Monodromy Alexander Modules of Trigonal Curves

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For an algebraic curve $C$ on the complex projective plane $\mathbb{P}^2$, the fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$ is of interest. The structure of the group is not understood in general: there are a few general restrictions known and a few specific curves for which it is explicitly computed. Analogously, for a curve $C$ on a Hirzebruch surface $\Sigma$ with exceptional section $E \subset \Sigma$, one considers the fundamental group $\pi_1(\Sigma \setminus (C \cup E))$. The Zariski-van Kampen theorem expresses the latter group in terms of generators and braid monodromy relations. However, it is not in general easy to simplify this presentation to obtain an explicit description of the group. A relatively simpler invariant of a curve $C$ (for both $\mathbb{P}^2$ and $\Sigma$) is the (conventional) Alexander module $A_C$ defined in terms of the linking coefficient epimorphism $\text{lk}: \pi_1 \to \mathbb{Z}_d$ where $d$ is the degree:

$$A_C = K/K', \quad K = \text{Ker} \text{lk}.$$  

$A_C$ is a module over the ring $\Lambda := \mathbb{Z}[t, t^{-1}]$. For $n$-gonal $C \subset \Sigma$, we define the monodromy Alexander module as $A^m_C = A(H_C)$ in terms of the group $H_C \subset GL(n-1, \Lambda)$ of monodromy actions, where $A(H)$ is defined as:

$$A(H) = \Lambda^{n-1}/\langle (h-1) \cdot u \mid u \in \Lambda^{n-1}, \, h \in H \rangle, \quad H \subset GL(n-1, \Lambda).$$

Due to the Zariski-van Kampen theorem, there is a canonical epimorphism $A_C^m \to A_C$, which is often (but not always) an isomorphism.

We recently classified the monodromy Alexander modules of non-isotrivial trigonal curves (the case $n = 3$) based on Degtyarev’s characterization of $H_C$ for these curves: the monodromy group of $C$ valued in the modular group $\Gamma := PSL(2, \mathbb{Z})$ is genus-zero, which imposes that $H_C$ is a genus-zero subgroup of the Burau group $Bu_3 \subset GL(2, \Lambda)$, i.e. it is a finite-index subgroup mapped to a genus-zero subgroup of $\Gamma$ under the canonical epimorphism $c: Bu_3 \to \Gamma$ defined by evaluation at $t = -1$. Moreover, there is an almost full converse statement.

In view of this, we found an explicit list of genus-zero subgroups of $Bu_3$ such that for any genus-zero $H \subset Bu_3$, there is unique $H' \subset Bu_3$ in the list with $A(H) = A(H')$. The list consists of two infinite families and a finite number of sporadic subgroups (there are approximately 100 of them).