Introduction to Extremal Set Theory

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Let \([n] = \{1, 2, \ldots, n\}\) be our underlying set and let \(\binom{n}{k}\) denote the family of its \(k\)-element subsets. We will consider families \(\mathcal{F} \subset 2^n\) of subsets of \([n]\). If all of these subsets have sizes \(k\) then we call it a \(k\)-uniform family, otherwise it is non-uniform. The main goal of our investigations is to find the maximum of \(|\mathcal{F}|\) under some conditions supposed on \(\mathcal{F}\). The conditions are mostly given in the form that certain configurations of subsets are forbidden in \(\mathcal{F}\).

The first such theorem was found by Sperner in 1928. He proved that the maximum of \(|\mathcal{F}|\) is \(\binom{n}{\lfloor n/2 \rfloor}\) under the condition that \(\mathcal{F}\) contains no pair of members \(F, G\) such that \(F \subset G\). (Such families will be called Sperner families in what follows.) The next results were found on intersecting families, containing no pair of members \(F, G\) such that \(F \cap G = \emptyset\). Erdős, Ko and Rado (1961) made the observation that the largest size of an intersecting family is \(2^{n-1}\). A more difficult result in the same paper says that the maximum size of an intersecting \(k\)-uniform family is \(\binom{n-1}{k-1}\), supposing \(k \leq n/2\).

The family \(\mathcal{F}\) is called \(t\)-intersecting if \(|F \cap G| \geq t\) holds for any two members \(F, G \in \mathcal{F}\). The largest \(t\)-intersecting families were determined in 1964. But the determination of the largest \(t\)-intersecting \(k\)-uniform families proved to be more difficult. This was a result of Ahlswede and Khachatrian in 1997. In the present lectures we will show some results, problems and methods along these lines. A recently popular direction of this theory is when a small poset \(P\) is fixed and the maximally sized family is sought under the condition that the family contains no members forming \(P\) with respect to the relation \(\subset\). E.g. if \(P\) has only two comparable elements, this condition means that the family has no member containing another one as a subset: the condition of the Sperner theorem.