nm_1m_2}), \quad b_2 = \omega''(I_{nm_1m_2})/2.$$

The functions $A(v; I_{nm_1m_2})$, $P_0(v; I_{nm_1m_2})$ and $Q_0(v; I_{nm_1m_2})$ are periodic in v with the least period equal to $2\pi/n$ [3, 7]. Therefore, the phase space of system (2.1) is the cylinder $\{v(\text{mod}(2\pi/n)), u\}$. A simple stable (unstable) equilibrium state of the system (2.1) corresponds to a stable (unstable) quasi-periodic resonance solution with periods $2\pi n/\omega_1$, $2\pi n/\omega_2$, and two-dimensional invariant tori with quasi-periodic motion in the original four-dimensional system.

The functions $A(v; I_{nm_1m_2})$ and $P_0(v; I_{nm_1m_2})$ can be represented as

$$\begin{aligned} A(v; I_{nm_1m_2}) &= \tilde{A}(v; I_{nm_1m_2}) + B_0(I_{nm_1m_2}), \quad P_0(v; I_{nm_1m_2}) = \tilde{P}_0(v; I_{nm_1m_2}) + B_1(I_{nm_1m_2}), \\ B_0(I_{nm_1m_2}) &= \langle A(v; I_{nm_1m_2}) \rangle_v = \frac{n}{2\pi} \int_0^{2\pi/n} A(v; I_{nm_1m_2}) dv, \\ B_1(I_{nm_1m_2}) &= \langle P_0(v; I_{nm_1m_2}) \rangle_v = \frac{n}{2\pi} \int_0^{2\pi/n} P_0(v; I_{nm_1m_2}) dv. \end{aligned}$$

Since we want to study the case, when $I_{nm_1m_2} = I_*$ ($B_0(I_*) = B_1(I_*) = 0$), it is necessary to take into account the terms $O(\mu^3)$. Neglecting the terms $O(\mu^4)$, we obtain the system

$$\begin{cases} \dot{u} = \mu \left(\tilde{A}(v; I_{nm_1m_2}) + B_0(I_{nm_1m_2}) \right) + \mu^2 \left(\tilde{P}_0(v; I_{nm_1m_2}) + B_1(I_{nm_1m_2}) \right) u + \\ \quad + \mu^3 P_1(v; I_{nm_1m_2}) u^2, \\ \dot{v} = \mu b_1 u + \mu^2 (b_2 u^2 + Q_0(v; I_{nm_1m_2})) + \mu^3 (b_3 u^3 + Q_1(v; I_{nm_1m_2}) u), \end{cases} \quad (2.2)$$

where

$$\begin{aligned} P_1(v; I_{nm_1m_2}) &= \frac{1}{8\pi^2 n^2} \int_0^{2\pi n} \int_0^{2\pi n} F''_{II} \left(I_{nm_1m_2}, v + \frac{m_1\theta_1 + m_2\theta_2}{n}, \theta_1, \theta_2 \right) d\theta_1 d\theta_2, \\ Q_1(v; I_{nm_1m_2}) &= \frac{1}{4\pi^2 n^2} \int_0^{2\pi n} \int_0^{2\pi n} G'_I \left(I_{nm_1m_2}, v + \frac{m_1\theta_1 + m_2\theta_2}{n}, \theta_1, \theta_2 \right) d\theta_1 d\theta_2, \\ b_3 &= \omega'''(I_{nm_1m_2})/6. \end{aligned}$$

The functions $P_1(v; I_{nm_1m_2})$ can be represented as $P_1(v; I_{nm_1m_2}) = \tilde{P}_1(v; I_{nm_1m_2}) + B_2(I_{nm_1m_2})$, $B_2(I_{nm_1m_2}) = B''_0(I_{nm_1m_2})/2$.

