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Book of Abstracts

2025

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Program

27 October, Monday

$09^{30} - 09^{50}$	REGISTRATION OF PARTICIPANTS
$09^{50} - 10^{00}$	OPENING OF THE CONFERENCE
$10^{00} - 10^{35}$	Valery Kozlov (ONLINE) *The Lagrange identity and dynamics in a potential Jacobi field*
$10^{35} - 11^{10}$	Andrey Mironov, Siyao Yin *First integrals of geodesic flows on cones*
$11^{10} - 11^{45}$	Sergey Kuksin *Two approaches to average stochastic perturbations of integrable systems*
$11^{45} - 12^{10}$	COFFEE-BREAK
$12^{10} - 12^{45}$	Alexey Glutsyuk *On geometry and dynamics of exotic rationally integrable planar dual billiards*
$12^{45} - 13^{20}$	Vladislav Kibkalo, Andrey Konyaev *Billiards and families of quadrics associated with integrable geodesic flows for geodesically equivalent metrics*
$13^{20} - 13^{55}$	Andrey Dymov, Lev Lokutsievskiy, and Andrey Sarychev *Energy dissipation in weakly damped Hamiltonian chains*
$13^{55} - 15^{20}$	LUNCH
$15^{20} - 15^{55}$	Olga Pochinka, Vlad Galkin *Topology of 4-manifolds that admit non-singular flows with saddle orbits of the same index*
$15^{55} - 16^{30}$	Andrey Il'ichev *Solutions describing particle trajectories in the field of soliton-like wave structures in a fluid beneath an ice cover*
$16^{30} - 17^{05}$	Vladimir Dragović (ONLINE) *Isoperiodic deformations of meromorphic differentials on Riemann surfaces, soliton equations, and SU(N) Seiberg–Witten theory*

28 October, Tuesday

$10^{00} - 10^{35}$	Božidar Jovanović *Integrable cases of a heavy rigid body with a gyrostat and contact magnetic geodesic and sub-Riemannian flows on* $V_{n,2}$
$10^{35} - 11^{10}$	Sergei Tabachnikov (ONLINE) *Cusps of caustics by reflection, results and conjectures*
$11^{10} - 11^{45}$	Sergey Bolotin *Chaotic dynamics in the two body problem on a sphere*
$11^{45} - 12^{10}$	COFFEE-BREAK
$12^{10} - 12^{45}$	Vladislav Sidorenko *Secular evolution of motions in the planetary version of the non-restricted three-body problem*
$12^{45} - 13^{20}$	Alain Albouy (ONLINE) *Central configurations by computer algebra*
$13^{20} - 13^{55}$	Boris Bardin, Badma Maksimov *On orbital stability of periodic solutions of Hamiltonian system with two degrees of freedom in resonant cases of degeneration*
$13^{55} - 15^{20}$	LUNCH
$15^{20} - 15^{55}$	Alexey Kazakov *On robustly chaotic attractors in the generalized Kuramoto model*
$15^{55} - 16^{30}$	Jair Koiller (ONLINE) *Rubber rolling of solids of revolution over the plane: the surprising miracle of the "Nose" function*
$16^{30} - 17^{05}$	Alexander Kilin *On the isomorphism of the problems of the rubber rolling of bodies of revolution and on the dynamics of a rubber torus*
$17^{05} - 17^{25}$	Evgenia Mikishanina *Problems and prospects in research on the dynamics of mechanical systems with servo-constraints*
17^{30}	GROUP PHOTO SESSION
18^{00}	FELLOWSHIP BANQUET

29 October, Wednesday

EXCURSION TO KRASNAYA POLYANA

30 October, Thursday

$10^{00} - 10^{35}$	Vladimir Dragović, Borislav Gajić, and Božidar Jovanović *Integrable magnetic flows on n-dimensional spheres and nonholonomic mechanics*
$10^{35} - 11^{10}$	Sergei Agapov *On integrability of magnetic geodesic flows on 2-surfaces at different energy levels*
$11^{10} - 11^{45}$	Sergey Gonchenko *On global resonances in area-preserving maps leading to infinitely many elliptic periodic points and Poincaré problems in celestial mechanics*
$11^{45} - 12^{10}$	COFFEE-BREAK
$12^{10} - 12^{45}$	Alexander Bufetov *Convergence of Random Measures*
$12^{45} - 13^{20}$	Luis García-Naranjo (ONLINE) *Affine nonholonomic rolling on the plane*
$13^{20} - 13^{40}$	Ivan Bizyaev *Dynamics of a rotating test body in the Schwarzschild metric*
$13^{40} - 15^{00}$	LUNCH
$15^{00} - 15^{35}$	Anna Chugainova *Nonclassical discontinuities for nonstrictly hyperbolic conservation laws*
$15^{35} - 16^{10}$	Ivan Polekhin *Metric geometry and forced oscillations in mechanical systems*
$16^{10} - 16^{30}$	Mikhail Garbuz *The stability of tumbling modes of heavy plate in a resisting medium*
$16^{30} - 17^{30}$	POSTERS*

31 October, Friday

$10^{00} - 10^{20}$	Anna Tsvetkova *On the motion and deformation of localized wave beams generated by the Bessel functions*
$10^{20} - 10^{40}$	Alexey Elokhin, Andrey Dymov, and Alberto Maiocchi *Method of quasisolutions applied to R. Peierls's theory of thermal conductivity*
$10^{40} - 11^{00}$	Evgenii Borisenko *The dynamics of an elastic string under the action of dry friction*
$11^{00} - 11^{20}$	Artem Alexandrov *Phase-locking phenomenon in dynamical systems and quantum mechanics*
$11^{20} - 11^{45}$	COFFEE-BREAK
$11^{45} - 12^{05}$	Ivan Shilin *Sparkling saddle loops of vector fields on surfaces and related issues*
$12^{05} - 12^{25}$	Anna Chugainova, Ruzana Polekhina *Orbital stability of overcompressed discontinuities of a hyperbolic* 2×2 *system of conservation laws*
$12^{25} - 12^{45}$	Elena Pivovarova *Dynamics of a homogeneous ball on a rotating cylinder*
$12^{45} - 13^{05}$	Ivan Mamaev *Vortex dynamics on nonsimply connected surfaces*
$13^{05} - 13^{40}$	Dmitry Treschev *Integrable perturbations of polynomial Hamiltonians*
$13^{40} - 13^{45}$	CLOSING OF THE CONFERENCE

*POSTERS

1. Pavel Aleshin, Anton Shiryaev
   Development of a regulator for orbital stabilization of the orientational motion of a nanosatellite by means of magnetic moment
2. Nurradin Adigozalov
   A semi-analytical approach to secular effects in the motion of Earth's quasi satellites
3. Tatiana Bogatenko, Konstantin Sergeev, and Galina Strelkova
   Constant and periodic forces in two coupled Hodgkin – Huxley neurons
4. Alexander Gonchenko
   On discrete Lorenz-like attractors of three-dimensional maps
5. Tatyana Ivanova, Alexander Kilin
   Analysis of the dynamics of the controlled motion of a three-link wheeled mobile robot within the framework of different friction models
6. Efrosiniia Karatetskaia
   On bifurcations leading to the instant creation of hyperchaotic attractors with two and three positive Lyapunov exponents
7. Vladislav Koryakin
   Robustly chaotic dynamics in a 3-level laser model with opticl pumping
8. Andery Mironov, Siyao Yin
   Billiard trajectories inside cones
9. Nataliia Nikishina, Ivan Kolesnikov, and Andrei Bukh
   Statistical properties of extreme events in the ring of FitzHugh – Nagumo neurons
10. Ivan Proskurnin
    Morsification of semihomogeneous functions with any possible number of critical points
11. Constantin Ruchkin
    Application of Poisson learning for the classification of solutions of Hamiltonian systems
12. Klim Safonov
    New cases of heteroclinic bifurcations resulting in the emergence of Lorenz-like chaotic attractors

13. Oleg Shilov, Alexey Kazakov
On different types of hyperbolic chaotic sets appearing as a result of the perturbations of Anosov map on a 2D torus

14. Oleg Sumenkov, Sergei Gusev
Robust control design of underactuated systems via a family of time-periodic sliding surfaces

15. Ivan Tarabukin, Sergei Gusev
Stabilization of the equilibrium position of mechanical systems using a drive that monitors a reference speed

16. Kirill Zaichikov
Shilnikov criteria in the extended Shimizu – Morioka system

A Semi-Analytical Approach to Secular Effects in the Motion of Earth's Quasi-Satellites

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A quasi-satellite of a planet is a small body that stays in the planet's vicinity for long intervals at heliocentric distances much smaller than the planet–Sun distance, yet always far outside the planet's Hill sphere [1]. Quasi-satellite motion (later will be denoted as QS motion) corresponds to the 1:1 mean-motion resonance of the object and the planet (Fig. 1). For Earth, several such objects are currently cataloged (e. g., Cardea, 2006 FV35, 2013 LX28, 2014 OL339, Kamo'oalewa, 2020 PP1, 2022 YG, 2023 FW13).

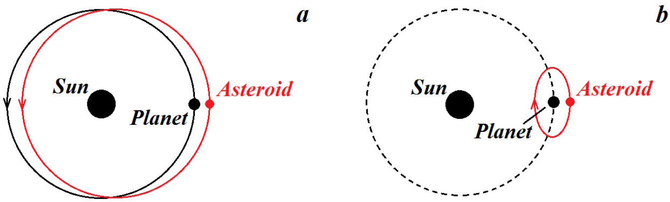

Figure 1. Orbital motion of a planet and its quasi-satellite: Panel **a** motion in a heliocentric reference frame whose axes keep a fixed orientation in inertial space; the planet moves around the Sun on a circular orbit, while the quasi-satellite follows an elliptical one; Panel **b** motion of the planet and the quasi-satellite in a rotating coordinate frame that keeps the Sun–planet direction unchanged [2].

The relative position of a quasi-satellite and Earth during their orbital motion is conveniently tracked by the difference of their mean longitudes $\varphi = \lambda - \lambda_3$, we will refer to it as the resonant phase considering that we are describing resonant motion. Due to the large distance from the planet, quasi-satellite's motion can be treated as a weakly perturbed heliocentric motion, so perturbation techniques are suitable for following how φ evolves over time (e. g., [2,3]). Averaging over the orbital motion of the quasi-satellite and those planets that exert the main influence leads to a compact evolution equation for φ:

$$3\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} + \mu\frac{\partial W}{\partial \varphi} = 0, \tag{1}$$

where μ is the Earth-to-Sun mass ratio, and W is the disturbing function that characterizes how the gravitational influences of Earth and the other planets to the quasi-satellite's motion. Fig. 2 shows examples of graphs illustrating the dependence of the value of the disturbing function W on the resonant phase φ for real quasi-satellites of the Earth.

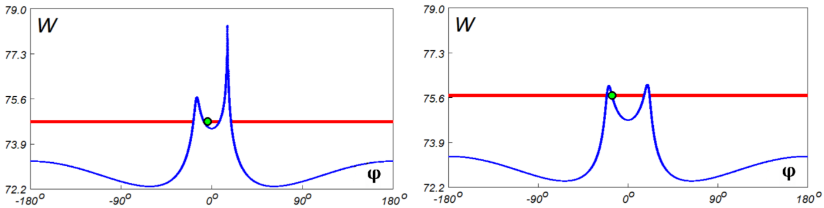

Figure 2. Averaged disturbing function for asteroids (164207) Cardea (left) and 2023 FW13 (right). The red lines show the value of energy integral of the evolution equation (1) that matches the observed motion of the asteroids. The dots mark the current values of the resonant phase.

Over time, an object's resonant regime can change, for instance, from QS regime to HS regime (corresponds to movement on horseshoe like orbit in rotating frame) or to combined QS+HS regime. Such changes may occur in a deterministic or in a probabilistic manner. Type of transition depends on the values of the problem's first integrals. Formulas for the probabilities of transitions between resonant states are given in [4], for the present case they can be written as follows:

$$P_{QS,HS} = \frac{\widehat{\Theta}_{QS,HS}}{\widehat{\Theta}_{QS} + \widehat{\Theta}_{HS} + \widehat{\Theta}}, \quad P_{QS+HS} = 1 - P_{QS} - P_{HS},$$

where $\widehat{\Theta}_{QS,HS} = \max\big(\Theta_{QS,HS},\, 0\big), \widehat{\Theta} = \max\big(-\Theta_{QS} - \Theta_{HS},\, 0\big)$. The quantities $\widehat{\Theta}_{QS,HS}$ represent the rates of change of the areas in the phase plane of equation (1) that are occupied by trajectories corresponding to the QS and HS regimes, respectively.

Described procedure was applied to Earth's quasi-satellites. Different scenarios of their evolution were established.

This work was carried out under the supervision of professor V. V. Sidorenko.

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On Integrability of Magnetic Geodesic Flows on 2-Surfaces at Different Energy Levels

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We consider geodesic flows in a magnetic field on 2-surfaces. In contrast with the standard geodesic flows (with zero magnetic field), dynamics of magnetic ones at different energy levels is different (e. g., see [1]). This is why an integrability of such flows at all energy levels simultaneously is a rather rare phenomena (e. g., see [2–4] where this problem has been studied on the 2-torus).

