B_w^u-function spaces and their interpolation

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Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. We denote by Q_r the open cube centered at the origin and sidelength $2r$, or the open ball centered at the origin and of radius r, that is,

$$Q_r = \left\{ y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n : \max_{1 \leqslant i \leqslant n} |y_i| < r \right\} \qquad \text{or}$$
$$Q_r = \{ y \in \mathbb{R}^n : |y| < r \}.$$

For each $r \in (0, \infty)$, let $E(Q_r)$ be a function space on Q_r with quasi-norm $\| \cdot \|_{E(Q_r)}$. Let $E_Q(\mathbb{R}^n)$ be the set of all measurable functions f on $\mathbb{R}^n$ such that $f|_{Q_r} \in E(Q_r)$ for all $r > 0$.

We assume the following *restriction property*:

$$f|_{Q_r} \in E(Q_r) \text{ and } 0 < t < r < \infty \quad \Longrightarrow \quad f|_{Q_t} \in E(Q_t) \text{ and } \|f\|_{E(Q_t)} \leqslant C_E \|f\|_{E(Q_r)}, \tag{1}$$

where C_E is a positive constant independent of r, t and f.

Definition. Let $w\colon (0, \infty) \to (0, \infty)$ be a weight function and let $u \in (0, \infty]$. We define function spaces $B_w^u(E) = B_w^u(E)(\mathbb{R}^n)$ and $\dot{B}_w^u = \dot{B}_w^u(E)(\mathbb{R}^n)$ as the sets of all functions $f \in E_Q(\mathbb{R}^n)$ such that $\|f\|_{B_w^u(E)} < \infty$ and $\|f\|_{\dot{B}_w^u(E)} < \infty$, respectively, where

$$\|f\|_{B_w^u(E)} = \|w(r)\|f\|_{E(Q_r)}\|_{L^u([1,\infty), dr/r)},$$
$$\|f\|_{\dot{B}_w^u(E)} = \|w(r)\|f\|_{E(Q_r)}\|_{L^u((0,\infty), dr/r)}.$$

In the above we abbreviated $\|f|_{Q_r}\|_{E(Q_r)}$ to $\|f\|_{E(Q_r)}$.

If $E = L^p$, then $\dot{B}_w^u(L^p)(\mathbb{R}^n)$ is the local Morrey-type space introduced by Burenkov and Guliyev [6], Example 5, below.

Here, we always assume that w has some decreasing condition. Note that, if $w(r) \to \infty$ as $r \to \infty$, then $B_w^u(E) = \dot{B}wu = \{0\}$.

In particular, if $w(r) = r^{-\sigma}$, $\sigma \geqslant 0$ and $u = \infty$, we denote $B_w^u(E)(\mathbb{R}^n)$ and $\dot{B}_w^u(E)(\mathbb{R}^n)$ by $B_\sigma(E)(\mathbb{R}^n)$ and $\dot{B}_\sigma(E)(\mathbb{R}^n)$, respectively, which were introduced recently by Komori-Furuya, Matsuoka, Nakai and Sawano [17]. These B_σ-function spaces unify several function spaces, see the following Examples 1–4.

Example 1. $B^p(\mathbb{R}^n)$, the dual of Beuling algebra $A^p(\mathbb{R}^n)$ (Beurling [2], Feichtinger [12]).

Example 2. The central mean oscillation space $\mathrm{CMO}^p(\mathbb{R}^n)$, the central bounded mean oscillation space $\mathrm{CBMO}^p(\mathbb{R}^n)$ (Chen and Lau [11] and García-Cuerva [13] , Lu and Yang [19], [20]).

Example 3. The central Morrey spaces, the λ-central mean oscillation space and the λ-central bounded mean oscillation space as an extension of $B^p(\mathbb{R}^n)$, $\dot{B}^p(\mathbb{R}^n)$, $\mathrm{CMO}^p(\mathbb{R}^n)$ and $\mathrm{CBMO}^p(\mathbb{R}^n)$ (García-Cuerva and Herrero [14] and Alvarez, Guzmán-Partida and Lakey [1]).