Making the change $u \rightarrow u - \mu Q_0(v; I_{nm_1m_2})/b_1$ in system (2.2) and passing to the slow time $\tau = \mu t$, we obtain the system

$$\begin{cases} \frac{du}{d\tau} = \tilde{A}(v; I_{nm_1m_2}) + B_0(I_{nm_1m_2}) + \mu [\tilde{\sigma}(v; I_{nm_1m_2}) + B_1(I_{nm_1m_2})] u + \mu^2 \left[\left(\tilde{P}_1(v; I_{nm_1m_2}) + \right. \right. \\ \quad \left. \left. + B_2(I_{nm_1m_2}) + \frac{b_2}{b_1} \frac{dQ_0(v; I_{nm_1m_2})}{dv} \right) u^2 - \tilde{P}_0(v; I_{nm_1m_2}) \frac{Q_0(v; I_{nm_1m_2})}{b_1} \right], \\ \frac{dv}{d\tau} = b_1 u + \mu b_2 u^2 + \mu^2 \left[b_3 u^3 + \left(Q_1(v; I_{nm_1m_2}) - 2 \frac{b_2}{b_1} Q_0(v; I_{nm_1m_2}) \right) u \right], \end{cases} \quad (2.3)$$

where $\tilde{\sigma}(v; I_{nm_1m_2}) = \tilde{P}_0(v; I_{nm_1m_2}) + dQ_0(v; I_{nm_1m_2})/dv$. This system determines the topology of individual resonance zones up to terms of order μ^3 . According to [3], a simple stable (unstable) equilibrium state of the averaged system (2.3) corresponds to a stable (unstable) quasi-periodic resonance solution of periods $\frac{2\pi n}{\omega_1}, \frac{2\pi n}{\omega_2}$ in system (0.1), or a two-dimensional stable (unstable) invariant torus in system (0.3). A rough limit cycle of the averaged system (2.3) with frequency ω_0 corresponds to a quasi-periodic resonance solution of periods $\frac{2\pi}{\omega_0}, \frac{2\pi n}{\omega_1}, \frac{2\pi n}{\omega_2}$ in system (0.1), or a three-dimensional invariant torus in system (0.3) of the same stability type as the cycle.

An important role in the study of the structure of the neighborhood of resonance levels $I = I_{nm_1m_2}$ is played by the function $\sigma(v; I_{nm_1m_2}) = \tilde{\sigma}(v; I_{nm_1m_2}) + B_1(I_{nm_1m_2})$, $B_1(I_{nm_1m_2}) = \langle \sigma(v; I_{nm_1m_2}) \rangle_v$. First of all, it is necessary to find out whether it depends on v or is constant [28, 29].

§ 3. Study of the averaged system

Put $B_0(I_{nm_1m_2}) = B_1(I_{nm_1m_2}) = 0$. Then system (2.3) without taking into account the conservative term $2\mu \frac{b_2}{b_1} \tilde{A}(v; I_{nm_1m_2}) \dot{v}$ is transformed to the form

$$\ddot{v} - b_1 \tilde{A}(v; I_{nm_1m_2}) = \mu \tilde{\sigma}(v; I_{nm_1m_2}) \dot{v} + \mu^2 [M(v; I_{nm_1m_2}) \dot{v}^2 + N(v; I_{nm_1m_2})], \quad (3.1)$$

where

$$M(v; I_{nm_1 m_2}) = \frac{1}{b_1} \left[\tilde{P}_1(\cdot) + Q_{1v}'(\cdot) + \frac{b_2}{b_1} \left(\tilde{P}_0(\cdot) + \tilde{\sigma}(\cdot) \right) + 3 \frac{b_3}{b_1} \tilde{A}(\cdot) \right] + \frac{1}{b_1} B_2(I_{nm_1 m_2}),$$

$$N(v; I_{nm_1 m_2}) = -\tilde{P}_0(\cdot) Q_0(\cdot) + \left(Q_1(\cdot) - 2 \frac{b_2}{b_1} Q_0(\cdot) \right) \tilde{A}(\cdot).$$

Here $(\cdot) \equiv (v; I_{nm_1 m_2})$.

