Sparse approximation with respect to the trigonometric system

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Sparse approximation with respect to dictionaries is a very important topic in the high-dimensional approximation. The main motivation for the study of sparse approximation is that many real world signals can be well approximated by sparse ones. Sparse approximation automatically implies a need for nonlinear approximation, in particular, for greedy approximation. We give a brief description of a sparse approximation problem and present a discussion of the obtained results and their relation to previous work. In a general setting we are working in a Banach space X with a redundant system of elements $\mathcal{D}$ (dictionary $\mathcal{D}$). There is a solid justification of importance of a Banach space setting in numerical analysis in general and in sparse approximation in particular. Let X be a real Banach space with norm $\|\cdot\| := \|\cdot\|_X$. We say that a set of elements (functions) $\mathcal{D}$ from X is a dictionary if each $g \in \mathcal{D}$ has norm one ($\|g\| = 1$), and the closure of span $\mathcal{D}$ is X. A symmetrized dictionary is $\mathcal{D}^{\pm} := \{\pm g : g \in \mathcal{D}\}$. For a nonzero element $g \in X$ we let F_g denote a norming (peak) functional for g:

$$\|F_g\|_{X^*} = 1, \qquad F_g(g) = \|g\|_X.$$

The existence of such a functional is guaranteed by the Hahn-Banach theorem.

An element (function, signal) $s \in X$ is said to be m-sparse with respect to $\mathcal{D}$ if it has a representation $s = \sum_{i=1}^m c_i g_i$, $g_i \in \mathcal{D}$, $i = 1, \dots, m$. The set of all m-sparse elements is denoted by $\Sigma_m(\mathcal{D})$. For a given element f we introduce the error of best m-term approximation

$$\sigma_m(f, \mathcal{D})_X := \inf_{s \in \Sigma_m(\mathcal{D})} \|f - s\|_X.$$

Let $t \in (0, 1]$ be a given nonnegative number. We define the Weak Chebyshev Greedy Algorithm (WCGA) that is a generalization for Banach spaces of Weak Orthogonal Greedy Algorithm .

Weak Chebyshev Greedy Algorithm (WCGA).
We define $f_0 := f$. Then for each $m \geqslant 1$ we inductively define

1) $\varphi_m \in \mathcal{D}$ is any element satisfying

$$|F_{f_{m-1}}(\varphi_m)| \geqslant t \sup_{g \in \mathcal{D}} |F_{f_{m-1}}(g)|;$$

2) define

$$\Phi_m := \operatorname{span}\{\varphi_j\}_{j=1}^m,$$

and define G_m to be the best approximant to f from Φ_m;

3) denote

$$f_m := f - G_m.$$

We demonstrated that the Weak Chebyshev Greedy Algorithm (WCGA) is very good for m-term approximation with respect to a special class of dictionaries, in particular, for the trigonometric system. The trigonometric system is a classical system that is known to be difficult to study. We studied among other problems the problem of nonlinear sparse approximation with respect to it. Let $\mathcal{RT}$ denote the real trigonometric system $1, \sin 2\pi x, \cos 2\pi x, \dots$ on $[0,1]$ and let $\mathcal{RT}_p$ to be its version normalized in $L_p([0,1])$. Denote $\mathcal{RT}_p^d := \mathcal{RT}_p \times \cdots \times \mathcal{RT}_p$ the d-variate trigonometric system. We need to consider the real trigonometric system because the algorithm WCGA is well studied for the real Banach space. We proved the following Lebesgue-type inequality for the WCGA.

THEOREM. *Let $\mathcal{D}$ be the normalized in L_p, $2 \leqslant p < \infty$, real d-variate trigonometric system. Then for any $f \in L_p$ the WCGA with weakness parameter t gives*

$$\|f_{C(t,p,d)m\ln(m+1)}\|_p \leqslant C\sigma_m(f,\mathcal{D})_p.$$

The above Lebesgue-type inequality guarantees that the WCGA works very well for each individual function f.