Dedicated to the memory of S. G. Mikhlin

SURVEY ON GRADIENT ESTIMATES FOR NONLINEAR ELLIPTIC EQUATIONS IN VARIOUS FUNCTION SPACES

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Very general nonvariational elliptic equations of $p$-Laplacian type are treated. An optimal Calderón–Zygmund theory is developed for such a nonlinear elliptic equation in divergence form in the setting of various function spaces including Lebesgue spaces, Orlicz spaces, weighted Orlicz spaces, and variable exponent Lebesgue spaces. The addressed arguments also apply to Morrey spaces, Lorentz spaces and generalized Orlicz spaces.

§1. Introduction

Solutions to various real problems realize minimal energy of suitable nonlinear functionals, and the central problem of the calculus of variations is to get the existence of such solutions and to study their qualitative properties. It is the machinery of the nonlinear functional analysis that plays a crucial role in that issue. On the other hand, each minimizer of a variational functional solves weakly the corresponding Euler–Lagrange equation and this fact allows us to employ the powerful theory of PDEs as an additional tool. The Euler–Lagrange equations are divergence form PDEs, usually elliptic and nonlinear, and their weak solutions (that are minimizers of the corresponding functional) possess some basic minimal smoothness. The regularity theory of general (not necessary variational) divergence form elliptic PDEs establishes how the smoothness of the data reflects on the regularity of the solution, already obtained under very general circumstances. Once better smoothness is ensured, powerful tools of functional analysis apply to infer more precise properties of the solution.

Key words: gradient estimate, nonlinear elliptic equation, $L^p$ space, weighted Lebesgue space, Orlicz space, BMO, Muckenhoupt weight, Reifenberg flat domain.

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