Let us rewrite equation (3.1) as

$$\begin{cases} \dot{u} = \tilde{A}(v; I_{nm_1 m_2}) + \mu \tilde{\sigma}(v; I_{nm_1 m_2}) u + \frac{\mu^2}{b_1} [M(v; I_{nm_1 m_2}) b_1^2 u^2 + N(v; I_{nm_1 m_2})], \\ \dot{v} = b_1 u. \end{cases} \quad (3.2)$$

Let us consider the case, when $\sigma(v; I_{nm_1 m_2}) = \text{const}$, therefore $\tilde{\sigma} = 0$. In this case the system (3.2) is invariant under the change $u \rightarrow -u, \tau \rightarrow -\tau$, therefore, the phase portrait of equation (3.1) is symmetric about the u axis. Phase space of system (3.2) is the cylinder $\{v(\text{mod}(2\pi/n)), u\}$. System (3.2) is close to Hamiltonian with the Hamiltonian function $\tilde{H}(v, u) = b_1 u^2/2 - V(v)$, $V(v) = \int \tilde{A}(v; I_{nm_1 m_2}) dv$. Denote by $(v_0, 0)$ the equilibrium state of the unperturbed system. System (3.2) has simple equilibrium states of two types: the center, if $b_1 \tilde{A}'_v(v_0) < 0$, and the saddle, if $b_1 \tilde{A}'_v(v_0) > 0$. To establish the relative position of the separatrices of the saddle $(v_0, 0)$ of the unperturbed system under the action of the perturbation, we use the Melnikov formula [32]: $\Delta_\mu = \mu^2 \Delta_1 + O(\mu^3)$, which determines (up to terms of order μ^3) the distance between the perturbed separatrices. Making the substitution $v = w + v_0$, moving the equilibrium state $(v_0, 0)$ of the saddle type of the unperturbed system to the origin, we obtain

$$\Delta_1 = \frac{1}{b_1} \int_{-\infty}^{\infty} [M(w + v_0) b_1^2 u^2 + N(w + v_0)] \frac{dw}{d\tau} d\tau, \quad (3.3)$$

where $w(\tau), u(\tau)$ is the solution of the unperturbed system on the separatrix. From the integral of the unperturbed system we find

$$u = \pm \sqrt{\frac{2}{b_1} (V(w, v_0) - V(0, v_0))}, \quad V(w, v_0) = \int \tilde{A}(w + v_0) dw. \quad (3.4)$$

Substituting (3.4) into (3.3), we obtain

$$\Delta_1 = \frac{1}{b_1} \int_0^{2\pi/n} [2b_1 M(w + v_0) (V(w, v_0) - V(0, v_0)) + N(w + v_0)] dw.$$

There are two possible cases: $\Delta_1 = 0$ and $\Delta_1 \neq 0$. In the first case, we have two separatrix loops of the saddle on the phase cylinder (on the upper and lower half-cylinders). This is a bifurcation case. In the second case, the separatrices are split, and the size of the gap depends on the relationship between the amplitude of the function $\tilde{M}(v; I_{nm_1 m_2}) = \tilde{P}_1(v; I_{nm_1 m_2}) + Q_{1v}'(v; I_{nm_1 m_2}) + \frac{b_2}{b_1} \tilde{P}_0(v; I_{nm_1 m_2}) + 3 \frac{b_3}{b_1} \tilde{A}(v; I_{nm_1 m_2})$ and the value $B_2(I_{nm_1 m_2})$. Possible phase portraits of system (3.2) on the phase cylinder $\{v(\text{mod}(2\pi/n)), u\}$ are shown in Fig. 1. The case of impassable resonance (synchronization of oscillations) is shown in Fig. 1, b: for any initial

Fig. 1. Possible phase portraits of system (3.2) for $\tilde{\sigma} = 0$

conditions of the phase space of the averaged system, the phase curves of the averaged system remain in a limited region of the phase space. The case of partly passable resonance is shown in Fig. 1, *a, c*: in the phase space of the averaged system, there are sets of initial conditions (inside the separatrix loop) that correspond to bounded phase curves, as well as sets of initial conditions that correspond to phase curves (outside the separatrix loop) that leave any bounded region of the phase space when $\tau \rightarrow \infty$.

