On a Volterra equation of the second kind with “incompressible” kernel

M. T. Jenaliyev[a], M. M. Amangaliyeva[a], M. T. Kosmakova[b], M. I. Ramazanov[c]

[a]*Institute of Mathematics and Mathematical Modeling*
[b]*Al-Farabi Kazakh National University*
[c]*E. A. Buketov Karaganda State University*

Solving the boundary value problems of the heat equation in noncylindrical domains degenerating at the initial moment leads to the necessity of research of the singular Volterra integral equations of the second kind, when the norm of the integral operator is equal to 1. The paper deals with the singular Volterra integral equation of the second kind, to which by virtue of 'the incompressibility' of the kernel the classical method of successive approximations is not applicable. It is shown that the corresponding homogeneous equation when $|\lambda| > 1$ has a continuous spectrum, and the multiplicity of the characteristic numbers increases depending on the growth of the modulus of the spectral parameter $|\lambda|$. By the Carleman-Vekua regularization method [1] the initial equation is reduced to the Abel equation. The eigenfunctions of the equation are found explicitly. Similar integral equations also arise in the study of spectral-loaded heat equations [2].

When solving model problems for parabolic equations in domains with moving boundary the singular integral equations of the following form arise:

$$\varphi(t) - \lambda \int\limits_0^t K(t,\tau)\varphi(\tau)\,d\tau = f(t), \qquad t > 0, \tag{1}$$

where

$$K(t,\tau) = \frac{1}{2a\sqrt{\pi}}\left\{\frac{t+\tau}{(t-\tau)^{3/2}}\exp\left(-\frac{(t+\tau)^2}{4a^2(t-\tau)}\right) + \frac{1}{(t-\tau)^{1/2}}\exp\left(-\frac{t-\tau}{4a^2}\right)\right\}.$$

The kernel $K(t,\tau)$ has the following properties:

1) $K(t,\tau) \geqslant 0$ and continuously at $0 < \tau < t < +\infty$;
2) $\lim_{t\to t_0}\int_{t_0}^t K(t,\tau)\,d\tau = 0$, $t_0 \geqslant \varepsilon > 0$;
3) $\lim_{t\to 0}\int_0^t K(t,\tau)\,d\tau = 1$, $\lim_{t\to +\infty}\int_0^t K(t,\tau)\,d\tau = 1$.

The kernel $K(t,\tau)$ is summable with weight function $t^{-1/2}$.

Problem. To find the solution $\varphi(t)$ of integral equation (1) satisfying the condition $\sqrt{t}\cdot\varphi(t) \in L_\infty(0,\infty)$ for any given function $\sqrt{t}\cdot f(t) \in L_\infty(0,\infty)$ and each given complex spectral parameter $\lambda \in \mathcal{C}$.

The following theorem holds.

Theorem. *The nonhomogeneous integral equation* (1) *is solvable in the class* $\sqrt{t}\cdot\varphi(t) \in L_\infty(0,\infty)$ *for any right-hand side* $\sqrt{t}\cdot f(t) \in L_\infty(0;\infty)$ *and for each* $|\lambda| > 1$. *The corresponding homogeneous equation has* $(N_1 + N_2 + 1)$ *eigenfunctions*

$$\varphi_k(t) = \frac{1}{\sqrt{t}}\exp\left(\frac{p_k}{t} - \frac{t}{4a^2}\right) + \frac{\lambda\sqrt{\pi}}{2a}\exp\left(\frac{\lambda^2-1}{4a^2}t - \frac{\lambda\sqrt{-p_k}}{a}\right)\cdot \operatorname{erfc}\left(\frac{2a\sqrt{-p_k} - \lambda t}{2a\sqrt{t}}\right),$$

and the general solution of integral equation (1) *can be written as*

$$\varphi(t) = F(t) + \frac{\lambda^2}{4a^2}\int_0^t \exp\biggl(\frac{\lambda^2(t-\tau)}{4a^2}\biggr) F(t)\, d\tau + \sum_{k=-N_1}^{N_2} C_k \varphi_k(t),$$

where

$$N_1 = \biggl[\frac{\ln|\lambda| + \arg\lambda}{2\pi}\biggr], \qquad N_2 = \biggl[\frac{\ln|\lambda| - \arg\lambda}{2\pi}\biggr],$$

$$F(t) = \widetilde{f}_2(t) - \frac{\lambda}{2a\sqrt{\pi}}\int_0^t \frac{\widetilde{f}_2(\tau)}{\sqrt{t-\tau}}\, d\tau,$$

and the function $\sqrt{t}\cdot\exp\{-t/(4a^2)\}\cdot\widetilde{f}_2(t) \in L_\infty(0,\infty)$ *is defined by equality*

$$\widetilde{f}_2(t) = \widetilde{f}(t) + \lambda\int_0^t r(t,\tau)\widetilde{f}(\tau)\, d\tau.$$

References

[1] I. N. Vekua, *Generalized analytic functions*, FIZMATLIT, Moscow, 1988.

[2] M. M. Amangaliyeva, D. M. Akhmanova, M. T. Jenaliyev, M. I. Ramazanov, “Boundary value problems for a spectrally loaded heat operator with load line approaching the time axis at zero or infinity”, *Differentsialniye uravneniya*, **47**:2 (2011), 231–243.