Due to this, it is very natural to study the question about integrability of such flows at a fixed energy level only. The case of an additional quadratic in momenta first integral was investigated in [5]. As shown in [5], this problem can be reduced to a certain semi-Hamiltonian system of PDEs [6]. In the talk we will discuss various methods of constructing smooth and analytical solutions to this system on the 2-torus [7, 8]. We also consider the case of an additional rational in momenta first integral and construct a family of local explicit integrable examples [9].

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Central Configurations by Computer Algebra

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The n-body problem of celestial mechanics has simple motions, the relative equilibria. The configuration of the bodies in such a motion is called central.

In 1996 I proved by using polynomial elimination on computer that there are only 3 types of symmetric noncollinear central configurations of 4 equal masses. My preceding paper had proved the symmetry of the central configurations with 4 equal masses.

In 2019 another computer assisted method, based on interval arithmetic, reproved these two results and the corresponding ones for 5, 6 and 7 bodies with equal masses [1].

In a central configuration with rational masses the ratio of two mutual distances is an algebraic number. In my problem one of these ratios is a root of an irreducible polynomial of degree 37 with integer coefficents. The first coefficent is 39858075. Only one root is relevant.

A main inconvenience with this method is the following. The polynomial cannot be further simplified. If the power law of force is changed, the set of solutions remains qualitatively unchanged, while the complexity of the polynomial drastically increases with the complexity of the power (which should be a rational number in this method). If the law of attraction is in fourth power of the inverse distance instead of second power, the degree is 109, the first coefficient is -51414728063385600, but there is again only one relevant root. In contrast, the first power of the inverse distance gives very simple polynomials. We do not know how to reduce the study to the case of this simpler power law. This method produces too complicated polynomials with many useless roots.

The above two computer assisted methods are less efficient when dealing with equations with parameters. In our question the important parameters are the masses of the 4 bodies. The thesis of Leandro (2003) under the direction of Moeckel gave a simple picture. A recent publication [2] tries another way to get the solution. But still the paper is 33 pages long, only for the simplest case.

I will show small ideas to improve the method, big polynomials characterizing some bifurcation sets, and small results from this material.

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On Orbital Stabilization of an Orientation of a Nano-satellite by a Magnetic Field of Earth

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In this note, a method for designing a robust feedback controller for orbital stabilization of a periodic trajectory (precisely, its orientation) of a nano-satellite using solely the Earth's magnetic field is presented and discussed. Such scenario assumes that

- the nano-satellite is equipped with magnetic coils and
- the re-orientation of a system is induced by an external torque that appears due to interaction of the coils with Earth's magnetic field.

Since for any position of a satellite the mentioned interaction torque (control variable) is at most two-dimensional while its orientation is defined by three quantities (e.g. three Euler angles), such control system is underactuated and its degree of under-actuation is equal to one. Such a structural feature imposes severe (continuum point-wise) constraints on system's dynamics, which, in turn, limits an immediate applicability of many classical motion planning and motion control methods, see e. g. [1], if accurate and robust reproduction of a targeted (nominal) orientation of the satellite along the orbit is required.

To suggest a constructive alternative for classical approaches, we consider only those nominal forced behaviors of the satellite whose partial stabilization can be obtained by stabilizing some of the transverse coordinates computed for the given nominal behavior [2]. In the note, we have explored the methodology when it is applied for orbital stabilization of the simplest nominal maneuver of the satellite on a circular orbit when it is oriented at each point of the orbit towards the center of the Earth. Following the procedure, we have introduced a set of transverse coordinates for the mentioned maneuver and derived the dynamics of variations of these quantities in a vicinity of the orbit. It resulted in an auxiliary linear time-varying control system with periodic coefficients, stabilization of which is achieved by solving the associated linear quadratic regulator problem. The linear feedback controller is converted into non-linear feedback that can be applied for the satellite accordingly. The effectiveness of the proposed feedback controller design is demonstrated through numerical simulations.

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Phase-Locking Phenomenon in Dynamical Systems and Quantum Mechanics

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The classical phase-locking phenomenon is a well-known fundamental property of nonlinear dynamical systems. It is familiar for the overdamped Josephson junction (JJ) driven by periodic external current (which is the famous RSJ model), systems of microparticles, superconducting nanowires, charge density waves, skyrmions and other nonlinear systems. Domains in the parameter space of the system where phase-locking takes place are usually called Arnold tongues. For dynamical systems on 2D torus, phase-locking is typically associated with rational values of the Poincaré rotation number ρ,

$$\rho = \lim_{t\to\infty} \frac{\phi(t) - \phi(0)}{t}, \tag{1}$$

where $\phi(t)$ is the phase variable that obeys equation $\dot{\phi} = f(\phi, t)$ with $f(\phi, t)$ is 2π-periodic in each variable. However, there exists a specific class of dynamical systems on the 2D torus, where the phase-locking occurs only for integer values of ρ. It is the so-called *rotation number quantization effect*, described in [1] and stuidied in details in [2]. We call this class of systems as *Möbius systems*. As was shown in [3, 4], Möbius systems are tightly related to the Hill equation, $\ddot{u} + V(t)u = 0$, where $V(t)$ is the T-periodic function. Specifically, the rotation number is given by

$$\rho = 2\pi \lim_{t\to\infty} \frac{N(t)}{t}, \tag{2}$$

where $N(t)$ is the number of zeros of the Hill equation solution $u(t)$. Moreover, the phase variable $\phi(t)$ can be constructed from $u(t)$ as $\phi(t) = \arg(u(t) - i\dot{u}(t))$. This correspondence provides a new insight to the classical-quantum correspondence. Introducing the frequency $\omega = 2\pi/T$ and dimensionless time $\tau = \omega t$, we can represent the Hill equation

$$\omega^2 \frac{d^2u}{d\tau^2} + V(\tau)u(\tau) = Eu(\tau), \tag{3}$$

where E is the real number, as a slow-fast system,

$$\dot{\Phi} = -E + \Phi^2 + V(\tau), \quad \dot{\tau} = \omega. \tag{4}$$

In this treatment, the slow manifold of the classical dynamical system corresponds to the energy surface of a certain quantum mechanical system. This fact allows us to apply the exact WKB approach and demonstrate that rotation number quantization is indeed quantum-mechanical quantization (known as Milne's quantization,. Moreover, for some cases this correspondence allows one to estimate width of Arnold tongues and gaps between them. In addition, some preliminary discussion of the relation between canard-type solutions and the WKB approach will be made. The talk is based on our joint work [5] with Alexey Glutsuyk and Alexander Gorsky.

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On Orbital Stability of Periodic Solutions of Hamiltonian System with Two Degrees of Freedom in Resonant Cases of Degeneration

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Periodic solutions of autonomous Hamiltonian systems usually form a family depending on one or more parameters. In the general case, the period of these solutions continuously depends on these parameters. The above circumstance leads to instability of periodic solutions in the sense of Lyapunov. On the other hand, the periodic solutions which are unstable in the sense of Lyapunov can be orbitally stable. The problem of orbital stability is of the great interest both from a general theoretical point of view and for applications in rigid body dynamics, celestial mechanics and satellite dynamics. Modern methods of the theory of dynamical systems, stability theory and the qualitative theory of differential equations allow to obtain a solution to this problem in a strictly nonlinear sense.

The most complete and rigorous conclusions on the orbital stability of periodic solutions have been obtained for autonomous Hamiltonian systems with two degrees of freedom. In particular, in cases, where the stability problem can be solved by terms of order no higher than fourth in the expansion of the Hamiltonian function in the neighborhood of a periodic solution, rigorous orbital stability criteria have been formulated and proven in [1–3] based on KAM theory methods. At present, in the problem of orbital stability of periodic solutions of an autonomous Hamiltonian system with two degrees of freedom, only special degenerate cases remain unexplored. In these cases, the analysis of fourth-order terms is insufficient to obtain rigorous conclusions about orbital stability and it is necessary to perform stability study by taking into account higher-order terms. In particular, such a degeneration arises in resonance cases.

In this work, we study the problem of orbital stability of periodic solutions of an autonomous Hamiltonian system with two degrees of freedom in previously unexplored cases of third, fourth and sixth order resonances, when, in order to obtain rigorous conclusions, it is necessary to take into account the terms of the sixth order in the Hamiltonian expansion in the neighborhood of the periodic solution. We obtain sufficient conditions of orbital stability and instability of the periodic solutions in a form of inequalities with respect to coefficients of Hamiltonian normal form calculated up to terms of sixth order. We apply the results of this study in the problem of orbital stability of periodic motions of heavy rigid body, whose principal moments of inertia satisfy the relation A=C=4B.

The work is supported by the grant of the Russian Science Foundation (project 24-11-00162, https://rscf.ru/en/project/24-11-00162/) and was carried out at the Moscow Aviation Institute (National Research University).

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Dynamics of a Rotating Test Body in the Schwarzschild Metric

Ivan Bizyaev

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Consider a test body that moves in space-time with a given metric $g_{\alpha\beta}$. Let p^α be the momentum of the test body and let $S^{\alpha\beta}$ be its tensor of angular momentum. The motion of this body in pole-dipole approximation is described by the Mathisson—Papapetrou equations [1,2]:

$$\frac{Dp^\alpha}{d\tau} = -\frac{1}{2} R^\alpha{}_{\lambda\mu\nu} u^\lambda S^{\mu\nu},$$
$$\frac{DS^{\alpha\beta}}{d\tau} = p^\alpha u^\beta - u^\alpha p^\beta,$$

where $R^\alpha{}_{\mu\nu\lambda}$ is the tensor of curvature and the covariant derivative is introduced:

$$\frac{Dp^\alpha}{d\tau} = \frac{dp^\alpha}{d\tau} + \Gamma^\alpha_{\mu\nu} p^\mu u^\nu, \quad \frac{DS^{\alpha\beta}}{d\tau} = \frac{dS^{\alpha\beta}}{d\tau} + \Gamma^\alpha_{\mu\nu} S^{\mu\beta} u^\nu + \Gamma^\beta_{\mu\nu} S^{\alpha\mu} u^\nu,$$

here $\Gamma^\alpha_{\mu\nu}$ is the Christoffel symbol and u^α is the velocity of the point in the body that can be defined from the Tulczyjew condition [3] $S^{\alpha\beta} p_\beta = 0$.

In the general case, this problem reduces to investigating a Hamiltonian system with six degrees of freedom. For the Schwarzschild metric, this system has additional integrals of motion and admits reduction by three or four degrees of freedom. The resulting reduced system possesses invariant manifolds on which it can be reduced to explicit quadratures. In this paper, a qualitative analysis of the resulting quadratures is carried out.

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Constant and Periodic Forces in Two Coupled Hodgkin – Huxley Neurons

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The study of neuronal behavior, both at the individual and collective levels, has become a topic of significant interest for researchers nowadays. Investigating the fundamental mechanisms governing interactions in single neurons and small ensembles can enhance our understanding of the structural and functional organisation of larger neuronal networks. Thus, the objective of this work is to fill in the gaps in our understanding of dynamics of small ensembles of Hodgkin – Huxley [1] neurons and establish the patterns of interaction between the two neurons under external forces [2].

While constant external current plays a significant role in the regime formation in the neurons, it has been found that it is possible to manage the regime by adjusting coupling strength value as well. It has also been shown that periodic external force is capable of inducing both regular and complex regimes in two synchronised Hodgkin – Huxley neurons (Fig. 1).

Numerical integration was conducted with the use of the 4th Runge – Kutta method, and the Pearson correlation coefficient was used to evaluate the collective behaviour of the neurons. A set of programs in the C language, Python3 and Gnuplot was used.

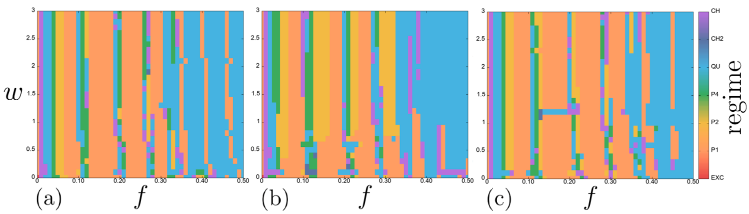

Figure 1. Regime maps for one of the Hodgkin – Huxley neurons for three values of its constant external current I_{ext}.

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Chaotic Dynamics in the Two Body Problem on a Sphere

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We prove the existence of chaotic trajectories for the two body problem on a sphere. The trajectories we construct encounter near collisions and are similar to the second species solutions of Poincaré of the classical 3 body problem. The construction uses a general result on Hamiltonian systems with Newtonian singularities.

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The Dynamics of an Elastic String under the Action of Dry Friction

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The presence of dry friction fundamentally complicates the analysis of mechanical systems. This complication is a consequence of the non-smooth nature of the forces involved: dry friction is not a continuous function of the velocity, it has a gap at points where relative motion ceases, i.e. velocity is zero. Unlike viscosity, Coulomb friction introduces a force that changes direction instantaneously upon reversal of motion but has magnitude determined by normal forces and dry friction coefficient only when sliding occurs. This mathematical ambiguity – "dry friction to be undetermined" – represents a critical point where standard assumptions about the well-posedness and smoothness of the equations governing mechanical systems break down.