Let us consider the case of an alternating function $\tilde{\sigma}(v; I_{nm_1m_2})$. Possible phase portraits of system (3.2) on the phase cylinder $\{v(\text{mod}(2\pi/n)), u\}$ in this case are shown in Fig. 2. The

Fig. 2. Possible phase portraits of system (3.2) for an alternating function $\tilde{\sigma}(v; I_{nm_1m_2})$

structure of impassable resonance zone is shown in Fig. 2, *a, b*: the phase point tends to a stable

equilibrium state at $\tau \rightarrow \infty$ for any initial point (a); for any initial point on the lower half-cylinder and on an unstable limit cycle located on the upper half-cylinder (b). The bifurcation case, when the limit cycle merges into the separatrix loop on the upper half-cylinder, is shown in Fig. 2, c. The question of limit cycles can be solved using the Poincaré–Pontryagin generating function, similarly to how it is done for a perturbed autonomous system (0.5). The structure of partly passable resonance zone is shown in Fig. 2, d.

We assign $B_0(I_{nm_1m_2}) = \mu\gamma_1$, $B_1(I_{nm_1m_2}) = \mu\gamma_2$. For nonzero parameters γ_1 and γ_2 , possible phase portraits of system (2.3) for $n = 3$ on the phase cylinder $\{v(\text{mod}(2\pi)), u\}$ in the case of a constant sign function $\sigma \neq 0$ are shown in Fig. 3. The focus will be on the phase cylinder of system (2.3) instead of the center, and there will also be an asymmetry in the formation of limit cycles on the upper and lower half-cylinders of different stability (shown in different colors). Therefore, a bifurcation case is possible when one of the limit cycles (on the upper or lower phase half cylinder) merges into the corresponding separatrix loop; as the parameter changes further, it disappears.

Fig. 3. Possible phase portraits of system (2.3) for $n = 3$ in the case of a constant sign function $\sigma \neq 0$

In the case of an alternating function σ , the existence of a limit cycle in the region of oscillatory motions is possible. Possible phase portraits of system (2.3) for $n = 3$ on the phase cylinder $\{v(\text{mod}(2\pi)), u\}$ in the case of an alternating function σ are shown in Fig. 4.

Cases of synchronization of oscillations are shown in Fig. 3, c, 4, c. Figures 3, a, 3, e, 4, a, 4, e show the structures of partly passable resonance zones. Bifurcation cases are shown in Fig. 3, b, 3, d, 4, b, 4, d.

In [30] the phenomenon of synchronization of oscillations was called restricted. This is due to the possibility of the existence of a limit cycle in the region of rotational motion of the pendulum;

Fig. 4. Possible phase portraits of system (2.3) for $n = 3$ in the case of an alternating function σ

therefore, not all phase points in the resonance zone tend to a stable periodic regime, which corresponds to a stable equilibrium state of the averaged system.

§ 4. Example for illustration

Let us consider the equation (0.6). The autonomous equation ($p_2 = 0$ and $p_3 = 0$) was studied in the paper [31]. It was found that the Poincaré–Pontryagin generating function for region of oscillatory motion (filled with closed phase curves $\dot{x}^2/2 - \cos x = h$, $h \in (-1, 1)$) of the unperturbed pendulum has the form

$$B_0 = B_0(k(h)) = \frac{8}{105\pi} \{ (1 - k^2)(105 + (128k^4 - 80k^2 + 3)p_1)\mathbf{K}(k) + (-105 + (2k^2 - 1)(128k^4 - 128k^2 + 3)p_1)\mathbf{E}(k) \}, \quad (4.1)$$

where $\mathbf{K}(k)$ and $\mathbf{E}(k)$ are complete elliptic integrals of the first and second kind respectively; $k = \sqrt{(1+h)/2} \in (0, 1)$ is their module. In [31] we found the bifurcation value $p_1 \approx -8.481$, for which there is a double limit cycle. In this case the Poincaré–Pontryagin generating equation $B_0(k) = 0$ has a double root $k = k_* \approx 0.759$.

