Feynman amplitudes and limits of heights

We investigate from a mathematical perspective how Feynman amplitudes appear in the low-energy limit of string amplitudes. In this paper, we prove the convergence of the integrands. We derive this from results describing the asymptotic behaviour of the height pairing between degree-zero divisors, as a family of curves degenerates. These are obtained by means of the nilpotent orbit theorem in Hodge theory.

Bibliography: 35 titles.

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À Jean-Pierre Serre, en témoignage d’admiration

§ 1. Introduction

This paper grew out of an attempt to understand from a mathematical perspective the idea we learned from physicists that Feynman amplitudes should arise in the low-energy limit $\alpha' \to 0$ of string theory amplitudes, cf. [33] and the references therein. Throughout we work in space-time $\mathbb{R}^D$ with a given Minkowski bilinear form $\langle \cdot, \cdot \rangle$.

String amplitudes are integrals over the moduli space $\mathcal{M}_{g,n}$ of genus $g \geq 1$ curves with $n$ marked points. They are associated to a fixed collection of external momenta $\mathbf{p} = (p_1, \ldots, p_n)$, which are vectors in $\mathbb{R}^D$ satisfying the conservation law $\sum_{i=1}^n p_i = 0$. Up to some factors carrying information about the physical process being studied, the string amplitude can be written as (see e.g. [35, p. 182])

$$A_{\alpha'}(g, \mathbf{p}) = \int_{\mathcal{M}_{g,n}} \exp(-i \alpha' \mathcal{F}) d\nu_{g,n}. \quad (1.1)$$

In this expression, $d\nu_{g,n}$ is a volume form on $\mathcal{M}_{g,n}$, independent of the momenta, $\alpha'$ is a positive real number, which one thinks of as the square of the string length, and $\mathcal{F}: \mathcal{M}_{g,n} \to \mathbb{R}$ is the continuous function defined at the point $[C, \sigma_1, \ldots, \sigma_n]$ of $\mathcal{M}_{g,n}$ by

$$\mathcal{F}([C, \sigma_1, \ldots, \sigma_n]) = \sum_{1 \leq i, j \leq n} \langle p_i, p_j \rangle g'_{\mathfrak{A}, C}(\sigma_i, \sigma_j),$$

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