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Singular Algebrogeometric Operators and Spectral Theory

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References: Novikov's Homepage
www.mi.ras.ru/~snovikov
click Publications, items 175,176,182, 184. New Results were published recently in the Journal of Brazilian Math Society, December 2013, special volume dedicated to the 60th Anniversary of IMPA and in the Journal Russian Math Surveys, 2014, n 5 (September-October)

Our motivation came from the following Problem: We know what is a right analog of Fourier Transform on Riemann surfaces. Is it isometric in some inner product? In the late 1980s Krichever and myself invented analogs of Fourier series to construct the interacting string theory based on the analogs of "creation and annihilation operators". This problem is methodological because quantization of strings was already done in the famous work of Polyakov (1981) using the Feinman path integral method. Broad use of the Singular Soliton Constructions appeared in our works. Grinevich and I invented continuous analog of that (the R-Fourier Transform). It has strong multiplicative properties.

Our goals here are following:

1. To extend the idea of Isospectral Deformation to Singular Solutions of KdV Equation and other Completely Integrable (1+1) and (2+1) PDE Systems.

2. To construct Spectral Theory on the real line for the corresponding real singular 1D Schrodinger Operators $L = -\partial_x^2 + u(x,t)$ and other singular formally self-adjoint OD operators producing Completely Integrable Systems.

3. To extend this results to such

(2+1) Completely Integrable Systems as KP and Nonstationary $(1+1)$ Schrodinger Operator. Singular $2D$ Stationary Schrodinger Operator is under investigation.

There are "weakly singular" solutions to KdV. For example, Tao proved about 10 years ago that KdV dynamics is well-posed in the Sobolev spaces H^{-s}, $s \leq 3/10$ at the real x-line (circle). No ideas of Inverse Scattering Transform were used, no claims about isospectral properties has been made. Kappeler and Topalov used finite-gap approximation and obtained better result– for $s \leq 1$,– including isospectral property. This is probably final limit for the ordinary spectral theory. However, a lot of exact singular ”multisoliton” and more general periodic and quasiperiodic singular ”algebrogeometric” solutions are known many years. They have stronger singularities. Simplest example: Stationary and elliptic solutions $u = 2/x^2, u = \wp(x - at + b)$ They have following property: All eigenfunctions $L\psi = \lambda\psi$ are x-meromorphi We call such operators "spectral meromorphic" (s-meromorphic).

Theory of Solitons (1970s): For all algebrogeometric solutions corresponding (AG) operators belong to this class.

Example: For $L = -\partial^2 + u$ singularities are: $u = \frac{n_k(n_k+1)}{(x-x_k)^2} + \sum_j b_{jk}(x - x_k)^{2j} + O((x - x_k)^{2n_k})$ for $j \geq 0$.

We consider Hermitian-symmetric OD operators with discrete set of such singularities–finite at every period for periodic case and finite at the whole line for the rapidly decreasing case. The spectral theory should be developed in the class of ψ-functions which are C^∞ plus isolated singularities, $x \in R$:

$\psi(x) = \Sigma_{j \le n_k} q_j (x - x_k)^{-n_k+2j} + o((x - x_k)^{n_k})$ for $j \ge 0$, nearby of every real singularity of potential u for given moment t. We call it $F_{x_1,\ldots,x_M;n_1,\ldots,n_M} = F_{X;N}$.

Similar spaces F we define for higher order s-meromorphic operators. The inner product in the space F is

$< \psi, \phi >= \int \psi(x)\bar{\phi}(\bar{x})dx$

Our theorem: It is well-defined if all singularities are the same as appear in the class of algebro-geometric operators L (i.e. admitting rank 1 commuting operator.) The inner product is indefinite. The number of negative squares m_F is equal to the sum of dimensions of the negative subspaces in the local variables $x - x_j$ in the space F for every singular point x_j at the period. For $L = -\partial_x^2 + u$ we have $F = F_{X;N}$ and $m_F = \Sigma_j[(n_j + 1)/2]$. We consider classes of functions rapidly decreasing at infinity ($T = \infty$) and quasiperiodic with condition

$\psi(x+T) = \varkappa\psi(x), \psi \in F_{X,N}(\varkappa)$

for $|\varkappa| = 1$. The number of negative squares of inner product in

the space $F_{X,N}(\varkappa)$ is the Integral of KdV dynamics, so the time deformation is isospectral.
This inner product is well-defined also for 2D case for the Non-stationary Schrodinger Operator $L = i\partial_y + \partial_x^2 + u(x,y)$. The case $L = -\Delta + u$ is under investigation .