This inherent discontinuity requires special analytical tools beyond classical theory. One significant approach is Filippov's theory of differential inclusions [5, 6]. This theory replaces the Newtonian equation $\dot{x}(t) = f(t, x(t))$ with a condition where the right-hand side is a measurable function. Then the derivative $\dot{x}(t)$ becomes an element of a set defined by f such that values of this function f on null sets, particularly at points of zero velocity, are insignificant. So it effectively capture the non-uniqueness or ambiguity in the friction force direction.

Alternatively, subdifferential operators offer another powerful mathematical concept for modeling dry friction [2, 4]. This tool allows describing dissipative effects without explicitly invoking Coulomb's law with its directional ambiguity. Systems driven by such subdifferential terms have been successfully analyzed to determine qualitative features of their long-term evolution.

While these generalized tools are analytically successful, much applied work relies on approximating the discontinuous friction force by a continuous function [1, 7]. While different approximations may introduce their own specific modeling errors [9], this approach facilitates numerical simulation as it inherently smooths out the sharp transitions present in dry friction.

This work addresses the core mathematical problem presented by dry friction, specifically targeting its impact on infinite-dimensional systems. We consider an oscillating string fixed on the rough table with a general case of dry friction coefficient. This mechanical problem can be formally expressed as a wave equation with a discontinuous term:

$$u_{tt}(t,x) = u_{xx}(t,x) + f(t,x) + \\ + \sigma(t,x,u,u_x,u_t)\left(-\chi_{\{u_t\neq 0\}}(t,x)\frac{u_t(t,x)}{|u_t(t,x)|} + \chi_{\{u_t=0\}}(t,x)\lambda(t,x)\right),$$

where u represents a displacement of string points, f is an external force acting on the string, σ is a nonnegative function that describes dry friction coefficient, χ is a characteristic function of the set ($\chi_B(t,x) = 1$ if $(t,x) \in B$ and 0 otherwise), and λ, $|\lambda| \leq 1$, is a static dry friction force. To solve it we suggest a functional differential interpretation of this problem to extend powerful inclusion theory mentioned above.

The primary contribution is a solution definition and a comprehensive existence proof for solutions governed by the suggested model. The solution in the sense of the suggested definition can be obtained as a passage to the limit in some preliminary approximations that replace the original problem with a sequence of smooth PDE. These smooth non-linear hyperbolic problems can be studied using methods discussed in [8]. Building upon these results important properties such as the uniqueness of the solution and Lipschitz continuity of the corresponding semiflow can be proved. This all establishes that the considered dry friction problem is well-posed.

This work was supported by the Russian Science Foundation under grant no.25-11-00114.

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Convergence of Random Measures

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The conference paper will present a simple general formalism which allows proving the convergence of random measures, with emphasis on application to the convergence of absolute values of random holomorphic functions to Gaussian multiplicative chaos.

Nonclassical Discontinuities for Nonstrictly Hyperbolic Conservation Laws

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Nonclassical discontinuities and their role in solving Riemann problem are studied. Solutions to a system of two hyperbolic equations representing conservation laws are investigated. On the one hand, this system of equations makes it possible to demonstrate the non-standard solutions to the Riemann problem, on the other hand, this system of equations describes longitudinal-torsional waves in elastic rods. We use the traveling wave criterion for admissibility of shocks as the additional jump condition. If the dissipation parameters included in each of the equations of the system are different, then there are undercompressive and overcompressive waves.

We have numerically studied the asymptotics of the main types of solutions to the Riemann problem for the system of equations describing nonlinear longitudinal–torsional waves in viscoelastic media [1]. The study has shown that the asymptotics of nonstationary solutions of the Riemann problem may contain undercompressive shock (nonclassical discontinuities). The solutions with a undercompressive shock are formed for a certain relation between the dissipation parameters appearing in the equations. If two dissipation parameters are identical (or their ratio is close to unity), then the asymptotics of the solution corresponds to the self-similar solution and does not contain undercompressive shocks. We have shown that, for different ratios of dissipation parameters, one can obtain different solutions to the Riemann problem for the same initial data.

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Orbital Stability of Overcompressed Discontinuities of a Hyperbolic 2×2 System of Conservation Laws

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This paper is devoted to the study of the orbital stability of overcompressed discontinuities (shocks) in a hyperbolic system of two conservation laws. The system describes longitudinal-torsional waves of small amplitude propagating in one direction in a nonlinear elastic rod and was originally derived in the works of [1, 2].

The standard method of viscous regularization is applied to the system. This regularization leads to the appearance of non-classical discontinuities — specifically, overcompressed shocks.

In the phase plane, an overcompressed shock is represented by a one-parameter family of integral curves (viscous profiles) connecting two nodal points. Any of these viscous profiles can represent the structure of the discontinuity. In this work the study of the stability of these viscous profiles is studied. The stability analysis is performed using the Evans function technique, which demonstrates that these viscous profiles are orbitally stable.

Furthermore, the Riemann problem is numerically solved for initial data corresponding to the overcompressed discontinuity. The computational results show that only one specific viscous profile from the continuous family emerges as the long-time asymptotics of the solution. The other profiles may also arise in non-stationary calculations, but only for specific initial conditions.

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Isoperiodic Deformations of Meromorphic Differentials on Riemann Surfaces, Soliton Equations, and SU(N) Seiberg – Witten Theory

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We study deformations of elliptic and hyperelliptic Riemann surfaces with an Abelian differential of the second or third kind, which preserve the periods of the differential with respect to a chosen canonical homology basis of the surface. We derive differential equations with rational coefficients governing the deformations. We apply these results to soliton theory, e. g. to the Boussinesq, KdV, and sine-Gordon equations, and to the Toda lattice, as well as to SU(N) Seiberg – Witten theories. The talk is based on new joint results with Vasilisa Shramchenko and the listed references.

The work is supported by the Simons Foundation grant no. 854861.

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Integrable Magnetic Flows on n-dimensional Spheres and Nonholonomic Mechanics

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We introduce and study the Chaplygin systems with gyroscopic forces. We put a special emphasis on the important subclass of such systems with magnetic forces. In a reduction, we construct Hamiltonian magnetic systems on spheres S^n. Recently, we provide a Lax representation of the equations of motion and prove complete integrability of those systems for any n. The integrability is provided via first integrals of degree one and two.

The research was supported by the Serbian Ministry of Science and the Simons Foundation grant no. 854861.

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Energy Dissipation in Weakly Damped Hamiltonian Chains

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We consider a Hamiltonian chain of $N \geq 2$ rotators (in general nonlinear) in which the first rotator is damped. We are interested in the dissipation rate of total energy of the chain when the energy is large.

This problem is motivated by nonequilibrium statistical mechanics of crystals where Hamiltonian chains of interacting particles, in which the first and last particles are driven by a stochastic perturbation and damped, are classical models. A fundamental question is whether such a chain is mixing, i.e. is it true that distributions of all solutions converge to a unique stationary measure when time goes to infinity? The chain of rotators gives an example when the mixing property is expected but is very hard to prove: despite many attempts undertaken in the last 20 years, it is established only when $N \leq 4$ [1–3]. The main difficulty is to get good control of the energy decay rate provided by the damping when the energy is large.

We show that time derivative of the total energy H is bounded by $-H^{-2N-3}$ once H is sufficiently large. This upper bound coincides with that obtained earlier in paper [4] on a different time scale, in which the authors also give numerical evidence that this estimate is optimal (so the energy indeed decays very slowly when $H \gg 1$, which explains the difficulty in proving the mixing property). The method employed in [4] is based on a KAM-like procedure, is technically complicated, and works only for very special initial conditions. This does not allow to apply it for proving the mixing property for the (stochastically driven) chains of significant length.

On the contrary, our proof is simple, short and holds for arbitrary initial conditions. We adopt completely different approach going back to Malisoff and Mazenec [5], relying on a method that allows under mild assumptions to explicitly construct a strict Lyapunov function once a non-strict one is given (in our case the latter is the Hamiltonian).

Unfortunately, the obtained Lyapunov function provides sufficiently good control only in absence of the stochastic perturbation. Constructing a Lyapunov function that controls the energy decay in the stochastically driven chain (and hence allows to prove the mixing property) is still an open problem.

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Method of Quasisolutions Applied to R. Peierls' Theory of Thermal Conductivity

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In his seminal 1929 paper [1], R. Peierls' described thermal conductivity in solids and provided a heuristic derivation of Fourier's law from microscopic dynamics of particles. By analyzing a lattice of nonlinearly coupled oscillators, he demonstrated that, in the limit of an infinite number of oscillators and vanishing nonlinearity, the energy distribution across Fourier modes satisfies a specific nonlinear kinetic equation. Although this result had motivated considerable efforts by mathematicians and mathematical physicists, a fully rigorous derivation remains an open problem.

Inspired by Peierls' kinetic theory, the field of wave turbulence theory emerged in the second half of the 20th century, focusing on the statistical properties of weakly nonlinear wave systems. Recent years have seen substantial progress in its rigorous justification.

In this talk, I will discuss ongoing work that revisits a setting similar to Peierls' original model, employing techniques from recent advances in wave turbulence. Starting from a d-dimensional lattice of oscillators with weak nonlinear coupling and stochastic perturbation, I will present the derivation of a kinetic equation for the energy spectrum of a "quasisolution" — an approximation that accurately captures the expected behavior of the true solution. I will provide an overview of the key steps in this derivation and discuss the technical challenges we have encountered in the process.

This work was supported by the Russian Science Foundation under grant no. 25-11-00114.

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The Stability of Tumbling Modes of Heavy Plate in a Resisting Medium

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We consider the free fall of a thin heavy plate in a fluid. The motion is described by Kirchhoff equations [1] with additional terms that correspond to viscous friction. In [2], among the possible steady-state movements, the so-called tumbling regimes were described. On these regimes, the center of mass descends on average in a straight line at a certain angle to the horizon, and the plate rotates around the lateral axis with a constant average angular velocity. The objective of this work is to study the stability of such modes with respect to possible displacements along the axis of rotation.

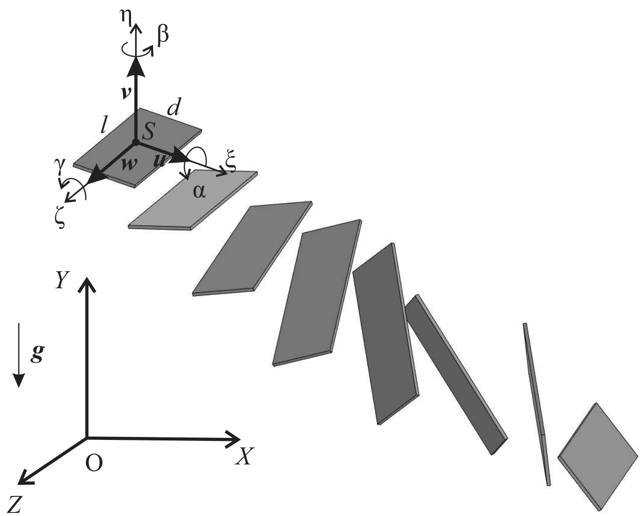

Figure 1. Position of the plate.

The problem will be reduced to studying the stability of the zero solution of the Hill equation: $\theta'' + p(t)\theta = 0$, where $p(t)$ is a π-periodic function. The stability condition will depend on the geometric and inertial characteristics of the plate.

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Affine Nonholonomic Rolling on the Plane

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We introduce a class of examples which provide an affine generalization of the nonholonomic problem of a convex body that rolls without slipping on the plane. These examples are constructed by taking as given two vector fields, one on the surface of the body and another on the plane, which specify the velocity of the contact point. We investigate dynamical aspects of the system such as existence of first integrals, smooth invariant measure, integrability and chaotic behavior, giving special attention to special shapes of the convex body and specific choices of the vector fields for which the affine nonholonomic constraints may be physically realized. We also discuss some remarkable behavior occurring when the body is a homogeneous sphere and the vector fields have discontinuities with specific symmetries.

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On Geometry and Dynamics of Exotic Rationally Integrable Planar Dual Billiards

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Planar dual billiards generalize usual billiards on the plane and on surfaces of constant curvature and outer planar billiards. They were introduced by S.Tabachnikov [3]. A planar *dual billiard* is a planar curve γ equipped with a family $(\sigma_P)|_{P\in\gamma}$ of projective involutions of the projective lines L_P tangent to γ at P that fix P. A dual billiard is called *rationally integrable,* if there exists a non-constant rational function $R(x, y)$ of two variables, called *first integral,* whose restriction to each tangent line L_P is σ_P-invariant. In the previous author's paper [1] it was shown that rationally integrable dual billiards exist only on conics punctured at k points, $0 \leq k \leq 4$. Their classification given there includes standard examples with quadratic integrals, defined by conical pencils, and an infinite family of exotic examples with minimal degree of integrals being any even number greater than two. The complexified involution family there is rational with at most four simple poles. For each integrable example on a conic γ, the *dual billiard map* is a birational map acting on the two-dimensional phase space: the complex algebraic surface consisting of pairs (Q, P), where $Q \in \mathrm{CP}^2$, $P \in \gamma$ and the line QP is tangent to γ at P. Namely, for every $Q \notin \gamma$ and $P \in \gamma$ at which the involution σ_P is well-defined, the pair (Q, P) is sent to the pair (Q^*, P^*), where $Q^* = \sigma_P(Q)$, Q^*P^* is the tangent line to γ different from Q^*P, and P^* is their tangency point. The phase space is fibered by invariant fibers, each being a ramified double cover over a level curve of the integral.

