Boas theorem for Lorentz spaces $\Lambda_q(\omega)$

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Let $0 < q \leqslant \infty$ and ω be a nonnegative function on $[0,1]$. The generalized Lorentz spaces $\Lambda_q(\omega)$ consists of the functions f on $[0,1]$ such that $\|f\|_{\Lambda_q(\omega)} < \infty$, where

$$\|f\|_{\Lambda_q(\omega)} := \begin{cases} \left(\int_0^1 (f^*(t)\omega(t))^q \frac{dt}{t}\right)^{\frac{1}{q}} & \text{for } 0 < q < \infty, \\ \sup\limits_{0\leqslant t\leqslant 1} f^*(t)\omega(t) & \text{for } q = \infty. \end{cases}$$

These spaces $\Lambda_q(\omega)$ coincide to the classical spaces L_{pq} in the case $\omega(t) = t^{\frac{1}{p}}$, $1 < p < \infty$ (see [1]).

Let $\mu = \{\mu(k)\}_{k\in\mathbb{N}}$ be a sequence of positive number and the space $\lambda_q(\mu)$ consists of all sequences $a = \{a_k\}_{k=1}^{\infty}$ such that $\|a\|_{\lambda_q(\mu)} < \infty$, where

$$\|a\|_{\lambda_q(\mu)} := \begin{cases} \left(\sum_{k=1}^{\infty} (a_k^*\mu(k))^q \frac{1}{k}\right)^{\frac{1}{q}} & \text{for } 0 < q < \infty, \\ \sup\limits_{k} a_k^*\mu(k) & \text{for } q = \infty. \end{cases}$$

Here $\{a_k^*\}_{k=1}^{\infty}$ is the nonincreasing rearrangement of the sequence $\{|a_k|\}_{k=1}^{\infty}$. Boas theorem was generalized also for more general Lorentz spaces $\Lambda_q(\omega)$ in 1974 by L.-E. Persson for the case when $\Phi = \{e^{2\pi ikx}\}_{k=-\infty}^{+\infty}$ is the trigonometric system (see [2]).

Let the function f be periodic with period 1 and integrable on $[0,1]$ and let $\Phi = \{\varphi_k\}_{k=1}^{\infty}$ be an orthonormal system on $[0,1]$. The numbers

$$a_k = a_k(f) = \int_0^1 f(x)\overline{\varphi_k(x)}dx, \qquad k \in \mathbb{N}$$

are called the Fourier coefficients of the functions f with respect to the system $\Phi = \{\varphi_k\}_{k=1}^{\infty}$.

We say that the orthonormal system $\Phi = \{\varphi_k\}_{k=1}^{\infty}$ is regular if there exists a constant B, such that

1) for every segment e from $[0,1]$ and $k \in \mathbb{N}$ it yields that

$$\left|\int_e \varphi_k(x)dx\right| \leqslant B\min(|e|, 1/k),$$

2) for every segment w from $\mathbb{N}$ and $t \in (0,1]$ we have that

$$\left(\sum_{k\in w} \varphi_k(\,\cdot\,)\right)^*(t) \leqslant B\min(|w|, 1/t),$$

where $\left(\sum_{k\in w}\varphi_k(\cdot)\right)^*(t)$ as usual denotes the nonincreasing rerrangement of the function $\sum_{k\in w}\varphi_k(x)$.

Examples of regular systems are all trigonometric systems, the Walsh system and Price's system. In [3], [4], [5] some results were obtained with respect to the regular system using network space.

Let $\delta > 0$ be a fixed parameter. Consider a nonnegative function $\omega(t)$ on $[0,1]$. We define the following classes:

$$A_\delta := \{\omega(t) : \omega(t)t^{-\frac{1}{2}-\delta} \text{ is an increasing function and } \omega(t)t^{-1+\delta} \text{ is a decreasing function}\},$$
$$B_\delta := \{\omega(t) : \omega(t)t^{-\delta} \text{ is an increasing function and } \omega(t)t^{-1+\delta} \text{ is a decreasing function}\},$$

Then the classes A, B can be defined as follows: $A = \bigcup_{\delta>0} A_\delta$, $B = \bigcup_{\delta>0} B_\delta$.

The main results of this work are the following generalizations of the Boas theorem.

Theorem 1. *Let* $1 \leqslant q \leqslant \infty$ *and* $\omega \in B$. *Let* $\Phi = \{\varphi_k\}_{k=1}^\infty$ *be a regular system and let* $f \overset{\text{a.e.}}{=} \sum_{k=1}^\infty a_k\varphi_k$. *If* f *is a nonnegative and a nonincreasing function, then*

$$\left(\int_0^1 (f(t)\omega(t))^q \frac{dt}{t}\right)^{\frac{1}{q}} \approx \left(\sum_{k=1}^\infty (a_k^*\mu(k))^q \frac{1}{k}\right)^{\frac{1}{q}},$$

where $\mu(k) = k\omega(1/k)$.

We say that a function f on $[0,1]$ is generalized monotone if there exists some constant $M > 0$ such that

$$|f(x)| \leqslant M\frac{1}{x}\left|\int_0^x f(t)\,dt\right|, \qquad x > 0.$$

Our next main result reads.

Theorem 2. *Let* $1 \leqslant q \leqslant \infty$ *and* $\omega \in A$. *Let* $\Phi = \{\varphi_k\}_{k=1}^\infty$ *be a regular system and let* $f \overset{\text{a.e.}}{=} \sum_{k=1}^\infty a_k\varphi_k$. *If* f *is a nonnegative and a generalized monotone function, then*

$$\|f\|_{\Lambda_q(\omega,[0,1])} \approx \left(\sum_{k=1}^\infty (a_k^*\mu(k))^q \frac{1}{k}\right)^{\frac{1}{q}},$$

where $\mu(k) = k\omega(1/k)$.

References

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