Now we choose such ω_1 and ω_2 so that the level $k = k_*$ coincides with the resonance level $k = k_{311}$. From the resonance condition $3\omega(k_*) = \omega_1 + \omega_2$, where $\omega(k_*) = \frac{\pi}{2\mathbf{K}(k_*)}$, we find, for example, $\omega_1 = 1$ and $\omega_2 \approx 1.448$.

Let us calculate system (2.3) for our case. This needs to be calculated:

$$\tilde{A}(v) = \frac{1}{36\pi^2} \int_0^{6\pi} \int_0^{6\pi} (p_2xy + p_3) \frac{\partial x}{\partial \theta} \cos \theta_1 \sin \theta_2 d\theta_1 d\theta_2, \quad (4.2)$$

$$\tilde{P}_0(v) = \frac{1}{36\pi^2} \int_0^{6\pi} \int_0^{6\pi} \frac{\partial}{\partial I} \left((p_2xy + p_3) \frac{\partial x}{\partial \theta} \right) \cos \theta_1 \sin \theta_2 d\theta_1 d\theta_2, \quad (4.3)$$

$$Q_0(v) = -\frac{1}{36\pi^2} \int_0^{6\pi} \int_0^{6\pi} [(-1 + p_1 \cos 3x + p_2x \cos \theta_1 \sin \theta_2)y + p_3 \cos \theta_1 \sin \theta_2] \frac{\partial x}{\partial I} d\theta_1 d\theta_2, \quad (4.4)$$

$$\tilde{P}_1(v) = \frac{1}{72\pi^2} \int_0^{6\pi} \int_0^{6\pi} \frac{\partial^2}{\partial I^2} \left((p_2xy + p_3) \frac{\partial x}{\partial \theta} \right) \cos \theta_1 \sin \theta_2 d\theta_1 d\theta_2, \quad (4.5)$$

$$Q_1(v) = -\frac{1}{36\pi^2} \int_0^{6\pi} \int_0^{6\pi} \frac{\partial}{\partial I} \left([(-1 + p_1 \cos 3x + p_2x \cos \theta_1 \sin \theta_2)y + p_3 \cos \theta_1 \sin \theta_2] \frac{\partial x}{\partial I} \right) d\theta_1 d\theta_2, \quad (4.6)$$

$$B_1(k) = \frac{dB_0(k)}{dI} = \frac{dB_0(k)}{dk} \frac{dk}{dh} \frac{dh}{dI} = \frac{\pi}{8k\mathbf{K}(k)} \frac{dB_0(k)}{dk}, \quad B_2(k) = \frac{d^2B_0(k)}{2dI^2} = \frac{\pi}{16k\mathbf{K}(k)} \frac{dB_1(k)}{dk}, \quad (4.7)$$

$$\tilde{\sigma}(v) = \frac{1}{36\pi^2} \int_0^{6\pi} \int_0^{6\pi} p_2x \cos \theta_1 \sin \theta_2 d\theta_1 d\theta_2, \quad (4.8)$$

$$b_1 = \frac{\pi}{8k\mathbf{K}(k)} \frac{d\omega(k)}{dk}, \quad b_2 = \frac{\pi}{16k\mathbf{K}(k)} \frac{db_1(k)}{dk}, \quad b_3 = \frac{\pi}{24k\mathbf{K}(k)} \frac{db_2(k)}{dk}, \quad (4.9)$$

where $x(k, \theta) = 2 \arcsin(ksn(2\mathbf{K}\theta/\pi, k))$, $y(k, \theta) = 2kcn(2\mathbf{K}\theta/\pi, k)$, $\theta = v + \frac{\theta_1 + \theta_2}{3}$ ($sn(u, k)$ is the Jacobi elliptic sine, $cn(u, k)$ is the Jacobi elliptic cosine). Here and below we put $k = k_* = k_{311}$.