Spectral Theory of Rapidly Decreasing and Periodic Schrodinger Operators L requires NONSINGULARITY of Potential $u(x)$ as well as physical derivation of KdV in the Theory of Solitons.

A number of other applications

of KdV theory was discovered later.

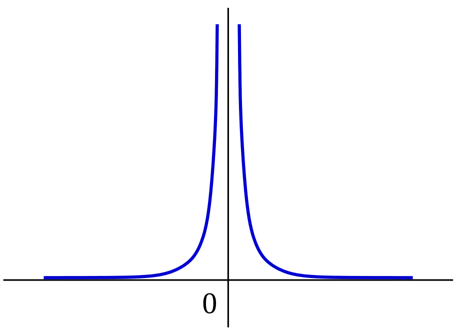
Huge Literature is dedicated to the singular KdV Solutions. Rational and Elliptic Solutions are especially popular.

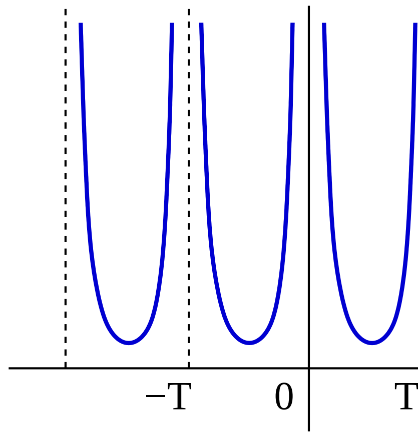
Example: Consider Real Rational and Elliptic Solutions

$u(x,t) = \sum_j 2/(x - x_j(t))^2$

$u(x,t) = \sum_j 2\wp(x - x_j(t))$

Let $u(x,0) = n(n+1)/x^2$





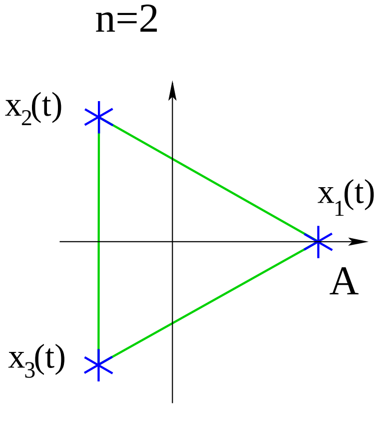
or $u(x,0) = n(n+1)\wp(x)$;
(the famous Lame' Potentials.)

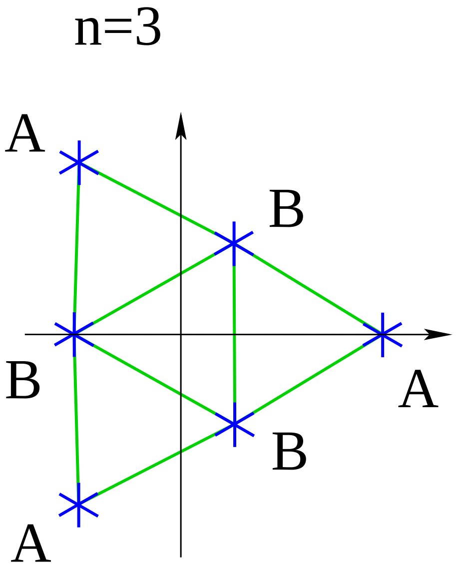
Hermit found Spectrum with Dirichlet boundary conditions for $x = 0, T$. Here T is a real period. No spectral theory was constructed on the real line.

