Sharp Pitt inequality and logarithmic uncertainty principle for Dunkl transform in L^2

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Let $\Gamma(t)$ be the gamma function, $\mathbb{R}^d$ be the real space of d dimensions, equipped with a scalar product (x,y) and a norm $|x| = \sqrt{(x,x)}$. Denote by $S(\mathbb{R}^d)$ the Schwartz space on $\mathbb{R}^d$ and by $L^2(\mathbb{R}^d)$ the Hilbert space of complex-valued functions endowed with a norm $\|f\|_2 = \bigl(\int_{\mathbb{R}^d} |f(x)|^2\,dx\bigr)^{1/2}$. The Fourier transform is defined by

$$\widehat{f}(y) = (2\pi)^{-n/2}\int_{\mathbb{R}^d} f(x)e^{-i(x,y)}\,dx.$$

W. Beckner [1] proved the Pitt inequality for the Fourier transform

$$\bigl\||y|^{-\beta}\widehat{f}(y)\bigr\|_2 \leqslant C(\beta)\bigl\||x|^{\beta}f(x)\bigr\|_2, \qquad f\in S(\mathbb{R}^d), \quad 0<\beta<\frac{d}{2}, \tag{1}$$

with sharp constant

$$C(\beta) = 2^{-\beta}\,\frac{\Gamma(\frac{1}{2}(\frac{d}{2}-\beta))}{\Gamma(\frac{1}{2}(\frac{d}{2}+\beta))}.$$

Noting that $\||y|^{-\beta}\widehat{f}(y)\|_2 = (2\pi)^{-\beta}\||(-\Delta)^{\beta/2}f|\|_2$, Pitt's inequality can be viewed as a Hardy–Rellich inequality; see the papers by D. Yafaev [2] and S. Eilertsen [3] for alternative proofs and extensions of (1).

For $\beta = 0$, (1) is the Plancherel theorem. If $\beta > 0$ there is no extremiser in inequality (1) and its sharpness can be obtained on the set of radial functions.

The proof of (1) in [1] is based on an equivalent integral realization as a Stein-Weiss fractional integral on $\mathbb{R}^d$. D. Yafaev in [2] used the following decomposition

$$L^2(\mathbb{R}^d) = \sum_{n=0}^{\infty} \oplus \mathfrak{R}_n^d, \tag{2}$$

where $\mathfrak{R}_0^d$ is the space of radial function, and $\mathfrak{R}_n^d = \mathfrak{R}_0^d \otimes \mathfrak{H}_n^d$ is the space of functions in $\mathbb{R}^d$ that are products of radial functions and spherical harmonics of degree n. Thanks to this decomposition it is enough to study inequality (1) on the subsets of $\mathfrak{R}_n^d$ which are invariant under the Fourier transform.

Following [2] and using similar decomposition of the space $L^2(\mathbb{R}^d)$ with the Dunkl weight, we prove sharp Pitt's inequality for the Dunkl transform.

Let $R\subset\mathbb{R}^d$ be a root system, R_+ be the positive subsystem of R, and $k\colon R\to\mathbb{R}_+$ be a multiplicity function with the property that k is G-invariant. Here $G(R)\subset O(d)$ is a finite reflection group generated by reflections $\{\sigma_a : a\in R\}$, where σ_a is a reflection with respect to a hyperplane $(a,x)=0$.

Let

$$v_k(x) = \prod_{a\in R_+} |(a,x)|^{2k(a)}$$

be the Dunkl weight, $d\mu_k(x) = c_k v_k(x)dx$, where

$$c_k^{-1} = \int_{\mathbb{R}^d} e^{-|x|^2/2} v_k(x)\,dx$$

is the Macdonald–Mehta–Selberg integral. Let $L^2(\mathbb{R}^d, d\mu_k)$ be the Hilbert space of complex-valued functions endowed with a norm

$$\|f\|_{2,d\mu_k} = \left(\int_{\mathbb{R}^d} |f(x)|^2\,d\mu_k(x)\right)^{1/2}.$$