It is well-known that for a standard integrable dual billiard defined by a generic conical pencil, a generic invariant fiber is a complex torus and the dual billiard map acts there as a translation [2].

In the talk we present the recent author's results on the structure of the exotic rationally integrable dual billiards. For each example we find the type of a generic level curve of the integral and of the corresponding fiber. We show that a generic level curve is rational in most of examples and elliptic in two examples. In the rational case we find explicit rational parametrizations of generic level curves. We present formulas for holomorphic differentials on the elliptic level curves. We describe the dynamics of the dual billiard map along invariant fibers in the phase space.

The research is supported by the MSHE "Priority 2030" strategic academic leadership program.

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On Discrete Lorenz-like Attractors of Three-dimensional Maps

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We observe main elements of the theory of discrete attractors of Lorenz type in the case of three-dimensional maps. We select, first of all, two different types of such attractors: the so-called standard discrete Lorenz attractors, which, as shown in [1–4], appears in nonlinear maps of general type, as well as symmetric discrete Lorenz attractors that are characteristic for maps with the axial symmetry [5], (i. e. the symmetry $x \to -x, y \to -y, z \to z$ which is the same as the symmetry in the flow Lorenz model). For various types of discrete Lorenz attractors, we describe their basic geometric and dynamical properties, and also present main phenomenological bifurcation scenarios in which they arise. We also consider specific examples of discrete Lorenz attractors of various types in three-dimensional quadratic maps, such as three-dimensional Hénon maps and quadratic maps with axial symmetry and constant Jacobian. For the latter, their normal forms are presented, i. e. universal maps, to which any map from a given class can be reduced by means of linear coordinate transformations. For these normal forms we study bifurcations leading to symmetric discrete Lorenz-like attractors of different types, including the so-called twisted Lorenz attractors [5].

The work was prepared within the framework of the project "International academic cooperation" HSE University.

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On Global Resonances in Area-Preserving Maps Leading to Infinitely Many Elliptic Periodic Points and Poincaré Problems in Celestial Mechanics

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We consider analytic (or C^r, $r \geq 3$) two-dimensional area-preserving maps that have a quadratic homoclinic tangency of invariant manifolds of a saddle fixed point. For such maps, we specify *homoclinic resonance conditions* under which the map has, in a small neighborhood of the orbit of homoclinic tangency, *a countable set of generic (KAM-stable) elliptic points of all successive periods* $k_0, k_0 + 1, \ldots,$ *starting from some* k_0.

The set of such trajectories forms a specific cluster of an (infinitely) large number of generic elliptic points. This cluster is not structurally stable as a whole even with respect to conservative perturbations. However, if we mean "any given finite number of points", then it becomes stable with respect to this property. Therefore, we can ask about a possibility of observing this phenomenon. The question of where this can be observed, as seems to us, can receive an extremely unexpected answer.

The fact is that the problem under consideration, if we consider it in retrospect, is directly related to those problems of celestial mechanics that H. Poincaré considered in his famous "New Methods of Celestial Mechanics" [1], and in particular, to his planar circular restricted problem of three bodies. (Note that this problem is cardinally simpler than the general problem of three bodies: instead of nine degrees of freedom, it has only two, and moreover can be reduced to the consideration of a two-dimensional area-preserving map). In this problem, each elliptic periodic orbit can be compared with some cosmic body, not necessarily large, it can be a stone, a piece of ice, etc. Individually, such a body can be "invisible", but when many of them gather in one place, then this can already be an observable cosmic object. Objects of this type are well known, the most famous being the ring of Saturn.

It seems clear that some global resonance must be the cause of such a phenomenon. What its nature is, if so many small bodies are involved in this resonance, is not very clear. But in light of this work, we can assume that this is something like the homoclinic resonance we have considered, which can arise in the problem of celestial mechanics that Poincaré once considered.

Note that the considered resonance condition has a *bifurcation codimension equal 2*, since it includes only two “equality-type” assumptions: that the homoclinic tangency is quadratic and that some invariant of the homoclinic structure is zero. Both of the latter conditions are effectively verifiable, since they involve calculating only certain coefficients at constant, linear, and quadratic terms in the Taylor series of the map at the homoclinic point.

General remarks.

1) *Mathematical.*

(A) We have found effectively verifiable conditions under which two-dimensional area-preserving maps have infinitely many generic elliptic periodic points.

(B) We have shown that such points form a certain cluster in which all its points are ordered by their orbits with periods $k_0, k_0+1, \ldots$

(C) All points are concentrated in one region: they run for a period along a neighborhood of the homoclinic orbit.

(D) *Proof outline.* The global resonance is manifested in the fact that all first-return maps T_k near a homoclinic point, where $k = k_0, k_0+1, \ldots$, are the return times to its neighborhood, in some (rescaled) coordinates have the same form of the quadratic Hénon map

$$\bar{x} = y,\ \bar{y} = M - x - y^2\ ,$$

up to asymptotically small as $k \to \infty$ terms. It is well known that this conservative Hénon map at $-1 < M < 3$ has an elliptic fixed point, which is KAM-stable at $M \neq 0; 3/5$ (i.e., except for 1:4 and 1:3 resonances). Automatically, the initial map under consideration will have infinitely many elliptic periodic orbits of all consecutive periods, starting with k_0.

(E) *Other examples*. It is interesting that such type global resonance, only of *bifurcation codimension 1*, takes place in the case of symmetric cubic homoclinic tangencies in the case of two-dimensional reversible maps. Here, the condition of global resonance holds automatically at the moment of tangency and all the first-return maps Tk in some (rescaled) coordinates will be asymptotically close to the cubic Hénon map

$$\bar{x} = y,\ \bar{y} = My - x + \alpha y^3\ ,$$

where $\alpha = \pm 1$ depending on the type of tangency. This map has always the fixed point $O(x = 0, y = 0)$ that is generic elliptic for $-2 < M < 2$ and $M \neq \{-1; 0\}$.

2) *Physical (celestial mechanics).* It seems that, following the approach of Poincaré, who combined Hamiltonian dynamics and Celestial mechanics,

we have proposed some very simple mechanism for the emergence of global resonances involving a large number of small cosmic bodies in one big cluster.

This is joint work with M. Gonchenko. Some results on this topic was published in [2, 3].

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Solutions Describing Particle Trajectories in the Field of Soliton-Like Wave Structures in a Fluid Beneath an Ice Cover: Approximation of Flows on a Central Manifold by Integrable Normal Forms

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A fluid layer of finite depth is described by Euler's equations governing the motions of the ideal fluid (water). The ice is assumed to be solid and it freely floats on the water surface. It is modeled by a geometrically nonlinear elastic Kirchhoff-Love plate. The trajectories of liquid particles under the ice cover are found in the field of different nonlinear surface traveling waves of small, but finite amplitude. These waves are: the classical solitary wave of depression, existing on the water-ice interface when the initial stress in the ice cover is large enough, the generalized solitary wave tending to the periodic wave at infinity [1], the envelope solitary wave and the so-called dark soliton [2]. The last two waves indicate the focusing or defocusing of nonlinear carier surface wave, the generalized solitary wave consists of solitary wave core and periodic asymptotic wave at spacial infinity, moreover for the algebraically small amplitude of the wave core of the generalized solitary wave, the amplitude of the mentioned above periodic wave is exponentially small. The consideration is based on explicit asymptotic expressions for solutions describing the mentioned wave structures on the water-ice interface, as well as asymptotic solutions for the velocity field in the liquid column corresponding to these waves. These expressions are given by the solution of the finite dimensional reduced equations, which are obtained by projection of the infinite dimensional differential equations, modelling the system in question, to the central manifold. The reduced equations are approximated to any algebraic order in the wave amplitude by integrable normal form equations, whose solutions may be found explicitly.

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Analysis of the Dynamics of the Controlled Motion of a Three-Link Wheeled Mobile Robot within the Framework of Different Friction Models

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The problem of the controlled motion of a three-link wheeled mobile robot is addressed. It is assumed that the wheeled vehicle consists of three equal platforms (links) connected to each other by means of cylindrical joints, with wheel pairs rigidly fastened to the platforms. The points of attachment of the platforms to each other lie on straight lines that pass through the centers of mass of the platforms and are perpendicular to the axes of the wheel pairs. The center of mass of each platform lies on a straight line that passes through the point where the platforms are connected to each other and the wheel pair is fastened. It is assumed that the system moves on a horizontal plane and that its motion is due to periodic oscillations of the platforms relative to each other. The turning angles of the lateral platforms relative to the central platform are prescribed periodic functions of time (control functions).

In this paper we consider the dynamics of a three-link mobile robot within the framework of three different models of motion, and compare them with experimental results. The first of the models is the kinematic nonholonomic model of motion. Nonholonomic constraints imposed on the system correspond to the no-slip constraint at the points of contact of the wheels with the plane of motion. It is shown that in this case there exist ranges of values of control functions (special configurations) in which the constraint reactions increase without bound regardless of the type of these controls. As the system passes these configurations, the constraint reactions generally begin to increase without bound, so that the nonholonomic model becomes inapplicable.

The second model is the stick-slip hybrid model induced by Coulomb friction, which is developed in [1] for the three-link mobile robot. In this model it is assumed that the lateral slipping (along the axis of the wheel pair) begins at the moment when the reaction forces go beyond the limits of the cone of friction defined by the law of dry Coulomb friction. In this case, a reverse transition to the nonholonomic rolling is possible at the moment when the velocity of slipping becomes zero.

The third model is the model of viscous friction, in which we assume that the motion occurs with lateral slipping, and that the force of viscous friction that is proportional to the velocity of lateral slipping acts at the points of contact. In this case it is necessary to take account of the degrees of freedom related to the rotation of the wheels, in contrast to the nonholonomic model in which the wheel pairs can be replaced with a knife edge (skate) located at its center of mass and prohibiting sliding in the transverse direction (relative to the plane of the wheels) [2] . In this model, we investigate the influence of the friction force on the dynamics under the assumption that forces of viscous friction with a Rayleigh function proportional to the square of the velocity of lateral slipping act at the points of contact of the wheels with the plane.

To estimate the adequacy of the above-mentioned models of motion and to determine the scope of their applicability, a series of experiments with a prototype of the three-link wheeled robot with periodic controls were carried out. For "slow" controls (a period larger than some period defined experimentally) the experimental results are in good qualitative and quantitative agreement with the nonholonomic model and the model of viscous friction. As the period of control functions decreases, one can experimentally observe a lateral slipping, for which the model of viscous friction also provides an adequate qualitative and quantitative description.

The work was carried out within the framework of the state assignment of the Ministry of Science and Higher Education of Russia (FEWS-2025-0004).

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Integrable Cases of a Heavy Rigid Body with a Gyrostat and Contact Magnetic Geodesic and Sub-Riemannian Flows on $V_{n,2}$

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The rank-two Stiefel variety $V_{n,2} = SO(n)/SO(n-2)$ is the variety of ordered sets of two orthogonal unit vectors $\mathbf{e}_1, \mathbf{e}_2$ in the Euclidean space $(\mathbf{R}^n, \langle\cdot,\cdot\rangle)$. It can be seen also as the unit sphere bundle T_1S^{n-1} with respect to the standard round sphere metric

$$T_1S^{n-1} = \{\mathbf{e}_2 \in T_{\mathbf{e}_1}S^{n-1} \,|\langle \mathbf{e}_2, \mathbf{e}_2\rangle = 1,\, \mathbf{e}_1 \in S^{n-1}\},$$
$$S^{n-1} = \{\mathbf{e}_1 \in \mathbf{R}^n \,|\, \langle \mathbf{e}_1, \mathbf{e}_1\rangle = 1\}.$$

Therefore, it carries the *standard contact form*, the restriction of the Liouville 1-form from $TS^{n-1} \cong T^*S^{n-1}$ to the unit tangent bundle $\alpha = -\mathbf{e}_2 d\mathbf{e}_1|_{V_{n,2}} = -\sum_{i=1}^n e_2^i de_1^i|_{V_{n,2}}$.

Let $\mathcal{H} = \ker\alpha \subset TV_{n,2}$ be the *standard contact distribution*. The fact that $(V_{n,2}, \alpha)$ is a contact manifold is equivalent to the non-degeneracy of the closed two form $\omega_{mag} = d\alpha = d\mathbf{e}_1 \wedge d\mathbf{e}_2|_{V_{n,2}} = \sum_{i=1}^n de_1^i \wedge de_2^j|_{V_{n,2}}$ restricted to $\mathcal{H}$, or, to the condition $\alpha \wedge \omega_{mag}^{n-2} \neq 0$. We refer to ω_{mag} as the *standard contact magnetic form* on $V_{n,2}$.

