Monodromy Alexander Modules of Trigonal Curves

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For an algebraic curve C on the complex projective plane $\mathbb{P}^2$, the fundamental group $\pi_1(\mathbb{P}^2 \smallsetminus C)$ is of interest. The structure of the group is not understood in general: there are a few general restrictions known and a few specific curves for which it is explicitly computed. Analogously, for a curve C on a Hirzebruch surface Σ with exceptional section $E \subset \Sigma$, one considers the fundamental group $\pi_1(\Sigma \smallsetminus (C \cup E))$. The Zariski-van Kampen theorem expresses the latter group in terms of generators and *braid monodromy* relations. However, it is not in general easy to simplify this presentation to obtain an explicit description of the group. A relatively simpler invariant of a curve C (for both $\mathbb{P}^2$ and Σ) is the (conventional) *Alexander module* A_C^{c} defined in terms of the linking coefficient epimorphism $\mathrm{lk}\colon \pi_1 \twoheadrightarrow \mathbb{Z}_d$ where d is the degree:

$$A_C^{\mathrm{c}} = K/K', \qquad K = \mathrm{Ker}(\mathrm{lk}).$$

A_C is a module over the ring $\Lambda := \mathbb{Z}[t, t^{-1}]$. For n-gonal $C \subset \Sigma$, we define the *monodromy Alexander module* as $A_C^{\mathrm{m}} = A(H_C)$ in terms of the group $H_C \subset GL(n-1, \Lambda)$ of monodromy actions, where $A(H)$ is defined as:

$$A(H) = \Lambda^{n-1}/\langle (h-1) \cdot u \mid u \in \Lambda^{n-1},\ h \in H \rangle, \qquad H \subset GL(n-1, \Lambda).$$

Due to the Zariski-van Kampen theorem, there is a canonical epimorphism $A_C^{\mathrm{m}} \twoheadrightarrow A_C^{\mathrm{c}}$, which is often (but not always) an isomorphism.

We recently classified the monodromy Alexander modules of *non-isotrivial* trigonal curves (the case $n = 3$) based on Degtyarev's characterization of H_C for these curves: the monodromy group of C valued in the modular group $\Gamma := PSL(2, \mathbb{Z})$ is genus-zero, which imposes that H_C is a *genus-zero* subgroup of the Burau group $\mathrm{Bu}_3 \subset GL(2, \Lambda)$, i.e. it is a finite-index subgroup mapped to a genus-zero subgroup of Γ under the canonical epimorphism $\mathrm{c}\colon \mathrm{Bu}_3 \twoheadrightarrow \Gamma$ defined by evaluation at $t = -1$. Moreover, there is an almost full converse statement.

In view of this, we found an explicit list of genus-zero subgroups of Bu_3 such that for any genus-zero $H \subset \mathrm{Bu}_3$, there is unique $H' \subset \mathrm{Bu}_3$ in the list with $A(H) = A(H')$. The list consists of two infinite families and a finite number of sporadic subgroups (there are approximately 100 of them).