In calculating integrals (4.2), (4.3), and (4.5), using the well-known Fourier series expansion for the Jacobi elliptic cosine [33], we obtain

$$\frac{\partial x}{\partial \theta} = \frac{y}{\omega} = \frac{4k\mathbf{K}}{\pi} cn(2\mathbf{K}\theta/\pi, k) = 8 \sum_{j=1}^{\infty} \frac{a^{j-1/2}}{1 + a^{2j-1}} \cos(2j-1)\theta,$$

$$\frac{\partial}{\partial I} \left(\frac{\partial x}{\partial \theta} \right) = \frac{\pi^3}{4k^2(1-k^2)\mathbf{K}^3} \sum_{j=1}^{\infty} \frac{(2j-1)a^{j-1/2}(1-a^{2j-1})}{(1+a^{2j-1})^2} \cos(2j-1)\theta,$$

$$\begin{aligned} \frac{\partial^2}{\partial I^2} \left(\frac{\partial x}{\partial \theta} \right) = & \\ = & \frac{\pi^4}{32k^4(1-k^2)^2\mathbf{K}^5} \left(((1+k^2)\mathbf{K} - 3\mathbf{E}) \sum_{j=1}^{\infty} \frac{(2j-1)a^{j-1/2}(1-a^{2j-1})}{(1+a^{2j-1})^2} \cos(2j-1)\theta + \right. \\ & \left. + \frac{\pi^2}{4\mathbf{K}} \sum_{j=1}^{\infty} \frac{(2j-1)^2 a^{j-1/2} (1-6a^{2j-1} + a^{2(2j-1)})}{(1+a^{2j-1})^3} \cos(2j-1)\theta \right), \end{aligned}$$

where $a = \exp \left(-\pi \frac{\mathbf{K}(\sqrt{1-k^2})}{\mathbf{K}(k)} \right)$. In calculating integrals (4.4) and (4.6), expansions for the functions $\frac{\partial x}{\partial I} = \frac{\pi (\operatorname{sn}(\cdot) \operatorname{dn}(\cdot) - \operatorname{cn}(\cdot) \operatorname{zn}(\cdot))}{4k(1-k^2)\mathbf{K}}$ (here the dot denotes the arguments $(2\mathbf{K}\theta/\pi, k)$; $\operatorname{dn}(u, k)$ is the delta amplitude, $\operatorname{zn}(u, k)$ is the Jacobi zeta function) and $\frac{\partial}{\partial I} \left(\frac{\partial x}{\partial I} \right)$ in a Fourier series

$$\frac{\partial x}{\partial I} = \frac{\pi^3}{4k^2(1-k^2)\mathbf{K}^3} \sum_{j=1}^{\infty} \frac{a^{j-1/2}(1-a^{2j-1})}{(1+a^{2j-1})^2} \sin(2j-1)\theta,$$

$$\begin{aligned} \frac{\partial}{\partial I} \left(\frac{\partial x}{\partial I} \right) = & \frac{\pi^4}{32k^4(1-k^2)^2\mathbf{K}^5} \left(((1+k^2)\mathbf{K} - 3\mathbf{E}) \sum_{j=1}^{\infty} \frac{a^{j-1/2}(1-a^{2j-1})}{(1+a^{2j-1})^2} \sin(2j-1)\theta + \right. \\ & \left. + \frac{\pi^2}{4\mathbf{K}} \sum_{j=1}^{\infty} \frac{(2j-1)a^{j-1/2}(1-6a^{2j-1} + a^{2(2j-1)})}{(1+a^{2j-1})^3} \sin(2j-1)\theta \right) \end{aligned}$$

were obtained. When calculating the integral (4.8), we have

$$x = 8 \sum_{j=1}^{\infty} \frac{a^{j-1/2}}{(2j-1)(1+a^{2j-1})} \sin(2j-1)\theta.$$