Thus, we can study the following three natural problems on the rank-two Stiefel variety:

- Magnetic geodesic flows with respect to the magnetic force defined by $\eta\,\omega_{mag}$. Here η is a real parameter representing the strength of the magnetic field, and for $\eta = 0$ we have the usual geodesic flows.
- Sub-Riemannian magnetic geodesic flows, with the sub-Riemannian structures defined on $\mathcal{H}$ and other $SO(n)$-invariant bracket generating distributions.
- Natural mechanical systems with influence of the magnetic field defined by $\eta\,\omega_{mag}$.

We prove the integrability of magnetic geodesic flows of $SO(n)$-invariant Riemannian metrics on the rank two Stefel variety $V_{n,2}$ with respect to the magnetic field $\eta\, d\alpha$, where α is the standard contact form on $V_{n,2}$ and

η is a real parameter. Also, we prove the integrability of magnetic sub-Riemannian geodesic flows for $SO(n)$-invariant sub-Riemannian structures on $V_{n,2}$. All statements in the limit $\eta = 0$ imply the integrability of the problems without the influence of the magnetic field. We also consider integrable pendulum-type natural mechanical systems with the kinetic energy defined by $SO(n) \times SO(2)$-invariant Riemannian metrics. For $n = 3$, using the isomorphism $V_{3,2} \cong SO(3)$, the obtained integrable magnetic models reduce to integrable cases of a motion of a heavy rigid body with a gyrostat around a fixed point: Zhukovskiy – Volterra gyrostat, the Lagrange top with a gyrostat, and the Kowalevski top with a gyrostat. As a by-product we obtain the Lax presentations for the Lagrange gyrostat and the Kowalevski gyrostat in the fixed reference frame (dual Lax representations).

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On Bifurcations Leading to the Instant Creation of Hyperchaotic Attractors with Two and Three Positive Lyapunov Exponents

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It is well kown that chaos in the parabola map $\bar{x} = 1 - ax^2$ appears via the cascade of period-doubling bifurcations [1]. In this report we present results of the study of the Kopel duopoly model [2]:

$$\bar{x} = (1-\rho)x + \rho\mu y(1-y), \quad \bar{y} = (1-\rho)y + \rho\mu x(1-x),$$

where x and y are output quantities of each firm, parameter μ measures the intensity of the effect that one firm's action has on the other firm, and ρ is the adjustment coefficient of the adaptive expectations of the firm. For this case our numerical experiments show that an infinite sequence of the degenerate period-doubling bifurcations (when both multipliers are equal to -1 at the bifurcation moment) accumulates to the so-called "double Feigenbaum point" [3] belonging to the boundary between periodic and hyperchaotic dynamics with two positive Lyapunov exponents.

We also extend our results to the case of hyperchaotic dynamics with three positive Lyapunov exponents by the example of the Kaneko map [4]:

$$\bar{x} = Ax + (1-A)(1-Dy^2), \quad \bar{y} = z, \quad \bar{z} = x.$$

This map depends on two parameters A and D. Along $A = 0$ at each period-doubling bifurcation we have three multipliers on the unit circle simultaneously: $\lambda_1 = -1$ and $\lambda_{2,3} = e^{\pm i\pi/3}$, i. e., a cascade of degenerate period-doubling bifurcations occurs. This cascade results in a hyperchaotic attractor with three positive Lyapunov exponents. We continue the region with such attractor on the (A, D)-plane.

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On Robustly Chaotic Attractors in the Generalized Kuramoto Model

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In this talk we consider the generalized Kuramoto model

$$\dot{\phi}_j = \omega + \frac{1}{N} \sum_{i=1}^{N} g(\phi_i - \phi_j), \quad j = 1, \dots, N, \tag{1}$$

where $\phi_j \in S^1$ is the phase of the j-th oscillator, N is the number of the oscillators, ω is the common frequency, and the 2π-periodic function $g(\phi)$ describes the coupling.

We show that for $N = 4$ this system can demonstrate, in certain regions of the parameter space, robustly chaotic attractors, with no stability windows. We give a rigorous proof of the existence of these regions and provide explicit conditions on the coupling function $g(\phi)$ which allow one to find them for arbitrary 2π-periodic function $g(\phi)$. We also show that the presence of the 4th harmonic is necessary for the existence of the robustly chaotic attractors.

The theoretical results are applied to an example of the coupling function with four Fourier modes: $g(\phi) = \sum_{k=1}^{4} A_k \sin(k\,\phi + \xi_k)$. In this case we find the regions with robustly and non-robustly chaotic attractors numerically. It is worth noting that the robustly chaotic attractors that we found remain chaotic under small time-dependent perturbations (periodic, quasiperiodic), and any network of weakly coupled, identical systems with attractors of this type is also robustly chaotic.

This is a joint work with E. Karatetskaia, K. Safonov, and D. Turaev.

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Billiards and Families of Quadrics Associated with Integrable Geodesic Flows for Geodesically Equavalent Metrics

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An operator field on a smooth manifold M^n is called a Nijenhuis operator field if its Nijenhuis torsion vanishes $\mathcal{N}_L(\xi, \eta) = [L\xi, L\eta] + L^2[\xi, \eta] - L[L\xi, \eta] - L[\xi, L\eta] = 0$.

Such operators are connected to various geometrical and algebraical constructions, arise in integrable hierarchies and finite-dimensional systems. Recent papers by A. Bolsinov, A. Konyaev and V. Matveev are dedicated to systematic study of their properties and applications [1].

In the talk we apply Nijenhuis geometry to integrability problem of geodesic flows and billiards on a plane. For the Euclidean metric case, well-known Bikhoff conjecture states (skipping some details) that a billiard is integrable if its domain is bounded by confocal quadrics. Several recent papers contain its proof under some additional assumptions, e.g. on the shape of the domain, properties of trajectories or first integral, see [2, 3]. Integrable billiards in confocal domains and their integrable generalizations (both classical and new one, like Vedyushkina billiard books or Fomenko billiards with slipping) turns out to form a "wide" class in the class of nondegenerate integrable systems [4] in the sense from the topological point of view, i.e. properties of their Liouville foliations, see also [5].

To describe new integrable geodesic flows and billiards in 2-dimensional case, we apply recent results [6] obtained from the Nijenhuis geometry and originated from the theory of projectively equivalent metrics, classical subject dated back to Dini.

Recall that two metrics $\tilde{g}, g$ on M^n are called *geodesically equivalent* if they have the same geodesics considered as unparametized curves. Thus operator $L = (\det \tilde{g} / \det g)^{1/(n+1)}\, \tilde{g}^{ik} g_{kj}$ is called *geodesically compatible* with both g and $\tilde{g}$. Condition $\mathcal{N}_L = 0$ is necessary but not sufficient.

Theorem 1. *Each pair of geodesically compatible flat metric g and operator L on R^2 belongs ti one of 15 classes. In appropriate flat coordinates metric g has one of the following forms $dx^2 + dy^2$, $dx^2 - dy^2$, $-dx^2 - dy^2$, $dxdy$ and operator L depends on one or several real parameters.*

The obtained classification of geodesically compatible pairs (g, L) generates classification of integrable geodesic flows with Hamiltonian $H = \frac{1}{2} g^{ij} p_i p_j$ and following first integral F

$$F = \frac{1}{2} K^i_q g^{qj} p_i p_j, \qquad K^i_q := L^i_q - \operatorname{tr} L \, \delta^i_q.$$

Theorem 2. *Geodesic flow of flat metric generated by a geodesically compatible pair (g, L) belongs to one of eight families I-IV and Ia-IVa of pairs (H, F). In the chosen flat coordinates (x, y), function H is described in Theorem 1 and quadratic integral F depends on real parameters.*

Separating coordinates for these systems can be constructed from analysis of eigenvalues λ_1, λ_2 of the operator L. Equation $\chi_l(\lambda) = 0$ determine a quadric on x, y which coefficients depend on λ and parameters of the family. For all values of λ except a finite number of them, such a quadric is nondegenerate for families I-IV and is a parabola for families $Ia - IVa$. Note that these family of quadrics is orthogonal with respect both quadratic forms H, F.

Family I corresponds to a confocal family on Euclidean plane $2H = p_1^2 + p_2^2$ for parameters $a \neq b$. For $a = b$, they are concentric circles and radii: instead of a levels of a constant eigenvalue $\lambda_2 = a$, second separating coordinate is determined by eigenvector of nonconstant eigenvalue $\lambda_1 = a - x^2 - y^2$. Family II determines family of quadrics confocal for pseudo-Euclidean metric. Billiards in such domains were studied by V. Dragovic and M. Radnovic [7]. Other nondegenerate families III and IV determine families of hyperbolas not familiar to us from literature. Hamiltonian has the form $H = p_1 p_2$ and first integrals are the following for $K = \pm 1, a, b \in \mathbb{R}$:

$$2F_{III} = -Kx^2 p_1^2 - 2(Kxy - a) p_1 p_2 + (1 - Ky^2) p_2^2;$$

$$2F_{IV} = -(a + Kx^2) p_1^2 - 2(Kxy - b) p_1 p_2 + (a - Ky^2) p_2^2.$$

Note that foci of the obtained hyperbolas belong to another hyperbola $y^2 - x^2 = 2/K$ for case III and $y^2 - x^2 = 2a/K$ for the case IV. It will be interesting to compare it with the well-known confocal property of quadrics and its pseudo-Euclidean analog.

The studied class of integrable geodesic flows can be generalized by adding appropriate potential fields U and V to H and F such that $\tilde{H} = H + U(x, y)$ and $\tilde{F} = F + V(x, y)$ Poisson commute. The obtained integrability condition on potentials develop the well-known Kozlov equation [8] for billiards in an ellipse to other flat metrics I-IV, Ia-IVa.

$$(a-b)V_{xy} + 3(yV_x - xV_y) + (y^2 - x^2)V_xy + xy(V_{xx} - V_{yy}) = 0.$$

Theorem 3. *For a geodesic flow (H, F) from families I-IV, Ia-IVa from Theorems 1-2, its integrable perturbations by analytic potentials can be described as following, where s_1, s_2 are functions on a real variable and λ_1, λ_2 are separating variables of the initial geodesic flow:*

$$U = \frac{s_2(\lambda_2) - s_1(\lambda_1)}{\lambda_2 - \lambda_1}, \qquad \mathrm{d}V = K^*\mathrm{d}U \text{ for } K^i_j = L^i_j - (\operatorname{tr} L)\,\delta^i_j.$$

The last point of this approach states that for each rectangular (in λ_1, λ_2 coordinates) domain such a geodesic flow (with or without potential field) generates an integrable billiard. First integrals $\tilde{H}$ and $\tilde{F}$ preserves after reflection from coordinate lines because their forms are diagonal one (in these coordinates) and thus value on a vector does not depend on signs of the latter components.

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On the Isomorphism of the Problems of the Rubber Rolling of Bodies of Revolution and on the Dynamics of a Rubber Torus

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Consider the rolling motion of a heavy rigid body of revolution on a plane. Assume that the body rolls on the plane without slipping (the velocity of the point of contact is zero) and without spinning (the projection of the angular velocity of the body onto the vertical is zero) and that it moves with only one point, P, in contact with the supporting plane.

The equations of motion are reduced to a conformally Hamiltonian system with two degrees of freedom with the Hamiltonian

$$H = \frac{1}{2}\left(\frac{J(\vartheta)^2}{B(\vartheta)}p_\vartheta^2 + \frac{p_\varphi^2}{\sin^2\vartheta}\right) + U(\vartheta)$$

and the equations

$$\dot{\vartheta} = \frac{1}{J(\vartheta)}\frac{\partial H}{\partial p_\vartheta}, \quad \dot{\varphi} = \frac{1}{J(\vartheta)}\frac{\partial H}{\partial p_\varphi}, \quad \dot{p}_\vartheta = -\frac{1}{J(\vartheta)}\frac{\partial H}{\partial \vartheta}, \quad \dot{p}_\varphi = -\frac{1}{J(\vartheta)}\frac{\partial H}{\partial \varphi}. \tag{1}$$

Here the following notation is used:

$$J(\vartheta) = \sqrt{i_1\cos^2\vartheta + i_3\sin^2\vartheta + m(\chi_1(\vartheta)\sin^2\vartheta + \chi_2(\vartheta)\cos\vartheta)^2},$$
$$B(\vartheta) = i_1 + m(\chi_1(\vartheta)^2\sin^2\vartheta + \chi_2(\vartheta)^2),$$

the potential energy $U = -mg(\boldsymbol{r}, \boldsymbol{\gamma})$ is expressed in terms of $\chi_1(\vartheta)$, $\chi_2(\vartheta)$ as follows:

$$U(\vartheta) = -mg(\chi_1(\vartheta)\sin^2\vartheta + \chi_2(\vartheta)\cos\vartheta),$$

and $\chi_1(\vartheta)$, $\chi_2(\vartheta)$ are arbitrary functions that define the form of the body of revolution and are related by

$$\frac{d\chi_2(\vartheta)}{d\vartheta} = -\sin\vartheta\chi_1(\vartheta) - \frac{\sin^2\vartheta}{\cos\vartheta}\frac{d\chi_1(\vartheta)}{d\vartheta}.$$

In the case of a body of revolution the variable φ is cyclic, and hence p_φ is an integral of motion. Thus, equations (1) admit two integrals of motion and an invariant measure (since they are conformally Hamiltonian). In addition, it is easy to show that the integral manifolds of the system under consideration are bounded. Consequently, by the Euler – Jacobi theorem (see, e.g., [1]), in nonsingular cases these manifolds are two-dimensional tori, and the system (1) can be reduced to quadratures.

