Moduli of the supporting convexity and the supporting smoothness

G. Ivanov

Moscow Institute of Physics and Technology (State University)
National Research University "Higher School of Economics"

In our talk we introduce the moduli of the supporting convexity and the supporting smoothness of a Banach space, which characterize the deviation of the unit sphere from an arbitrary supporting hyperplane.

Let X be a real Banach space. We use $\langle p, x\rangle$ to denote the value of a functional $p \in X^*$ at a vector $x \in X$.

Let

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \ \|x\| = \|y\| = 1, \ \|x-y\| \geqslant \varepsilon\right\}$$

and

$$\rho_X(\tau) = \sup\left\{\frac{\|x+y\|}{2} + \frac{\|x-y\|}{2} - 1 : \|x\| = 1, \ \|y\| = \tau\right\}.$$

The functions $\delta_X(\,\cdot\,)\colon [0,2] \to [0,1]$ and $\rho_X(\,\cdot\,)\colon \mathbb{R}^+ \to \mathbb{R}^+$ are referred to as the moduli of convexity and smoothness of X respectively.

We say that y is *quasiorthogonal* to the vector $x \in X \setminus \{o\}$ and write $y \urcorner x$ if there exists a functional p such that $\|p\| = 1$, $\langle p, x\rangle = \|x\|$, $\langle p, y\rangle = 0$.

Define the *modulus of local supporting convexity* as

$$\lambda^-_X(r) = \inf\{\lambda \in \mathbb{R} : \|x\| = \|y\| = 1, \ y \urcorner x, \ \|x + ry - \lambda x\| = 1\}.$$

Define the *modulus of local supporting smoothness* as

$$\lambda^+_X(r) = \sup\{\lambda \in \mathbb{R} : \|x\| = \|y\| = 1, \ y \urcorner x, \ \|x + ry - \lambda x\| = 1\}.$$

We show that the modulus of supporting smoothness and the modulus of smoothness are equivalent at zero, the modulus of supporting convexity is equivalent at zero to the modulus of convexity.

Theorem 1. *For any $r \in [0;1]$ we have that $\delta_X(r) \leqslant \lambda^-_X(r) \leqslant \delta_X(2r)$.*

Theorem 2. *For any $r \in [0, \frac{1}{2}]$ we have that $\rho_X\left(\frac{r}{2}\right) \leqslant \lambda^+_X(r) \leqslant \rho_X(2r)$.*

We prove a Day–Nordlander type result for these moduli. The Day–Nordlander theorem (see [7]) asserts that $\delta_X(\varepsilon) \leqslant \delta_H(\varepsilon)$ for $\varepsilon \in [0,2]$, where H denotes an arbitrary Hilbert space.

Theorem 3. *Let X be an arbitrary Banach space. Then*

$$\lambda^-_X(r) \leqslant \lambda^-_H(r) = 1 - \sqrt{1-r^2} = \lambda^+_H(r) \leqslant \lambda^+_X(r) \qquad \forall r \in [0,1].$$

If at least one of these inequalities turns into equality, then X is a Hilbert space.

In the paper [1] J. Banaś defined and studied some new modulus of smoothness. Namely, he defined

$$\delta_X^+(\varepsilon) = \sup\left\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1,\ \|x-y\| \leqslant \varepsilon\right\}, \qquad \varepsilon \in [0,2].$$

The function $\delta_X^+(\cdot)$ is called the *Banaś modulus.*

In the papers [1], [2], [3], [4] several interesting results concerning this modulus were obtained. Particulary, in [1], J. Banaś proved that a space is uniformly smooth iff $\frac{\delta_X^+(\varepsilon)}{\varepsilon} \to 0$ as $\varepsilon \to 0$. However, from the definition a space is uniformly smooth if and only if $\frac{\rho_X(\varepsilon)}{\varepsilon} \to 0$ as $\varepsilon \to 0$. This leads to the question: are the modulus of smoothness and the modulus of Banaś equivalent at zero? It is easy to check that there exist positive constant a, b such that $\delta_X^+(t) \leqslant a\rho_X(bt)$, but the lower estimate of the modulus of Banaś in terms of the modulus of smoothness is unknown. In the next theorem we prove that the modulus of Banaś and the modulus of supporting smoothness are equivalent at zero, so Theorem 2 answers the above question.

Theorem 4. *Let X be an arbitrary Banach space. Then the following inequalities hold*:

$$\delta_X^+(2r) \leqslant \lambda_X^+(r) \leqslant 2\delta_X^+(3r) \qquad \forall r \in \left[0, \frac{2}{3}\right].$$

References

[1] J. Banaś, "On moduli of smoothness of Banach spaces", *Bull. Pol. Acad. Sci., Math.*, **34** (1986), 287–293.

[2] J. Banaś, K. Fraczek, "Deformation of Banach spaces ", *Comment. Math. Univ. Carolinae*, **34** (1993), 47–53.

[3] J. Banaś, A. Hajnosz, S. Wedrychowicz, "On convexity and smoothness of Banach space", *Commentationes Mathematicae Universitatis Carolinae*, **31**:3. (1990), 445–452.

[4] J. Banaś, B. Rzepka, "Functions related to convexity and smoothness of normed spaces", *Rendiconti del Circolo Matematico di Palermo*, **46**:3 (1997), 395–424.

[5] M. Baronti, P. Papini, "Convexity, smoothness and moduli", *Nonlinear Analysis: Theory, Methods & Applications*, **70**:6 (2009), 2457–2465.

[6] J. Diestel, *Geometry of Banach Spaces. Selected Topics*, Lecture Notes in Math., **485**, Springer-Verlag, Berlin, 1975.

[7] G. Nordlander, "The modulus of convexity in normed linear spaces", *Arkiv för Matematik*, **4**:1 (1960), 15–17.