The evolution of Lame' Potentials $u(x,0) = n(n+1)\wp(x)$ or $u(x,0) = n(n+1)/x^2$ is singular

Important Question:

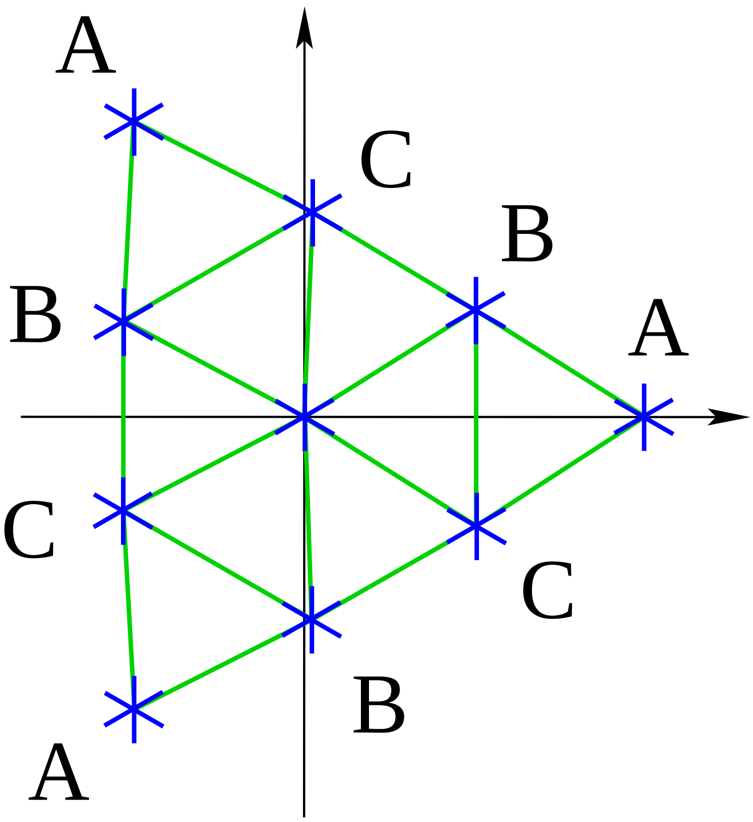
How many real poles these solutions have for $t > 0$?





The orbits of group $Z/3Z$ are marked here. We have $1, 1, 2, 2, 3, \ldots$ real poles for $n = 1, 2, 3, 4, 5 \ldots$.

n=4 $\qquad \frac{n(n+1)}{2} = 10$



$$x_j \sim r_j t^{1/3}$$

The symmetry group $Z/3Z$ acts here

$r_j \to \zeta r_j, \zeta^3 = 1$

Our Result: The number of real poles is equal to $[(n+1)/2]$. This number is equal to the number of negative squares for the Inner Product in the Spaces of functions on the real line where the operator $L = -\partial_x^2 + u(x,t)$ is symmetric. Arkad'ev, Polivanov and Pogrebkov constructed some kind of Scattering Theory for the potentials with singularities like $2/(x - x_k)^2$. No spectral theory was discussed.

Consider real (may be singular) "finite-gap" periodic operator with spectral curve (Riemann Surface) Γ:

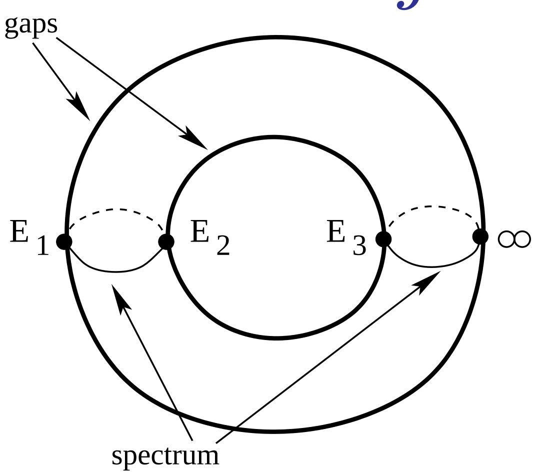
$\mu^2 = (E-E_1)\cdots(E-E_{2g+1})$ with involution $\sigma(E,\mu) = (E,-\mu)$ and poles of Ψ-function $D = \gamma_1, ..., \gamma_g$.