This paper addresses the question of the existence of different axisymmetric bodies whose foliations on invariant manifolds are identical.

This is a joint work with Elena Pivovarova.

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Rubber Rolling of Solids of Revolution over the Plane: the Surprising Miracle of the "Nose" Function

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Recall that "rubber rolling" means both no-slip and no-twist non-holonomic constraints. In this presentation we comment upon a very interesting result that appeared in Borisov and Mamaev [1] and Borisov, Mamaev and Bizyaev [2]. We use the *almost symplectic* approach to re-obtain their reduction to 1 DoF for surfaces of revolution that are rubber rolling over the plane. These are $SE(2)$-Chaplygin systems with an additional symmetry. We work in the space frame with Euler angles ϕ (yaw), θ (nutation) and ψ (roll).

A "miracle" happened, which makes this problem somewhat special. We do the reduction in two stages: first, the almost symplectic reduction by the group $SE(2) = \{(x, y, \phi)\}$ to the Poisson sphere $T^*S^2 = \{(\theta, \psi)\}$. It produces a 2 DoF conformally Hamiltonian system. The almost symplectic 2-form is

$$\Omega_{NH} = dp_\theta \wedge d\theta + dp_\psi \wedge d\psi + J \cdot K,$$

with the semi-basic term $J \cdot K = -p_\psi \, (d \log \big(N(\theta)\big) \wedge d\psi$, with

$$N(\theta) = \big(I_1 \, \cos^2\theta + I_3 \, \sin^2\theta + m \, z_C^2(\theta)\big)^{1/2}$$

which we call the "Nose" function), where $z_C(\theta)$ the height of the center of mass C. [Why we used the name "Nose"? We felt that what happened reminds a short story by Gogol.]

The conserved quantity due to the S^1 symmetry about the body axis is

$$\ell = N(\theta) \, \sin^2\theta \, \dot{\psi},$$

yielding the desired reduction to (θ, p_θ). Further simplification results by taking the new time $dt = \sqrt{B} \, d\tau$, with

$$B = I_1 + m \, |CP|^2 \quad \text{where } P \text{ is the point of contact.}$$

One gets $H = \frac{1}{2}\tilde{p}_\theta^2 + V(\theta)$, $V(\theta) = \ell^2/2\sin^2\theta + mg \, z_C(\theta)$ with $\tilde{p}_\theta = p_\theta/\sqrt{B}$ and usual symplectic form $d\tilde{p}_\theta \wedge d\theta$. The moments of inertia I_1, I_3 reappear in the reconstruction. We take the torus as a concrete example, and make some basic observations. Full reconstruction is in order.

We thank Alexander Kilin and Elena Pivovarova for many informations and references. We are submitting the paper to a volume in tribute to Alexey Borisov.

The work is supported by a UERJ fellowship.

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Robustly Chaotic Dynamics in a 3-Level Laser Model with Opticl Pumping

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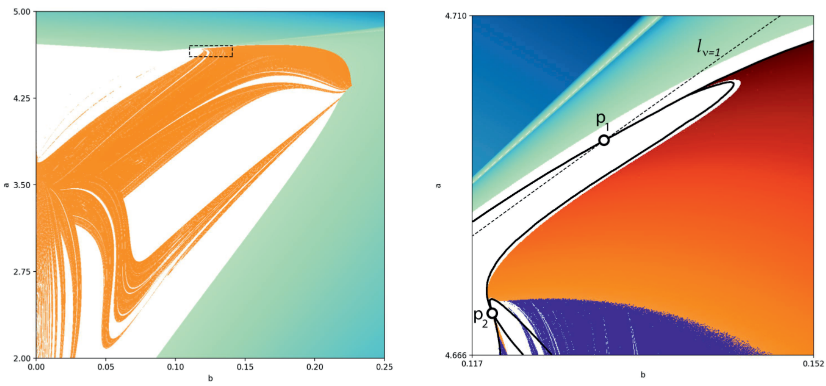
We present our recent results of a numerical study of a six-dimensional system describing the dynamics of the optically pumped three-level laser model. For the first time, we demonstrate that this system exhibits robustly chaotic attractors in an open region of the parameter space. Our results are based on the successive verification of all required pseudohyperbolicity conditions, as well as on checking the conditions of the Shilnikov criteria [1,2]. In particular, we found two Shilnikov points (denoted as p_1 and p_2 in the figure below), where an additional degeneracy occurs for the pair of homoclinic loops. The region with a robustly chaotic attractor (colored in yellow in the figure below) originates from points the p_1 and p_2 and spreads quite far from these points. The boundary of these regions is associated with the curve $l_A = 0$, where the first foliation tangency occurs along the homoclinic



(a) (b)

Figure 1. Two-parameter chart of dynamical regimes: a) chart of the top Lyapunov exponent; b) Enlarged fragment from Fig.1 a. Diagram of the minimum angle between subspaces. Robustly chaotic attractors exist in orange region where pseudohyperbolicity conditions for the corresponding 6D laser model are met. This regions originate from the Shilnikov points p_1 and p_2. In blue region a chaotic attractor exists but is not robust. The curve $l_{\nu=1}$ is a neutral saddle curve.

loops. The regions where the attractor is not robustly chaotic are colored in blue.

This is a joint work with K. Zaichikov.

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The Lagrange Identity and Dynamics in a Potential Jacobi Field

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A Jacobi field is a potential force field whose potential is a homogeneous function of degree -2. The problem of the motion of a particle in such a field admits an additional integral quadratic in velocities. It can be used to reduce the number of degrees of freedom and to pass to the study of a reduced system with spherical configuration space. These results are extended to the more general case of the motion of a particle in spaces of constant curvature. An analysis is made of particle motion on a cone whose vertex coincides with the singular point of the Jacobi potential. A lower estimate of the distance from the moving particle to the vertex of the cone is given. This approach is also applicable to a more general case where the charged particle is additionally located in the magnetic field of a monopole. A billiard inside the cone with a particle bouncing elastically off its boundary is considered.

Two Approaches to Average Stochastic Perturbations of Integrable Systems

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In symplectic space $(\mathbf{R}^{2n}_{x,y}, dx \wedge dy)$ we consider a Birkhoff-integrable Hamiltonian system — its Hamiltonian H depends only on the actions $I_j = (x_j^2 + y_j^2)/2$. Introducing complex coordinates $v_j = x_j + iy_j$ we write this system in the convenient complex form:

$$\frac{\partial}{\partial t} v_k(t) = i\nabla_k H(I) v_k, \quad k = 1, \dots, n, \quad I = I(v), \quad v(t) \in \mathbf{C}^n. \tag{1}$$

Note that $I_j = \frac{1}{2}|v_j|^2$. Study of small perturbations of this system for large values of time, either globally in $\mathbf{R}^n$, or locally near an equilibrium, is a classical problem of dynamical systems starting the end on XVIII century. There are two main settings:

A) the perturbation is a small smooth vector field,

B) the perturbation is a small Hamiltonian field.

We will consider a third case and examine an ε-small stochastic perturbation of the integrable system (1) for for $0 \le t \le T\varepsilon^{-1}$:

$$\frac{\partial}{\partial t} v_k(t) = i\nabla_k H(I) v_k + \varepsilon P_k(v) + \sqrt{\varepsilon} \sum_{j=1}^{n_1} B_{kj}(v) \dot{\beta}^c_j(t),$$
$$k = 1, \dots, n, \qquad v(0) = v^0,$$

where $\{\beta^c_k(t), 1 \le k \le n\}$ are standard independent complex Wiener processes. That is, $\beta^c_k(t) = \beta^+_k(t) + i\beta^-_k(t)$, and $\beta^\pm_k(t)$ are standard independent real Wiener processes. Passing to the slow time $\tau = \varepsilon t$ and using vector notation we re-write this system as

$$\dot{v}(\tau) = i\varepsilon^{-1} diag\{\nabla_k H(I)\} v + P(v) + B(v)\dot{\beta}^c(\tau), \quad v(0) = v^0, \tag{2}$$

where $v(\tau) \in \mathbf{C}^n$ is a complex $n \times n_1$ matrix and $0 \le \tau \le T$. We impose some mild regularity assumptions on coefficients of this equation and assume that the matrix $B(v)$ is non-degenerate: its rank is n for all v (so $n_1 \ge n$).

A solution of this equation will be denoted $v^\varepsilon(\tau)$. The goal is to study the behaviour of the vector of actions $I^\varepsilon(\tau) = (I_1^\varepsilon, \dots, I_n^\varepsilon)(\tau)$, where $I_j^\varepsilon(\tau) = I\big(v_j^\varepsilon(\tau)\big)$, when $\varepsilon \to 0$, on time-intervals $0 \le \tau \le T$. Following the works [1, 2] we will discuss two approaches to do that.

In the first approach we introduce in $\mathbf{C}^n = \{v = (v_1, \dots, v_n)\}$ the usual action-angle coordinates (I, φ), where $I = (I_1, \dots, I_n) \in \mathbf{R}_+^n$ and $\varphi = (\varphi_1, \dots, \varphi_n) \in \mathbf{T}^n = \mathbf{R}^n/(2\pi\mathbf{Z})^n$: $I_j = \frac{1}{2}|v_j|^2$, $\varphi_j = Arg\, v_j$, $\quad j = 1, \dots, n$ (so $v_j = \sqrt{2I_j}\exp i\varphi_j$). Then v-equation (2) may be re-written in terms of these coordinates. The equations for actions are:

$$\dot I_k^\varepsilon(\tau) = \Re(\bar v_k^\varepsilon\, P_k(v^\varepsilon)) + \sum_l |B_{kl}(v^\varepsilon)|^2 + \sum_{j=1}^{n_1} \Re\big(\bar v_k^\varepsilon B_{kj}(v^\varepsilon)\big)\dot\beta_j(\tau), \tag{3}$$

where $k = 1, \dots, n$ and $\beta_1(\tau), \dots, \beta_{n_1}(\tau)$ are independent standard real Wiener processes. The φ- equations are:

$$\begin{aligned} &\dot\varphi_k^\varepsilon(\tau) = \\ &= \varepsilon^{-1}\nabla_k H(I^\varepsilon) + \text{term of order one with strong singularity at } \{I_k^\varepsilon = 0\}, \end{aligned}$$

$1 \le k \le n$. We have got a *slow-fast system*, where the actions I_k^ε's are *slow variables* and the angles φ_k^ε's are *fast variables*. Then, by the usual logic of averaging in fast-slow systems, we in Eq. (3) write the coefficients of the equations as functions of actions and and angles, and next average them in angles (the matrix-function $B(I, \varphi) = \{B_{kl}\}(I, \varphi)$ has to be averaged by the rules of stochastic averaging). This provides us with averaged equations for actions. For this approach it is shown that the averaged equations for actions describes the limiting dynamics of actions of solutions $v^\varepsilon(\tau)$ as $\varepsilon \to 0$, in sense of distribution, if in (2) $P(v)$ is C^2-smooth and B is a constant matrix. It is very likely that these restrictions are necessary, see [2].

In the second approach we build an effective equation for Eq. (3), following the following procedure: in the Eq. (3) we

– dropp its first term,

– suitably average the other two terms with respect to the action of the torus $\mathbf{T^n} = \{\omega\}$ on $\mathbf{C^n}$ by means of the rotation-operators $\Phi_\omega = diag\left(e^{i\omega_1}, \dots, e^{i\omega_n}\right)$
(again, the matrix-function $B(v) = \{B_{kl}\}(v)$ has to be averaged by the rules of stochastic averaging). Then, without any restrictions on equation (3), we prove that as $\varepsilon \to 0$ the actions of solutions for the effective equation approximate those of solutions $v^\varepsilon(\tau)$, in sense of distributions.

Moreover, it turns out that if the first few moments of the norms $|v^\varepsilon(\tau)|$ are bounded uniformly in $\varepsilon > 0$ and $\tau \ge 0$, then this approximation is uniform in time. We provide easy sufficient conditions which imply these uniform bounds for the moments.

At the end of the talk we will compare these results with what is known for classical perturbative questions A) and B), given at the beginning of this note.

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Vortex Dynamics on Nonsimply Connected Surfaces

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This conference paper is concerned with motions of an ideal incompressible fluid on a two-dimensional manifold which can be reduced to finite-dimensional dynamical systems, for example, systems of point vortices. An analysis is made of the case where on the nonsimply connected surface there is a field of constant vorticity

$$\operatorname{rot} v = \omega_0 = \text{const.}$$

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Problems and Prospects in Research on the Dynamics of Mechanical Systems with Servo-Constraints

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The development of robotics is stimulating the interest of scientists in the field of mechanical system control. This report discusses the prospects and challenges associated with the study and control of mecanical systems with servo-constraints. These are the problem of steering a mechanical system into a motion regime that complies with servo-constraints, the problem of stability of servo-controlled motion and others.

In the scientific literature, there are several approaches to constructing equations of motion and determining control actions for systems with servo-constraints. This study uses an approach based on the generalized D'Alembert – Lagrange principle and program control [1–3].