Thus, calculating (4.2)–(4.9), we obtain an averaged system of the following form

$$\begin{cases} \dot{u} = p_3 \widetilde{A}_1 \sin 3v + p_2 \widetilde{A}_2 \cos 3v + B_0 + \mu (p_2 \widetilde{\sigma}_1 \cos 3v + B_1) u + \\ \quad + \mu^2 \left[\left(p_3 \left(\widetilde{P}_{11} - \frac{3b_2}{b_1} Q_{01} \right) \sin 3v + p_2 \left(\widetilde{P}_{12} + \frac{3b_2}{b_1} Q_{02} \right) \cos 3v + B_2 \right) u^2 - \right. \\ \quad \left. - \frac{1}{b_1} \left(p_3 \widetilde{P}_{01} \sin 3v + p_2 \widetilde{P}_{02} \cos 3v \right) (p_3 Q_{01} \cos 3v + p_2 Q_{02} \sin 3v) \right], \\ \dot{v} = b_1 u + \mu b_2 u^2 + \\ \quad + \mu^2 \left[b_3 u^3 + \left(p_3 \left(Q_{11} - \frac{2b_2}{b_1} Q_{01} \right) \cos 3v + p_2 \left(Q_{12} - \frac{2b_2}{b_1} Q_{02} \right) \sin 3v \right) u \right], \end{cases} \quad (4.10)$$

where

$$\begin{aligned} \widetilde{A}_1 &= -2 \frac{a^{3/2}}{1+a^3}, \quad \widetilde{A}_2 = \frac{1}{4\pi\omega} \int_0^{2\pi} xy^2 \sin 3\theta d\theta, \\ \widetilde{P}_{01} &= \frac{\pi^3}{16k^2(1-k^2)\mathbf{K}^3} \frac{3a^{3/2}(a^3-1)}{(1+a^3)^2}, \quad \widetilde{P}_{02} = \frac{1}{4\pi\omega} \int_0^{2\pi} \frac{\partial}{\partial I} (xy^2) \sin 3\theta d\theta, \\ \widetilde{P}_{11} &= \frac{\pi^4}{1024k^4(1-k^2)^2\mathbf{K}^6} \frac{3a^{3/2}}{(1+a^3)^3} (4((1+k^2)\mathbf{K} - 3\mathbf{E})\mathbf{K}(a^6-1) - 3\pi^2(a^6-6a^3+1)), \end{aligned}$$