Introduced by C. F. Dunkl, a family of differential–difference operators (Dunkl's operators) associated with G and k are given by

$$D_j f(x) = \frac{\partial f(x)}{\partial x_j} + \sum_{a\in R_+} k(a)(a, e_j)\,\frac{f(x) - f(\sigma_a x)}{(a, x)}, \qquad j = 1, \dots, d.$$

The Dunkl kernel $e_k(x, y) = E_k(x, iy)$ is the unique solution of the joint eigenvalue problem for the corresponding Dunkl operators:

$$D_j f(x) = iy_j f(x), \quad j = 1, \dots, d, \qquad f(0) = 1.$$

Let us define the Dunkl transforms as follows

$$\mathcal{F}_k(f)(y) = \int_{\mathbb{R}^d} f(x)\overline{e_k(x, y)}\,d\mu_k(x), \qquad \mathcal{F}_k^{-1}(f)(x) = \mathcal{F}_k(f)(-x),$$

where $\mathcal{F}_k(f)$ and $\mathcal{F}_k^{-1}(f)$ are the direct and inverse transforms correspondingly (see, e.g., [4]). For $k \equiv 0$ we have $\mathcal{F}_0(f) = \widehat{f}$.

Our goal is to study Pitt's inequality for the Dunkl transform

$$\||y|^{-\beta}\mathcal{F}_k(f)(y)\|_{2,d\mu_k} \leqslant C(\beta, k)\||x|^{\beta} f(x)\|_{2,d\mu_k}, \quad f \in S(\mathbb{R}^d), \tag{3}$$

with the sharp constant $C(\beta, k)$.

Let us first recall some known results on Pitt's inequality for the Hankel transform. Let $\lambda \geqslant -1/2$. Denote by $J_\lambda(t)$ the Bessel function of degree λ and by $j_\lambda(t) = 2^\lambda\Gamma(\lambda+1)t^{-\lambda}J_\lambda(t)$ the normalized Bessel function. Setting

$$b_\lambda = \left(\int_0^\infty e^{-t^2/2} t^{2\lambda+1}\,dt\right)^{-1} = \frac{1}{2^\lambda\Gamma(\lambda+1)}$$

and $d\nu_\lambda(r) = b_\lambda r^{2\lambda+1}\,dr$, we define $\|f\|_{2,d\nu_\lambda} = \left(\int_{\mathbb{R}_+} |f(r)|^2\,d\nu_\lambda(r)\right)^{1/2}$.

The Hankel transform is defined by

$$\mathcal{H}_\lambda(f)(\rho) = \int_{\mathbb{R}_+} f(r) j_\lambda(\rho r)\,d\nu_\lambda(r).$$

Note that $\mathcal{H}_\lambda^{-1} = \mathcal{H}_\lambda$.

Pitt's inequality for the Hankel transform is written as

$$\|\rho^{-\beta}\mathcal{H}_\lambda(f)(\rho)\|_{2,d\nu_\lambda} \leqslant c(\beta, \lambda)\|r^{\beta} f(r)\|_{2,d\nu_\lambda}, \qquad f \in S(\mathbb{R}_+), \tag{4}$$

where $c(\beta,\lambda)$ is the sharp constant in (4) and $S(\mathbb{R}_+)$ is the the Schwartz space on $\mathbb{R}_+$. Note that if $f\in\mathfrak{R}_0^d$, a study of the Hankel transform is of special interest since the Fourier transform of a radial function can be written as the Hankel transform.

L. De Carli [5] proved that $c(\beta,\lambda)$ is finite only if $0\leqslant\beta<\lambda+1$. For $\lambda=d/2-1$, $d\in\mathbb{N}$, the constant $c(\beta,\lambda)$ was calculated by D. Yafaev [2], and in the general case by S. Omri [6]. The proof of Pitt's inequality in [6] is rather technical and uses the Stein–Weiss type estimate for the so-called B-Riesz potential operator. Following [2], we give a direct and simple proof of inequality (4).