Example 4. If $E = L_{p,\lambda}$ (Morrey space) or $\mathcal{L}_{p,\lambda}$ (Campanato space), then the function spaces $B_\sigma(L_{p,\lambda})(\mathbb{R}^n)$, $\dot{B}_\sigma(L_{p,\lambda})(\mathbb{R}^n)$, $B_\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)$ and $\dot{B}_\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)$ unify the function spaces in above examples and the usual Morrey-Campanato and Lipschitz spaces. Actually, if $\lambda = -n/p$, then $L_{p,\lambda} = L^p$. If $\sigma = 0$, then

$$B_0(L_{p,\lambda})(\mathbb{R}^n) = \dot{B}_0(L_{p,\lambda})(\mathbb{R}^n) = L_{p,\lambda}(\mathbb{R}^n),$$
$$B_0(\mathcal{L}_{p,\lambda})(\mathbb{R}^n) = \dot{B}_0(\mathcal{L}_{p,\lambda})(\mathbb{R}^n) = \mathcal{L}_{p,\lambda}(\mathbb{R}^n).$$

If $\lambda = 0$, then $\mathcal{L}_{p,\lambda}(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$ for all $p \in [1,\infty)$ (John and Nirenberg [15]). If $\lambda = \alpha \in (0,1]$, then $\mathcal{L}_{p,\lambda}(\mathbb{R}^n) = \mathrm{Lip}_\alpha(\mathbb{R}^n)$ for all $p \in [1,\infty)$ (Campanato [10], Meyers [22], Spanne [24]). B_σ-Morrey-Campanato spaces were investigated in [16], [17], [18], [21].

Example 5. Local Morrey-type space $LM_{p\theta,w}(\mathbb{R}^n)$ with the (quasi-)norm

$$\|f\|_{LM_{p\theta,w}} = \|w(r)\|f\|_{L^p(Q_r)}\|_{L^\theta(0,\infty)},$$

(Burenkov and Guliyev [6]). $LM_{p\theta,\widetilde{w}}(\mathbb{R}^n)$ is expressed by $\dot{B}^u_w(E)(\mathbb{R}^n)$ with $E = L^p$ and $\widetilde{w}(r) = w(r)/r$. For recent progress of local Morrey-type spaces, see [3], [4]. See also [5], [9] for interpolation spaces for local Morrey-type spaces.

In this talk, we treat the interpolation property of B^u_w-function spaces. To do this, we also the following decomposition property: For any $f \in E_Q(\mathbb{R}^n)$ and for any $r > 0$, there exists a decomposition $f = f_0^r + f_1^r$ such that

$$\|f_0^r\|_{E(Q_t)} \leqslant \begin{cases} C_E\|f\|_{E(Q_t)} & (0 < t < r), \\ C_E\|f\|_{E(Q_{ar})} & (r \leqslant t < \infty), \end{cases} \tag{2}$$

and

$$\|f_1^r\|_{E(Q_t)} \leqslant \begin{cases} 0 & (0 < t < cr), \\ C_E\|f\|_{E(Q_{bt})} & (cr \leqslant t < \infty), \end{cases} \tag{3}$$

where C_E, a, b, c are positive constants independent of r, t and f.

Theorem. *Assume that a family* $\{(E(Q_r), \|\cdot\|_{E(Q_r)})\}_{0<r<\infty}$ *has the restriction and decomposition properties above. Let* $u_0, u_1, u \in (0,\infty]$, $w_0, w_1 \in \mathcal{W}^\infty$, *and*

$$w = w_0^{1-\theta} w_1^\theta.$$

Assume also that, for some positive constant ϵ, $(w_0(r)/w_1(r))r^{-\epsilon}$ *is almost increasing, or,* $(w_1(r)/w_0(r))r^{-\epsilon}$ *is almost increasing. Then*

$$(\dot{B}^{u_0}_{w_0}(E)(\mathbb{R}^n), \dot{B}^{u_1}_{w_1}(E)(\mathbb{R}^n))_{\theta,u} = \dot{B}^u_w(E)(\mathbb{R}^n),$$

and

$$(B^{u_0}_{w_0}(E)(\mathbb{R}^n), B^{u_1}_{w_1}(E)(\mathbb{R}^n))_{\theta,u,[1,\infty)} = B^u_w(E)(\mathbb{R}^n).$$

Here, $(A_0, A_1)_{\theta,u}$ is the usual K-real interpolation space, and we define the quasi-norm of $(A_0, A_1)_{u,[1,\infty)}$ as

$$\|a\|_{(A_0,A_1)_{u,[1,\infty)}} = \left[\int_1^\infty \left(\frac{K(t,a;A_0,A_1)}{t^\theta}\right)\frac{dt}{t}\right]^{1/u}$$

As applications of the interpolation property, we also give the boundedness of linear and sublinear operators. It is known that the Hardy–Littlewood maximal operator, fractional maximal operators, singular and fractional integral operators are bounded on B_σ-Morrey–Campanato spaces, see [16], [17], [18], [21]. Interpolate these function spaces, the we get the boundedness of these operators on $B^u_w(L_{p,\lambda})$,$\dot{B}^u_w(L_{p,\lambda})$, $B^u_w(\mathcal{L}_{p,\lambda})$ and $\dot{B}^u_w(\mathcal{L}_{p,\lambda})$, which are also generalization of the results on the local Morrey-type spaces $LM_{pu,w}(\mathbb{R}^n)$.

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