A weighted Hardy-type inequality for $0 < p < 1$ with sharp constant

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Let Ω be a Lebesgue measurable set in $\mathbb{R}^n$, u be a non-negative Lebesgue measurable function on Ω (weight function), and $0 < p < \infty$. We denote by $L_{p,u}(\Omega)$ the space of all Lebesgue measurable functions f on Ω for which

$$\|f\|_{L_{p,u}(B_r)} = \left(\int_\Omega |f(x)|^p u(x)\,dx\right)^{\frac{1}{p}} < \infty,$$

and by H the n-dimensional Hardy operator.

THEOREM. *Let $C_1 > 0$, $0 < p < 1$ and u, v be weight functions on $\mathbb{R}^n$, $(0, \infty)$ respectively. Suppose that*

$$\int_{B_r} u^{\frac{1}{1-p}}(x)\,dx = \infty \qquad \textit{for some} \quad r > 0 \tag{1}$$

and

$$V(r) := \int_r^\infty v(\rho)\rho^{-np}\,d\rho < \infty \qquad \textit{for all} \quad r > 0. \tag{2}$$

Consider the set of all Lebesgue measurable functions f on $\mathbb{R}^n$ satisfying the inequality

$$|f(x)| \leqslant C_1 u^{\frac{1}{1-p}}(x)\|f\|_{L_{p,u}(B_{(|x|)}.)} \tag{3}$$

for almost all $x \in \mathbb{R}^n$.

Then for all functions f in this set

$$\|Hf\|_{L_{p,v}(0,\infty)} \leqslant C_2\|f\|_{L_{p,w}(\mathbb{R}^n)} \tag{4}$$

where

$$w(x) = u(x)V(|x|), \qquad x \in \mathbb{R}^n,$$

and

$$C_2 = v_n^{-1}pC_1^{1-p}.$$

If, in addition,

$$\int_{B_{r_2}\setminus B_{r_1}} u^{\frac{1}{1-p}}(x)\,dx < \infty \qquad \textit{for all} \quad 0 < r_1 < r_2 < \infty, \tag{5}$$

and

$$\int_0^1 \exp\left(-C_1^p \int_{B_1\setminus B_{|x|}} u^{\frac{1}{1-p}}(y)\,dy\right) v(r)r^{-np}dr < \infty, \tag{6}$$

then the constant C_2 is sharp and there exists a functions $f \in L_{p,w}(\mathbb{R}^n)$ not equivalent to 0, satisfying inequality (3) and such that there is equality in inequality (4).

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