Norm convolution inequalities in L_p

E. D. Nursultanov[a], S. Yu. Tikhonov[b], N. T. Tleukhanova[c]

[a] *Kazakhstan Branch of Lomonosov Moscow State University*

[b] *Centre de Recerca Matematica*

[c] *L. N. Gumilyov Eurasian National University*

Let $1 \leqslant p \leqslant \infty$, $L_p \equiv L_p(\mathbb{R})$ and let the convolution operator be given by

$$(Af)(x) = (K * f)(x) = \int_{\mathbb{R}} K(x-y)f(y)dy, \qquad K \in L_{\mathrm{loc}}.$$

Let $d > 0$ and let

– M_1 be the set of all intervals of length $\leqslant d$;
– M_2 be the set of all measurable sets $e \subset [-d, d]$ such that $\operatorname{diam}(e) = \sup_{x,y\in e} |x-y| \leqslant d$;
– W_1 be the set of all finite arithmetic progressions of integer numbers;
– W_2 be the set of all finite sets $w \subset \mathbb{Z}$ such that $\min_{i,j\in w} |i-j| \geqslant 2$.

Now we define the sets $\mathfrak{L}_d, \mathfrak{U}_d, \mathfrak{V}_d$ as follows:

$$\mathfrak{L}_d = \left\{ E = \bigcup_{k\in w} (e + kd) : e \in M_1,\ w \in W_1 \right\},$$

$$\mathfrak{U}_d = \left\{ E = \bigcup_{k\in w} (e_k + kd) : e_k \in M_2,\ w \in W_2,\ |e_k| = |e_j|,\ k, j \in w \right\},$$

$$\mathfrak{V}_d = \left\{ E = \bigcup_{x\in e} (x + w(x)d) : e \in M_2,\ w(x) \in W_2,\ |w(x)| = |w(y)|,\ x, y \in e \right\},$$

where $|e|$ is the measure of a set $e \in M_i$ and $|w|$ is the number of elements of $w \in W_i$. Note that $\mathfrak{L}_d \subset \mathfrak{U}_d \cap \mathfrak{V}_d$. If $E \in \mathfrak{L}_d$, then $|E| = |e||w|$, where e, w are the sets from the representation of E. Similarly, this property holds for $E \in \mathfrak{U}_d$ and $E \in \mathfrak{V}_d$.

Theorem 1. *Let $1 < p < q < \infty$. If for some $d > 0$ we have either*

$$\sup_{E\in\mathfrak{U}_d} \frac{1}{|E|^{1/p-1/q}} \int_E |K(x)|\, dx \leqslant D$$

or

$$\sup_{E\in\mathfrak{V}_d} \frac{1}{|E|^{1/p-1/q}} \int_E |K(x)|\, dx \leqslant D,$$

then the operator $Af = K * f$ is bounded from $L_p(\mathbb{R})$ to $L_q(\mathbb{R})$ and

$$\|A\|_{L_p\to L_q} \leqslant C(p,q)D,$$

where $C(p,q)$ depends on p and q.

Theorem 2. *Let $1 < p < q < \infty$, $d > 0$, and the operator $Af = K * f$ be bounded from $L_p(\mathbb{R})$ to $L_q(\mathbb{R})$. If for any $B > 0$ we have*

$$\sup_{\substack{E\in\mathfrak{L}_d\\|E|\leqslant B}} \frac{1}{|E|^{1/p-1/q}}\left|\int_E K(x)\,dx\right| \leqslant C(B) < \infty,$$

then

$$\sup_{E\in\mathfrak{L}_d} \frac{1}{|E|^{1/p-1/q}}\left|\int_E K(x)\,dx\right| \leqslant C(p,q)\|A\|_{L_p\to L_q}.$$

Corollary. *Let $1 < p \leqslant q < \infty$ and $\lambda = 1 - (\frac{1}{p} - \frac{1}{q})$. Let also*

$$\mathcal{K}(x) = \frac{e^{i|x|^a}}{|x|^b},$$

where $a \neq 0$, $a \neq 1$, and $b \neq \lambda$. If

$$\max(q, p') > \frac{a}{\lambda - b} > 0,$$

then the operator $Af = \mathcal{K} * f$ is not bounded from L_p to L_q.

References

[1] R. O'Neil, "Convolution operators and $L(p,q)$ spaces", *Duke Math. J.*, **30** (1963), 129–142.

[2] V. D. Stepanov, *Some topics in the theory of integral convolution operators*, Dalnauka, Vladivostok, 2000.

[3] P. Sjölin, "Regularity of solutions to the Schrödinger equation", *Duke Math. J.*, **55**:3 (1987), 699–715.