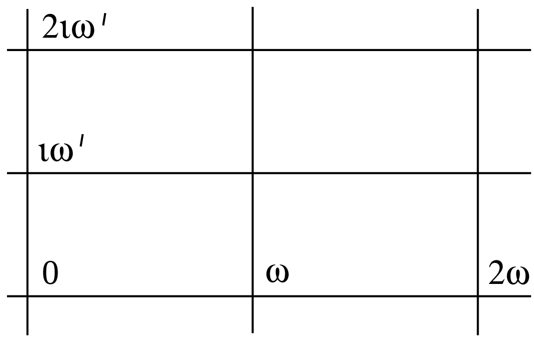
Real case corresponds to the data where Γ and poles are invariant under conjugation $\tau(E,\mu) = (\bar{E},\bar{\mu})$.

The spectrum of operator (see below) is equal to the projection on the complex λ-plane of the τ-invariant Canonical Contour κ_0.

Example 1: Let $g = 1$ (Γ is a torus)

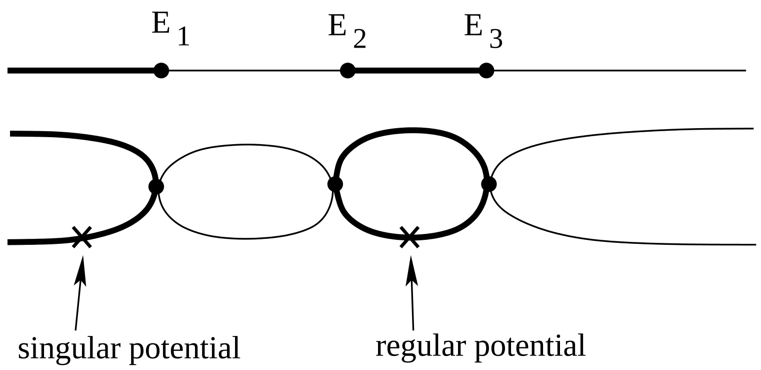
and all E_j are real, $j = 1, 2, 3$:





The lattice of periods of the Weierstrass $\wp$-function in this case is **rectangular with periods** $2\omega, 2i\omega'$.

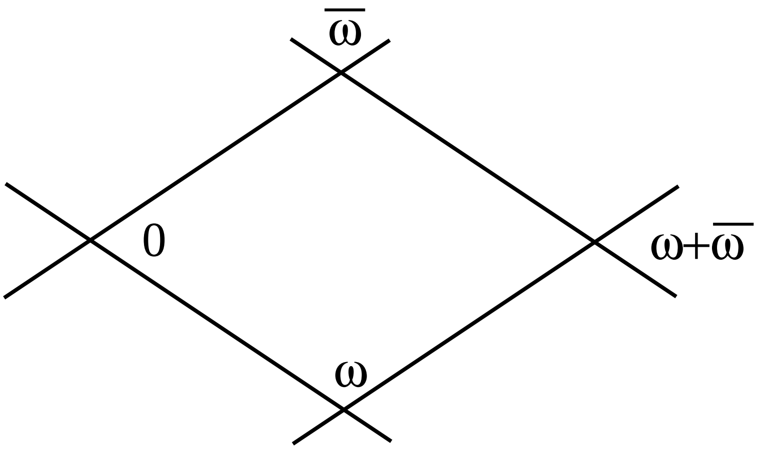
The spectrum is real , and spectral gaps are $[-\infty, E_1]$ and $[E_2, E_3]$, $\tau = id$ at κ_0