Let $|k|=\sum_{a\in R_+}k(a)$ and $\lambda_k=d/2-1+|k|$. For a radial function $f(r)$, $r=|x|$, Pitt's inequality for the Dunkl transform (3) corresponds to Pitt's inequality for the Hankel transform (4) with $\lambda=\lambda_k$. Therefore the condition

$$0\leqslant\beta<\lambda_k+1 \tag{5}$$

is necessary for $C(\beta,k)<\infty$. Our goal is to show that in fact $C(\beta,k)=c(\beta,\lambda_k)$ if condition (5) holds.

Note that for the one-dimensional Dunkl weight

$$v_\lambda(t)=|t|^{2\lambda+1},\qquad d\mu_\lambda(t)=\frac{v_\lambda(t)\,dt}{2^{\lambda+1}\Gamma(\lambda+1)},\qquad \lambda\geqslant-\frac{1}{2},$$

and the corresponding Dunkl transform

$$\mathcal{F}_\lambda(f)(s)=\int_{\mathbb{R}}f(t)\overline{e_\lambda(st)}\,|t|^{2\lambda+1}\,d\mu_\lambda(t),\qquad e_\lambda(t)=j_\lambda(t)-ij'_\lambda(t),$$

F. Soltani [7] proved Pitt's inequality that can be equivalently written as

$$\||s|^{-\beta}\mathcal{F}_\lambda(f)(s)\|_{2,d\mu_\lambda}\leqslant\max\{c(\beta,\lambda),c(\beta,\lambda+1)\}\,\||t|^{\beta}f(t)\|_{2,d\mu_\lambda} \tag{6}$$

for $f\in S(\mathbb{R})$ and $0\leqslant\beta<\lambda+1$. Since $c(\beta,\lambda)\geqslant c(\beta,\lambda+1)$ (see [2]), then in fact (6) holds with the constant $c(\beta,\lambda)$ and therefore, we have in this case $C(\beta,k)=c(\beta,\lambda_k)$.

Finally, we remark that Pitt's inequality in L^2 for the multi-dimensional Dunkl transform has been recently established in [8] in the case of $\lambda_k-1/2<\beta<\lambda_k+1$. The obtained constant is not sharp.

Let $\mathbb{S}^{d-1}$ be the unit sphere in $\mathbb{R}^d$, $x'\in\mathbb{S}^{d-1}$, and dx' be the Lebesgue measure on the sphere. Set $a_k^{-1}=\int_{\mathbb{S}^{d-1}}v_k(x')\,dx'$, $d\omega_k(x')=a_kv_k(x')\,dx'$, and $\|f\|_{2,d\omega_k}=\big(\int_{\mathbb{S}^{d-1}}|f(x')|^2\,d\omega_k(x')\big)^{1/2}$.

Let us denote by $\mathfrak{H}_n^d(v_k)$ the subspace of k-spherical harmonics of degree $n\in\mathbb{Z}_+$ in $L^2(\mathbb{S}^{d-1},d\omega_k)$. Let $\Delta_kf(x)=\sum_{j=1}^dD_j^2f(x)$ be the Dunkl Laplacian and $\mathfrak{P}_n^d$ be the space of homogeneous polynomials of degree n in $\mathbb{R}^d$. Then $\mathfrak{H}_n^d(v_k)$ is the restriction of $\ker\Delta_k\cap\mathfrak{P}_n^d$ to the sphere $\mathbb{S}^{d-1}$.

If l_n is the dimension of $\mathfrak{H}_n^d(v_k)$, we denote by $\{Y_n^j\colon j=1,\dots,l_n\}$ the real-valued orthonormal basis $\mathfrak{H}_n^d(v_k)$ in $L^2(\mathbb{S}^{d-1},d\omega_k)$. A union of these bases forms orthonormal basis in $L^2(\mathbb{S}^{d-1},d\omega_k)$ consisting of k-spherical harmonics, i.e., we have

$$L^2(\mathbb{S}^{d-1},d\omega_k)=\sum_{n=0}^{\infty}\oplus\mathfrak{H}_n^d(v_k). \tag{7}$$

