On spherical functions connected with a general PDE of the second order in the unit ball

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The report is devoted to a connection between the Dirichlet problem in the unit ball for a general PDE of the second order and spherical functions which are zero on null-variety of the PDE-symbol.

Let $L = L(x,D) = \sum_{|\alpha|\leqslant 2} a_\alpha D^\alpha$ be a general linear differential operation with constant coefficients, which can be complex-valued or matrix, and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$.

Let us consider the Dirichlet problem

$$Lu = f, \qquad u|_{\partial\Omega} = 0 \tag{1}$$

in the Sobolev space $W_2^2(\Omega)$. We extend functions f and u by zero: $\widetilde{u} = u$ in Ω, $\widetilde{u} = 0$ outside of Ω. Then

$$L\widetilde{u} = \widetilde{f} + L_1 u \delta_{\partial\Omega}, \tag{2}$$

where $L_1 u$ is a linear differential expression on ψ and $u'_\nu \langle \delta_{\partial\Omega}, \varphi\rangle = \int_{\partial\Omega} \overline{\varphi}\, ds$. Let the domain Ω be defined by means of the inequality $P(x) > 0$ where $P \in \mathbb{R}[x]$ is a polinomial, $|\nabla P|_{P=0} \neq 0$. We multiply equality (2) by $P(x)$ and apply the Fourier transform. We obtain

$$P(-D_\xi)[L(\xi)F(\widetilde{u})(\xi)] = g(\xi) \tag{3}$$

with a known function g. Here $L(\xi)$ is the symbol and $L_2(\xi)$ is the major symbol.

Statement 1. *The solvability of the last equation in some classes of entire functions is equivalent to the solvability of problem* (1).

If the domain is the unit ball, then $P(-D_\xi) = \Delta_\xi$ and if, moreover, the right-hand side $f = 0$, then $g = 0$ and for the uniqueness problem in problem (1) we obtain the equivalent problem of the following form: $(\Delta_\xi + 1)[L_2(\xi)v(\xi)] = 0$. Now for lowest term $v_m(\xi)$ of the power series for v we have the equation $\Delta_\xi[L_2(\xi)v_m(\xi)] = 0$.

The application of this methods gives, in particular, the following results. Let us consider

$$Lu = u_{x_1x_1} + \cdots + u_{x_kx_k} - a^2(u_{x_{k+1}x_{k+1}} + \cdots + u_{x_nx_n}).$$

Statement 2. *Problem* (1) *with* $f = 0$ *has a nontrivial solution in* $W_2^2(\Omega)$ *if and only if there exist natural numbers* m, i, j, $i + j \leqslant m$ *such that*

1) $m - i - j$ *even and*

$$P^{(\frac{n-k}{2}+j-1,\, i+\frac{k}{2}-1)}_{\frac{m-i-j}{2}+1}\left(\frac{a^2-1}{a^2+1}\right) = 0$$

or

2) $m+n-k-i+j$ *even and*

$$P^{(1-j-\frac{n-k}{2},i+\frac{k}{2}-1)}_{\frac{m+n-k-i+j}{2}}\left(\frac{a^2-1}{a^2+1}\right)=0$$

or

3) $m+n+i+j$ *even and*

$$P^{(1-j-\frac{n-k}{2},1-i-\frac{k}{2})}_{\frac{m+n+i+j}{2}-1}\left(\frac{a^2-1}{a^2+1}\right)=0$$

or

4) $m+k+i-j$ *even and*

$$P^{(\frac{n-k}{2}+j-1,1-i-\frac{k}{2})}_{\frac{m+k+i-j}{2}}\left(\frac{a^2-1}{a^2+1}\right)=0,$$

where $P^{(\alpha,\beta)}_N(x)$ *is the Jacoby polynomial.*

For the case $n=2$ the result conforms with the well-known result for the string equation. There is also an application of these results to problems of the interal geometry.