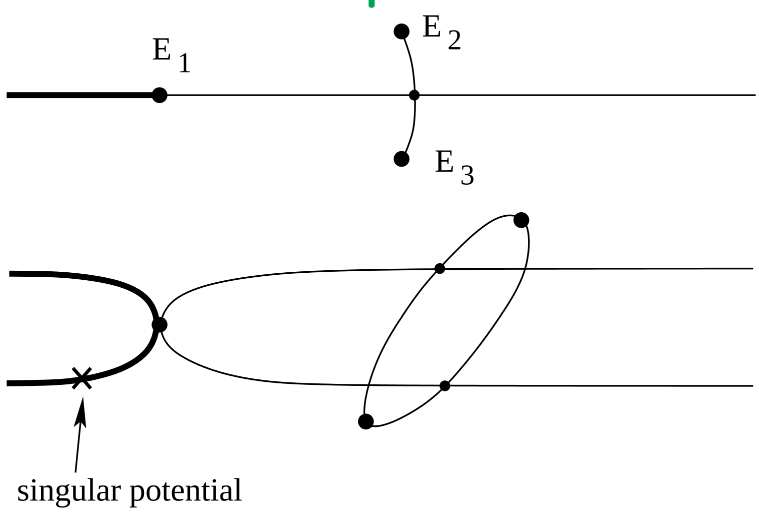
κ_0 is represented by fine lines.

The contour κ_0 has 2 components here: infinite and finite. There is only one pole γ: For Regular Case it belongs to the finite gap, for the Singular Case it belongs to the infinite gap (They both are the shifted Hermit-Lame Operators but in regular case the shift is imaginary, in singular case the shift is real).

Example 2. Let $g = 1, E_1 \in R, E_3 = \overline{E_2}$:



The lattice of periods is



rombic. κ_0 given by fine lines.

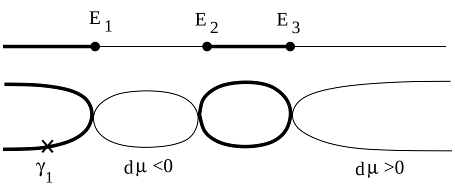
The spectrum on the whole line coincides with the projection of the contour κ_0 on the $E-plane$. It contains complex arc joining $E_2, \bar{E}_2$ and $\tau \neq id$ at κ_0

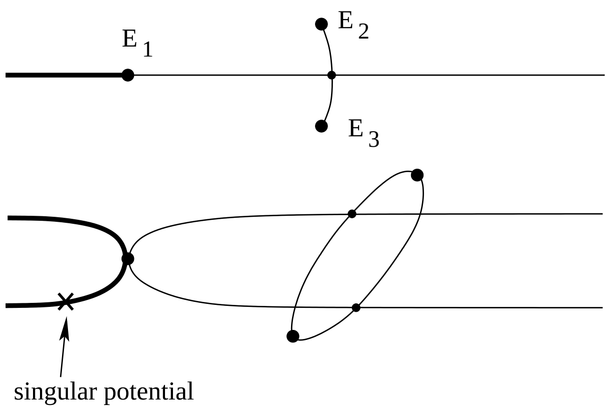
Example 1. All branching points are real: τ acts identically on κ_0, the "Spectral Measure" $d\mu$ is negative in some finite components of spectrum. For the case of "R-Fourier Transform" (All poles are concentrated at infinity) we have: $d\mu^F/dp > 0$ exactly in every second component starting from the infinite one; So we have $[(g+1)/2]$ "negative" finite components.

Example 2. Some pair of branching points is complex adjoint: τ

is not identity in the nonreal components of κ_0; So the inner product is nonlocal and therefore indefinite.

We proved Completeness Theorem in the spaces $F_{X,N}(\varkappa)$ which are similar to the Pontryagin-Sobolev spaces





Remark: Singular Bloch-Floquet eigenfunctions are known for the $k+1$-particle Moser-Calogero operator with Weierstrass pairwise potential if coupling constant is equal to $n(n+1), n \in Z$. They form a k-dimensional complex algebraic variety. No one function is known for $k > 1$ serving the discrete spectrum in the space $\mathcal{L}_2$ of the bounded domain inside of poles. Our case is $k = 1$. We believe

that for all $k > 1$ this family of eigenfunctions also serves spectral problem in some indefinite inner product in the proper space of functions defined in the whole space R^k.