$$\begin{aligned}
\widetilde{P}_{12} &= \frac{1}{8\pi\omega} \int_0^{2\pi} \frac{\partial^2}{\partial I^2} (xy^2) \sin 3\theta d\theta, \quad Q_{01} = \frac{\widetilde{P}_{01}}{3}, \quad Q_{02} = \frac{1}{4\pi} \int_0^{2\pi} xy \frac{\partial x}{\partial I} \cos 3\theta d\theta, \\
Q_{11} &= \frac{2\widetilde{P}_{11}}{3}, \quad Q_{12} = \frac{1}{4\pi} \int_0^{2\pi} \frac{\partial}{\partial I} (xy) \frac{\partial x}{\partial I} \cos 3\theta d\theta, \quad \tilde{\sigma}_1 = \frac{2a^{3/2}}{3(1+a^3)}, \\
B_1 &= -\frac{1}{15} \left[15 + (128k^4 - 144k^2 + 31) p_1 - (256k^4 - 256k^2 + 46) p_1 \frac{\mathbf{E}}{\mathbf{K}} \right], \\
B_2 &= \frac{p_1 \pi}{120k^2(1-k^2)\mathbf{K}^3} \left[(384k^6 - 656k^4 + 295k^2 - 23) \mathbf{K}^2 - \right. \\
&\quad \left. - (768k^6 - 1280k^4 + 558k^2 - 46) \mathbf{K}\mathbf{E} - (128k^4 - 128k^2 + 23) \mathbf{E}^2 \right], \\
b_1 &= \frac{\pi^2 (k^2 - 1) \mathbf{K} + \mathbf{E}}{16 k^2 (k^2 - 1) \mathbf{K}^3}, \quad b_2 = -\frac{\pi^3 (k^2 - 1) \mathbf{K}^2 - 2(k^2 - 2) \mathbf{K}\mathbf{E} - 3\mathbf{E}^2}{256 k^4 (k^2 - 1)^2 \mathbf{K}^5}, \\
b_3 &= \frac{\pi^4 (k^4 + 2k^2 - 3) \mathbf{K}^3 - (2k^4 + 3k^2 - 13) \mathbf{K}^2 \mathbf{E} + 5(k^2 - 5) \mathbf{K}\mathbf{E}^2 + 15\mathbf{E}^3}{6144 k^6 (k^2 - 1)^3 \mathbf{K}^7},
\end{aligned}$$

B_0 is determined by formula (4.1).

Possible phase portraits of the averaged system (4.10) obtained using the WInSet software [34, 35] are shown in Figures 1–3.

Conclusion

The problem of the action of time-quasiperiodic perturbations on a self-oscillating system plays an important role in the theory of oscillations. As in the case of time-periodic perturbations, it leads to the phenomenon of synchronization of oscillations, first studied by Andronov and Witt [36] on the example of the Van der Pol equation in the quasi-linear case. Later, this phenomenon was studied by Morozov and Shilnikov [37] for systems close to arbitrary two-dimensional Hamiltonian ones (with time-periodic perturbation).

The existence of quasi-periodic solutions in systems close to Hamiltonian ones has been considered in many papers. However, studies of the effect of quasi-periodic perturbations on the self-oscillating system appeared for the first time, apparently, in the papers of Morozov and his students [1, 3, 7, 8, 11].

In this paper, the problem of the effect of quasi-periodic perturbations on systems close to arbitrary two-dimensional Hamiltonian in the case when perturbed autonomous systems have a double limit cycle was studied. Averaged systems are obtained for the case when the resonance level is close to the level from which a double limit cycle appears under the influence of an autonomous perturbation. These systems are not amenable to investigation in the general case. This means the need to consider new examples and study the behavior of solutions of these averaged systems in each case.

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О двухчастотных квазипериодических возмущениях систем, близких к двумерным гамильтоновым с двойным предельным циклом

Ключевые слова: квазипериодические возмущения, двойной предельный цикл, резонансы, усредненные системы.

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Изучается задача о воздействии двухчастотных квазипериодических возмущений на системы, близкие к произвольным нелинейным двумерным гамильтоновым в случае, когда соответствующие возмущенные автономные системы имеют двойной предельный цикл. Ее решение имеет важное значение как для теории синхронизации колебаний, так и для теории бифуркаций динамических систем. В случае соизмеримости собственной частоты невозмущенной системы с частотами квазипериодического возмущения имеет место резонанс. Выводятся усредненные системы, позволяющие установить структуру резонансной зоны, то есть описать поведение решений в окрестностях индивидуальных резонансных уровней. Исследование этих систем позволяет установить возможные бифуркации, возникающие при отклонении резонансного уровня от уровня невозмущенной системы, порождающего двойной предельный цикл в возмущенной автономной системе. Полученные теоретические результаты применяются при исследовании двухчастотного квазипериодически возмущенного уравнения маятникового типа и иллюстрируются при помощи численных вычислений.

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