Using (7) and the following Funk-Hecke formula for k-spherical harmonic $Y\in\mathfrak{H}_n^d(v_k)$

$$\int_{\mathbb{S}^{d-1}}Y(y')\overline{e_k(x,y')}\,d\omega_k(y')=\frac{(-i)^n\Gamma(\lambda_k+1)}{2^n\Gamma(n+\lambda_k+1)}\,Y(x')r^nj_{n+\lambda_k}(r),\qquad x=rx',$$

similarly to (2) we have the direct sum decomposition of $L^2(\mathbb{R}^d, d\mu_k)$:

$$L^2(\mathbb{R}^d, d\mu_k) = \sum_{n=0}^{\infty} \oplus \mathfrak{R}_n^d(v_k), \qquad \mathfrak{R}_n^d(v_k) = \mathfrak{R}_0^d \otimes \mathfrak{H}_n^d(v_k),$$

and that the space $\mathfrak{R}_n^d(v_k)$ is invariant under the Dunkl transform.

The next result provides a sharp constant in the Pitt inequality for the Dunkl transform (3).

Theorem 1. *Let $\lambda_k = d/2 - 1 + |k|$ and $0 \leqslant \beta < \lambda_k + 1$, then for $f \in S(\mathbb{R}^d)$ we have*

$$C(\beta, k) = 2^{-\beta} \frac{\Gamma(\frac{1}{2}(\lambda_k + 1 - \beta))}{\Gamma(\frac{1}{2}(\lambda_k + 1 + \beta))}.$$

Sharpness of this inequality can be seen by considering radial functions.

W. Beckner in [1] proved the logarithmic uncertainty principle for the Fourier transform using Pitt's inequality (1): if $f \in S(\mathbb{R}^d)$, then

$$\int_{\mathbb{R}^d} \ln(|x|)|f(x)|^2\,dx + \int_{\mathbb{R}^d} \ln(|y|)|\widehat{f}(y)|^2\,dy \geqslant \left(\psi\left(\frac{d}{4}\right) + \ln 2\right) \int_{\mathbb{R}^d} |f(x)|^2\,dx,$$

where $\psi(t) = d \ln \Gamma(t)/dt$ is the ψ-function.

For the Hankel transform the logarithmic uncertainty principle reads as follows (see [6]): if $f \in S(\mathbb{R}_+)$ and $\lambda \geqslant -1/2$, then

$$\int_{\mathbb{R}_+} \ln(t)|f(t)|^2 t^{2\lambda+1}\,dt + \int_{\mathbb{R}_+} \ln(s)|\mathcal{H}_\lambda(f)(s)|^2 s^{2\lambda+1}\,ds \geqslant \left(\psi\left(\frac{\lambda+1}{2}\right) + \ln 2\right) \int_{\mathbb{R}_+} |f(t)|^2 t^{2\lambda+1}\,dt.$$

For the one-dimensional Dunkl transform of functions $f \in S(\mathbb{R})$, F. Soltani [7] has recently proved that

$$\int_{\mathbb{R}} \ln(|t|)|f(t)|^2|t|^{2\lambda+1}\,dt + \int_{\mathbb{R}} \ln(|s|)|\mathcal{F}_\lambda(f)(s)|^2|s|^{2\lambda+1}\,ds \geqslant \left(\psi\left(\frac{\lambda+1}{2}\right) + \ln 2\right) \int_{\mathbb{R}} |f(t)|^2|t|^{2\lambda+1}\,dt.$$

Using Pitt's inequality (3) we obtain the logarithmic uncertainty principle for the multidimensional Dunkl transform.

Theorem 2. *Let $\lambda_k = d/2 - 1 + |k|$ and $f \in S(\mathbb{R}^d)$. We have*

$$\int_{\mathbb{R}^d} \ln(|x|)|f(x)|^2\,d\mu_k(x) + \int_{\mathbb{R}^d} \ln(|y|)|\mathcal{F}_k(f)(y)|^2\,d\mu_k(y) \geqslant \left(\psi\left(\frac{\lambda_k+1}{2}\right) + \ln 2\right) \int_{\mathbb{R}^d} |f(x)|^2\,d\mu_k(x).$$

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