Formal equivariant $\hat{A}$ class, splines and multiplicities of the index of transversally elliptic operators

Let $G$ be a connected compact Lie group acting on a manifold $M$ and let $D$ be a transversally elliptic operator on $M$. The multiplicity of the index of $D$ is a function on the set $\hat{G}$ of irreducible representations of $G$. Let $T$ be a maximal torus of $G$ with Lie algebra $t$. We construct a finite number of piecewise polynomial functions on $t^*$, and give a formula for the multiplicity in terms of these functions. The main new concept is the formal equivariant $\hat{A}$ class.

§ 1. Introduction

Let $G$ be a compact Lie group acting on a manifold $M$ of dimension $d$. As described in the monograph [1], Atiyah–Singer have associated to any $G$-transversally elliptic symbol $\sigma$ on $M$ a virtual trace class representation of $G$.

Let $\text{Index}_G(\sigma)(g)$ be its trace:

$$\text{Index}_G(\sigma)(g) = \sum_{\lambda \in \hat{G}} \text{mult}_G(\sigma)(\lambda)\chi_{\lambda}(g).$$

Thus $\text{Index}_G(\sigma)(g)$ is a $G$-invariant (generalized) function on $G$, and the right hand side of the above formula is its Fourier expansion in terms of the traces $\chi_{\lambda}$ of the unitary irreducible representations $V_{\lambda}$ of $G$. If $D$ is a transversally elliptic operator with principal symbol $\sigma$, we write indifferently $\text{Index}_G(D)$ or $\text{Index}_G(\sigma)$, $\text{mult}_G(D)$ or $\text{mult}_G(\sigma)$. The computation of $\text{mult}_G(\sigma)(\lambda)$ is important. For example, if $D$ is a transversally elliptic operator with principal symbol $\sigma$, the multiplicity of the trivial representation in $\text{Index}_G(\sigma)$ is the (virtual) dimension of the space of $G$-invariant (virtual) solutions of $D$.

In this article, we will restrict ourselves to the case when $G$ is connected.

Let $g$ be the Lie algebra of $G$, and $g^*$ its dual vector space. Our aim is to construct a canonical $G$-invariant function $m_G(\sigma)$ on $g^*$ which extends the multiplicity function $\text{mult}_G(\sigma)$ on $\hat{G} \subset g^*/G$.

The first instance of such a relation between the multiplicity function on $\hat{G}$ and functions on $g^*$ is the formula for the Kostant partition function in terms of derivatives of spline functions (that is, piecewise polynomial functions) [9], [8]. Similarly, Heckman’s result [17] on branching rules relates asymptotically multiplicities to spline functions. For example, if $T$ is the maximal torus of $G$, the asymptotic