Special attention is also devote to the stability of motion using the example of a system consisting of two spheres, where one sphere rolls over the surface of the other sphere (Fig. 1). To study the stability problem when one sphere moving along a circular path on the surface of the another one, a method based on equations in variations is used. When one sphere is held on the surface of the another one in an upper or lower position, the stability problem reduces to studying the stability of equilibrium positions.

Linearization of equations of motion in the neighborhood of the equilibrium positions does not answer the question of whether equilibrium positions are stable because all eigenvalues of the matrix of the linearized system have a zero real part. Therefore, a different approach is required. The existence of a complete set of first integrals and a measure makes the system of motion

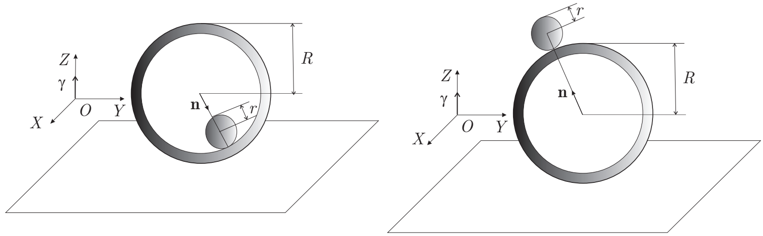

Figure 1. Systems of two spherical bodies.

equations integrable by the Euler – Jacobi theorem. To study the stability of equilibrium positions, we proceed to the study of the potential energy of the reduced system.

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First Integrals of Geodesic Flows on Cones

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We study the behavior of geodesics on cones over arbitrary C^3-smooth Riemannian manifolds. We show that the geodesic flow on such cones admits first integrals whose values uniquely determine almost all geodesics except for cone generatrices.

Billiard Trajectories inside Cones

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The report discusses billiard trajectories inside an n-dimensional cone over a strictly convex closed manifold M. It is shown that if M is a C^3-smooth manifold, then every trajectory has a finite number of reflections and, in this case, the billiard admits first integrals whose values uniquely determine all billiard trajectories. At the same time, there exists a C^2-smooth manifold M and a billiard trajectory in the cone such that this trajectory has infinitely many reflections in finite time.

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Statistical Properties of Extreme Events in the Ring of FitzHugh – Nagumo Neurons

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Extreme events in society and nature, such as pandemic outbreaks, abnormal waves, or structural failures, can have catastrophic consequences. An extreme event is an event in which the value under consideration shows significant deviations from the average value, several times higher than the standard deviation from the average value, and with an outbreak of activity leads to a change in the normal functioning of the system. This phenomenon is also observed with the extreme work of brain neurons during epileptic seizures [1]. Therefore, this kind of research is important and relevant to study extreme events in neural networks.

The most common biologically significant and simple neuron model is the FitzHugh – Nagumo oscillator [2]. A network consisting of this neuron model can show how the brain works, and also helps simplify the study of the effects of various kinds of phenomena and external influences on the dynamics of neurons. In this study, Poisson impulses and Lévy noise were used as external influences, which make it possible to control processes in neural systems.

A system of coupled neurons was used to study extreme events in the ring of FitzHugh – Nagumo neurons:

$$\begin{aligned} \varepsilon\dot{x}_i &= x_i\left(1-\frac{u^3}{3}\right) - y_i + F(t), \\ \dot{y}_i &= x_i + a, \end{aligned} \tag{1}$$

where x_i and y_i are dynamic variables that describe the temporal dynamics of the activator (fast variable) and the inhibitor (slow variable), respectively; $i = 1, 2, \ldots, N$ is the element number, and $N = 10$ is the total number of oscillators in the network.

$F(t)$ is responsible for the external effect on the network of neurons. In the case when Poisson pulses are used as an external influence, $F(t) = A\xi^f(t)$, where f is the frequency parameter, and A in (1) is the amplitude of the signal. Poisson pulses are characterized by an independent time between the moments of pulse delivery and have an exponential distribution. If Lévy noise is used as an external influence, then $F(t) = \eta_i^{\alpha^L,\beta^L}(t, \sigma^L)$, where the parameter α is the stability index showing the rate of narrowing of the tails of the distribution. The parameter $\beta^L \in [-1; 1]$ defines the skewness of the distribution, and σ defines the width of the distribution. Couplings between neurons are subject to periodic boundary conditions ($i + kN = i$ for any integer k).

Extreme events exceeding the three standard deviations have a greatest range of probability of observation. When the coupling strength is attractive, it affects the behavior of the system, bringing the distribution of spike amplitude closer to normal in some optimal range of coupling strengths. With a high repulsive coupling, leakage is observed in areas with a lower probability, while the entire parameter plane has a higher probability. Statistical characteristics of extreme events under external influence were also considered. The analysis of the relationship of statistical characteristics with the behavior of the system is carried out.

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Dynamics of a Homogeneous Ball on a Rotating Cylinder

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In this work we consider the problem of a homogeneous ball rolling on the inner surface of a circular cylinder. The system is placed in a gravity field parallel to the axis of the cylinder. The cylinder's axis moves with plane-parallel motion along a fixed circle, while the cylinder does not rotate about its own axis.

The equations of motion of the system considered are integrated by quadratures, and the possibility of the ball moving downward or upward along the cylinder's axis is examined in detail.

It turns out that, by making the supporting cylinder execute circular motions, it is impossible to achieve an unbounded elevation of the ball along the cylinder. In the general case, the ball will execute quasi-periodic motions without drifting in the vertical direction either upward or downward (see Fig. 1, left). However, in contrast to a fixed (or rotating) cylinder, there exist resonances at which the ball moves on average downward with constant acceleration (see Fig. 1, right).

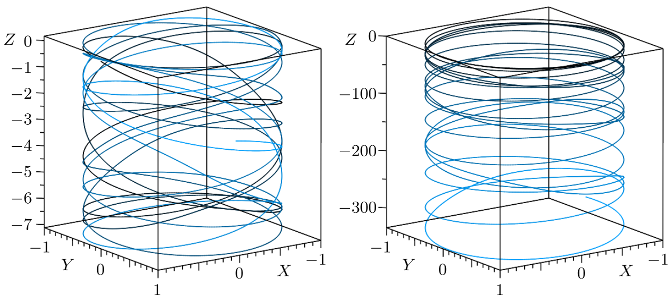

Figure 1. An example of the trajectory of the center of the ball in the general case (left) and in the resonant case (right).

The results of the work were published in [1].

This is a joint work with Alexander Kilin.

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Topology of 4-Manifolds that Admit Non-Singular Flows with Saddle Orbits of the Same Index

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This paper studies regular topological flows f^t defined on closed topological manifolds M^n. The chain recurrent set of such a flow consists of a finite number of topologically hyperbolic fixed points and periodic orbits. Like their smooth analogues — Morse – Smale flows [3] — regular flows possess a continuous Morse-Bott function that decreases outside the chain recurrent set and is constant on the chain components of the flow. This circumstance leads to a close connection between such flows and the topology of the carrying manifold. In particular, the ambient manifold for non-singular flows (regular flows without fixed points), by the Poincare – Hopf formula, has a zero Euler characteristic. The latter property is a criterion for a manifold M^n to admit a non-singular flow in all dimensions except dimension $n = 3$ [1, 2]. Thus, in higher dimensions, any odd-dimensional manifold admits a non-singular flow, and the list of even-dimensional manifolds is quite broad; at the very least, it includes all manifolds of the form $M^{n-1} \times S^1$, where M^{n-1} is any closed $(n-1)$-manifold.

A surprising result of the present paper is the proof of the fact that in dimension 4, all this variety of carrying spaces can only be achieved if the flow has saddle orbits of different Morse indices. Specifically, for dimensional reasons, the Morse index of a saddle orbit of a non-singular flow $f^t : M^4 \to M^4$ can only take two values, 1 or 2. We prove that non-singular 4-flows with saddle orbits of the same Morse index exist only on skew or direct products of the 3-sphere and the circle, i.e., $M^4 \cong S^3 \tilde{\times} S^1$ or $M^4 \cong S^3 \times S^1$.

The work was supported by project "Symmetry. Information. Chaos" within the framework of Program of fundamental research of HSE in 2025.

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Metric Geometry and Forced Oscillations in Mechanical Systems

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Let M be a Riemannian manifold with metric T. Equivalently, we can say that a Lagrangian mechanical system with configuration space M and kinetic energy T is given. The kinetic energy is a positive definite quadratic form on the generalized velocities:

$$T = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(q)\dot{q}_i\dot{q}_j, \quad a_{ij}(q) = a_{ji}(q) \text{ for all } q \in M, 1 \leqslant i, j \leqslant n.$$

Let Q be a generalized force (possibly depending on the generalized velocity $\dot{q}$: $Q = Q(q, \dot{q})$; $Q(q, \dot{q})$ is a covector, for any given q and $\dot{q}$). If we consider the generalized force to be non-autonomous, we should add the time dependence: $Q = Q(q, \dot{q}, t)$.

The Lagrange equations can be written in the usual way:

$$[T] = Q,$$

where $[\,\cdot\,]$ denotes the Lagrangian derivative.

Let Q now be a τ-periodic in t generalized force, i.e., for any q, $\dot{q}$ and $t \in \mathbb{R}$, the following is satisfied

$$Q(q, \dot{q}, t) = Q(q, \dot{q}, t + \tau).$$

We will study the existence of τ-periodic solutions in the system $[T] = Q$.

Even in the simplest case of a harmonic oscillator when $M = \mathbb{R}$, $T = \dot{q}^2/2$ and $Q = -q + \sin t$ there may be no forced oscillations in the system. Moreover, in this case all solutions are unbounded.

If M is a closed manifold and the generalized forces are small (in the sense that the parameter ε is present in the system and the generalized forces have the form εQ), then for any field Q there exists a τ-periodic solution provided that $\varepsilon > 0$ is small and the Euler characteristic $\chi(M)$ is not zero. The proof of this statement follows from the existence of a positive injectivity radius for the exponential mapping of the geodesic flow.

In the talk, we discuss connections between the behavior of geodesics (i. e., solutions of the equation $[T] = 0$) and the existence of forced oscillations for the equation $[T] = Q$. The main method we use is to consider some modified system that is obtained from $[T] = Q$ by adding dissipative terms defined only for sufficiently large generalized velocities. Under some additional weak assumptions, this allows us to prove the existence of a forced oscillation in the modified system. Further we use the property of the system $[T] = Q$ that for large velocities the trajectories of its solutions are close to those of the system $[T] = 0$ (provided that the right-hand side grows slower than quadratically with the growth of the generalized velocity). This allows us to show that the found solution of the modified system cannot pass through the points where we introduced friction, i. e., the corresponding periodic solution exists in the original system as well.

Morsifications of Semihomogeneous Functions with Any Possible Number of Critical Points

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A deformation of an analytic function is called a *morsification* if it has only nondegenerate critical points. It is well-known that any morsification of a germ of an analytic function has the same number of complex critical points, known as a Milnor number (or multiplicity) of a germ. By contrast, different real morsifications of a real analytic function can have different amounts of real critical points. Constructions of morsifications with controlled number and locations of critical points were considered by various authors (e. g. [1,3–8]). Such morsifications have applications in various areas of singularity theory, such as the Picard-Lefschetz theory (where a specific type of morsification can be used for monodromy calculation [1,2]) and the theory of Lagrangian singularities [4]. V.Vassiliev has constructed morsifications with all possible numbers of real critical points for all simple real singularities [4] and some real singularities of modality one [5]. Here we would like to present explicit constructions of morsifications with all topologically permissible numbers of critical points for real semihomogeneous function germs of two variables with real components of zero level set.

The work is supported by the Ministry of Education and Science of the Russian Federation as part of the program of the Moscow Center for Fundamental and Applied Mathematics under the agreement no. 075-15-2022-284.

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Application of Poisson Learning for the Classification of Solutions of Hamilton Systems

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In this work, we consider Hamiltonian systems and investigate their behavior using modern machine learning techniques — specifically, graph-based semi-supervised Poisson learning. The general objective of this study is to develop an algorithmic framework that enables automatic classification of Hamiltonian system solutions in real time. The analytical foundation of the research is based on the Poincaré section method, which provides an informative two-dimensional projection of the system's phase space. The structure of the resulting phase space consists of multiple trajectories represented by periodic curves and/or chaotic point clouds. The trajectories are constructed through numerical integration of differential equations with given initial conditions in real time. However, this process is time-limited due to accumulated numerical error, which also poses a major challenge for full automation of the classification procedure. As a result, after a fixed simulation period, the two-dimensional Poincaré sections yield trajectory flows with distinct properties that must be classified into separate categories within a limited time window.

To address this problem — the classification of trajectory projections of phase-space flows on two-dimensional manifolds — we propose the use of a graph-based semi-supervised Poisson learning (PL) approach. Graph-based semi-supervised learning methods represent a modern machine learning framework for solving data classification problems when the use of fully supervised methods is impractical or impossible. The goal of this approach is to achieve high classification performance with a minimal number of labeled data samples. Poisson learning assumes that labels are known for a subset of data points, and these labels are then propagated to the remaining nodes using a harmonic function defined by the Poisson equation. This method has a variational formulation leading to the minimization of the Dirichlet energy functional. The corresponding optimization problem can be solved iteratively, for instance, using the Jacobi method. In the context of semi-supervised machine learning, such an iterative propagation process can be interpreted as a random walk on graphs. In this work, we also introduce a modified iterative random-walk solution designed to accelerate the learning and classification process. Thus, the proposed approach enables real-time automatic classification of Hamiltonian system solutions [1] while accounting for numerical accumulation errors.

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New Cases of Heteroclinic Bifurcations Resulting in the Emergence of Lorenz-like Chaotic Attractors

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Since the paper of L. P. Shilnikov [1], it became clear that bifurcations of the homoclinic butterfly can lead to the emergence of the Lorenz attractor if the homoclinic butterfly has an additional degeneracy. Nowadays, three main cases of such bifurcations are known: inclination flip, orbit flip, and bifurcation with a neutral saddle. In this talk, we discuss two new similar bifurcations lying on the boundary of the existence of Lorenz-like attractors in 3D systems of differential equations. These bifurcations are heteroclinic connections of saddle equilibria with a one-dimensional unstable manifold and a two-dimensional stable manifold. The difference is that the connection contains two equilibria in one case and three equilibria in the other. Besides, in both cases, one of the saddles has multiple stable eigenvalue. In the first case, splitting the heteroclinic connection leads to the emergence of the Lorenz attractor. In the second case, the splitting leads to the emergence of a pair of Lorenz attractors or a new four-winged Lorenz-like attractor.

Our study is motivated by numerical studies of the 3D Henon map presented in [2–4]. The authors found discrete analogues of Lorenz-like attractors. They associated the region of the existence of the attractors with local bifurcations of fixed points with multipliers $(-1, -1, 1)$ and $(-1, i-, -i)$. These local bifurcations can be studied by considering the bifurcations of a triple degenerate equilibrium state in 3D systems invariant with respect to certain Z_2- and Z_4-symmetry. We show that our heteroclinic bifurcations occur in the corresponding normal forms. Thus, we obtain a complete analytical proof of the existence of Lorenz-like attractors near a triple degenerate equilibrium state in Z_2- and Z_4-symmetric systems.

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Sparkling Saddle Loops of Vector Fields on Surfaces and Related Issues

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If an orientation-preserving separatrix loop of a hyperbolic saddle of a vector field on a two-dimensional surface is accumulated by a separatrix of another saddle, sparkling saddle connections appear when the loop is unfolded in a one-parameter family of vector fields. If it is the "free" unstable separatrix of the same saddle that accumulates to the loop, we get sparkling saddle loops instead. This can occur only on surfaces different from S^2 and $\mathbb{RP}^2$, of course.

The corresponding generic semi-local bifurcations differ dramatically depending on whether the path along the free separatrix preserves the local orientation or reverses it.

In the first case, the parameter values for which the vector field has a saddle loop are the endpoints of the gaps in a Cantor set. A thin tubular neighborhood of the unstable manifold of our saddle is homeomorphic to a torus with a disk removed. By cutting out this neighborhood and gluing a disk to it, we obtain a family of vector fields on the torus. The remaining points of the aforementioned Cantor set in the parameter space correspond to the Cherry flows on this torus: the Poincaré map has irrational rotation number, and hence the flow has nontrivial recurrent trajectories.

In the case when the path along the "free" unstable separatrix reverses the orientation, there are only two alternating convergent sequences that correspond to separatrix loops involving different unstable separatrices. Assuming the saddle is dissipative, flip-cycles are born and destroyed when these loops are unfolded, and the two sequences split the one-sided neighborhood of the critical parameter value into intervals where either one or two such cycles are present.

The configuration in which an orientation-reversing saddle loop of a dissipative saddle is accumulated by its free unstable separatrix does not appear in generic C^1 one-parameter families of vector fields.

On Different Types of Hyperbolic Chaotic Sets Appearing as a Result of the Perturbations of Anosov Map on a 2D Torus

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The report is devoted to the study of a two-parameter diffeomorphism obtained by perturbing the Anosov map using the Möbius map on a two-dimensional torus. The Möbius map depends on two parameters which are responsible for the dissipativity and shift of coordinates [1]. In the parameter space of the studied diffeomorphism, three regions were identified depending on the structure of the non-wandering set: (1) the non-wandering set covers the entire torus; (2) the non-wandering set consists of a fixed point and a one-dimensional hyperbolic set; (3) the non-wandering set consists of stable and completely unstable fixed points and a zero-dimensional chaotic set. The second region is divided into two subregions: (2.1) the fixed point is a repeller, and the one-dimensional hyperbolic set is an attractor; (2.2) the fixed point is an attractor, and the one-dimensional hyperbolic set is a repeller. To study the hyperbolicity of the map and the structure of the non-wandering set numerically, we use the calculation of Lyapunov exponents and angles between tangent subspaces [2, 3]. The transition from the first region to the second was described in detail in [4], so this work focuses on the study of the third region. In this parameter region, a zero-dimensional chaotic set is generated, which is neither an attractor nor a repeller, so in order to approximate this invariant set, the method described in the work [5] is used. The hyperbolicity of all the described sets is checked, and the bifurcations that occur during transitions between different regions are described.

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Secular Evolution of Motions in the Planetary Version of the Non-Restricted Three-Body Problem

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We consider a system of three celestial bodies consisting of a “star” and two “planets” orbiting around it. The masses of the planets are significantly less than the mass of the star. The evolution of the planets’ orbital motion is studied within the framework of a double averaged non-restricted three-body problem. The main attention is paid to coplanar configurations, when the star and planets move in a certain plane that preserves a constant position. Various variants of secular evolution are described in detail. In particular, the possibility of reversing the orbital motion of the inner planet is noted. Also we analyzed apsidal resonances representing stationary solutions of the averaged motion equations in which the positions of the lines of the apsides of the planets’ orbits coincide. Bifurcation diagrams are constructed that characterize the dependence of the number of stationary solutions and their stability properties on the values of the problem parameters. The realization of apsidal resonances in real exoplanetary systems is discussed.

Robust Control Design of Underactuated Systems via a Family of Time-Periodic Sliding Surfaces

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The paper addresses the problem of robust control design for orbital stabilization of periodic trajectories in mechanical systems affected by parametric uncertainties and matched bounded disturbances.

Based on a feedback that stabilizes a nominal linear system with periodic coefficients (in particular, the transversal linearization), using the solution of a modified Riccati differential equation a periodic family [1] of subspaces — sliding surfaces, is derived.

A first-order sliding mode controller is then designed to ensure finite-time convergence to the selected family of surfaces in the presence of matched uncertainties and external disturbances. This choice of sliding surfaces guarantees the asymptotic orbital stability of the desired periodic trajectory.

The effectiveness of the proposed approach is demonstrated through simulation on the orbital stabilization of a periodic trajectory of a spherical robot [2], implemented using the servoconstraint approach [3,4].

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Cusps of Caustics by Reflection, Results and Conjectures

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The “Last Geometric Statement of Jacobi” asserts that the conjugate locus of a non-umbilic point on a triaxial ellipsoid has exactly four cusps. Proved only in this century, this result conjecturally holds for the loci of the 2nd, 3rd, etc., conjugate points as well.

I shall discuss a billiard version of this problem: the surface is replaced by a plane oval, and the conjugate loci is replaced by the 1st, 2nd,... caustics by reflection with the radiant point located inside the oval. For every oval, the caustics by reflection have at least four cusps, and if the oval is an ellipse then, conjecturally, this number is exactly four. I shall present a partial result in this direction.

This problem has many extensions, for example, to Finsler billiards associated with a projective Finsler metric (the subject of Hilbert’s 4th Problem) and to magnetic billiards. I shall explain why each caustic by reflection in the former case has at least four cusps. The four cusp result in the case of magnetic billiards follows from still another 4-point theorem, a conjectural strengthening of the classical 4-vertex theorem of Mukhopadhyaya.

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Stabilization of the Equilibrium Position of Mechanical Systems using a Drive that Monitors a Reference Speed

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The report is devoted to the task of stabilizing the equilibrium position of underactuated mechanical systems. In numerous publications devoted to the management of such systems, the force or moment of force applied by the actuator is used as a control action. In practice, the control action is usually the linear or angular velocity of the axis of rotation of the servo. So on industrial manipulators, the control signal is the linear or angular speeds of the links.

The paper proposes a method for stabilizing control of mechanical systems, where servos provide tracking of a given speed, rather than applied forces or moments of force.

The theoretical results are illustrated by an experiment with a ball-and-disk system. The task is to stabilize the ball on the disk in an unstable equilibrium position. The disk is attached to the last link of the ABB IRB 1600 manipulator. The control is the speed of rotation of the manipulator link.

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Integrable Perturbations of Polynomial Hamiltonians

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Let $H = H(x, y)$ be a polynomial of degree N. Here $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are canonical coordinates and momenta. We assume that the corresponding Hamiltonian system has an elliptic equilibrium at the origin. We prove the following theorem.

Suppose that frequency vector of small oscillations is nonresonant. Then there exists a real-analytic function $F : \mathbb{R}^{2n} \to \mathbb{R}$, $F = O(|x|^{N+1} + |y|^{N+1})$ such that the system with Hamiltonian $H + F$ is completely integrable on $\mathbb{R}^{2n}$.

On the Motion and Deformation of Localized Wave Beams Generated by the Bessel Functions

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The three-dimensional Helmholtz equation

$$\Delta u + k^2 u = 0, \quad \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial z^2}$$

has an exact solution of the form

$$u = e^{i\beta z} J_0 \left(\sqrt{x_1^2 + x_2^2} \right), \quad \beta^2 = k^2 - 1,$$

where J_0 is a Bessel function. But such a beam carries infinite power. To obtain the asymptotics of the localized solution generated by a Bessel function, a two-dimensional Lagrangian manifold in a four-dimensional phase space

$$\Lambda_0 = \{p = n(\psi),\, x = \tau n(\psi) | \tau \in R,\, \psi \in S^1\}, \quad n = (\cos\psi, \sin\psi)^T$$

is constructed. The Maslov canonical operator on it defines a section of a solution under consideration by the plane $z = 0$. Then, the dynamics of this manifold along the trajectories of the Hamiltonian system with the Hamiltonian determined by the Helmholtz equation is studied. It turns out that for each fixed z the resulting manifold contains the same type of singularities as the initial one – a singularity in the form of a degenerate fold. This allows one to obtain a global asymptotics of the considered beam.

We also consider a three-dimensional Lagrangian manifold in a six-dimensional phase space

$$\Lambda = \{p = \omega(\theta, \psi),\, x = \tau\omega(\theta, \psi) | \tau \in R,\, (\theta, \psi) \in S^2\},$$
$$\omega = (\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta)^T,$$

which has a similar parametrization to the one studied earlier. The canonical operator on such a manifold is defined by the Bessel function $J_{1/2}$ and is localized in a neighborhood of zero. However, when this manifold is shifted along the trajectories of a Hamiltonian vector field with a Hamiltonian corresponding to the wave equation, the type of singularities on the manifold changes. In particular, the solution spreads out. We will describe the types of singularities that arise in this case, and also construct the asymptotics of the solution in terms of special functions corresponding to each of this types of singularities.

The talk is based on the joint works with S. Yu. Dobrokhotov and V. E. Nazaikinskii.

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Shilnikov Criteria in the Extended Shimizu – Morioka System

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The Shimizu – Morioka model

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x - \lambda y - xz, \\ \dot{z} = -\alpha z + x^2 \end{cases} \tag{1}$$

is a classical example of a system possessing the Lorenz attractor. The study of this system is important since it appears as a normal form for certain classes of codimension-3 bifurcations of equilibrium states and periodic orbits. Like the Lorenz system, system (1) has $\mathbb{Z}2$-symmetry $(-x, -y, z) \mapsto (x, y, z)$. In contrast to the Lorenz system, system (1) has a very complicated boundary of the Lorenz attractor existence region. The boundary contains two important codimension-2 points [1] where conditions of Shilnikov criteria [2] are met. The pair of homoclinic loops of a neutral saddle is observed at the first Shilnikov point, while vanishing of the so-called separatrix value A along the pair of homoclinic loops occurs at the second point. It is important to note that the Lorenz attractor existence region lies on one side from the neutral saddle curve in system (1).

In this work we study the extension of the Shimizu – Morioka system [3]

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x - \lambda y - xz - Bx^3, \\ \dot{z} = -\alpha z + x^2. \end{cases} \tag{2}$$

System (2) also possesses the same symmetry and can be viewed as a normal form for certain classes of codimension-3 bifurcations of equilibrium states and periodic orbits. We show that for $B = 0.2$ the neutral saddle curve intersects the Lorenz attractor existance region and, as result, two additional homoclinic curves, give rise to 2 more bifurcation points satisfying the conditions of the first Shilnikov criterion. At the first point, the separatrix value lies in the range $0 < A < 1$, so only the region with the Lorenz attractor emerges. At the second point, where $1 < A < 2$, two regions arise simultaneously: one contains the Lorenz attractor, the other contains the Rovella attractor [4].

The Rovella attractor is not pseudohyperbolic, since the corresponding saddle is contracting. We also show that as the parameter B increases, the points satisfying the second Shilnikov criterion, which give rise to the so-called Shilnikov flames, are gradually replaced by points corresponding to the first criterion.

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