SOBOLEV SPACES, GEOMETRIC FUNCTION THEORY and APPLICATIONS

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The talk is devoted to the memory of S. M. Nikolskiĭ

Lay-out of talk

Classical results on inverse mappings

Let $D \subset \mathbb{R}^n$ be an open set, $\varphi : D \to \mathbb{R}^n$ be C^1-injective mapping, and $\det D\varphi(x) \neq 0$. Then $\varphi(D) = D'$ is an open set, $\psi = \varphi^{-1} : D' \to D$ belongs to C^1 and $\det D\varphi^{-1}(y) \neq 0$.

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The inverse mapping to a quasiisometric one is also quasiisometric.

In another words, the inverse to a homeomorphism $\varphi : D \to D'$ of $W^1_{\infty,\mathrm{loc}}$ with $|D\varphi(x)| \leq M < \infty$ and $|J(x, \varphi)| \geq \mu > 0$ a. e., has the same properties.

General problem:

What properties has the inverse mapping to some Sobolev homeomorphism?

Sobolev space $L^1_p(D)$, $D \subset \mathbb{R}^n$, $p \in [1,\infty]$,

consists of locally integrable functions $f : D \to \mathbb{R}$ having the first generalized derivatives $\frac{\partial f}{\partial x_i}(x)$:

$$\int\limits_D \frac{\partial f}{\partial x_i}(x)\varphi(x)\,dx = -\int\limits_D f(x)\frac{\partial \varphi}{\partial x_i}(x)\,dx \quad \forall \varphi \in C_0^\infty(D),$$

$i = 1, \dots, n$, and the finite seminorm $\|f \mid L^1_p(D)\| = \|\nabla f \mid L_p(D)\|$, $\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$.

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$W^1_p(D) = L_p(D) \cap L^1_p(D)$ with the finite norm $\|f \mid W^1_p(D)\| = \|f \mid L_p(D)\| + \|\nabla f \mid L_p(D)\|$.

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$f \in W^1_{p,\mathrm{loc}}(D) \Longleftrightarrow f \in W^1_p(\Omega)$ for any $\Omega \Subset D$.

Two distortion functions.

For mapping $\varphi : D \to \mathbb{R}^n$, $D \subset \mathbb{R}^n$, $n \geq 2$, of $W^1_{1,\mathrm{loc}}(D)$ with the finite codistortion: $J(y,\psi) = 0$ implies $\operatorname{adj} D\psi(y) = 0$ a.e., define

$$D \ni x \mapsto \mathcal{K}_{\varphi,p}(x) = \begin{cases} \dfrac{|\operatorname{adj} D\varphi(x)|}{|J(x,\varphi)|^{\frac{n-1}{p}}} & \text{at } x \in D \setminus Z, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Recall that $\operatorname{adj} D\varphi(x) \cdot D\varphi(x) = \det D\varphi(x) \cdot E$.

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For mapping $\psi : D' \to \mathbb{R}^n$, $D \subset \mathbb{R}^n$, $n \geq 2$, of $W^1_{1,\mathrm{loc}}(D')$ with the finite distortion: $J(y,\psi) = 0$ implies $D\psi(y) = 0$ a.e., define

$$D' \ni y \mapsto K_{\psi,q'}(y) = \begin{cases} \frac{|D\psi(y)|}{|J(y,\psi)|^{\frac{1}{q'}}} & \text{at } y \in D' \setminus Z', \\ 0 & \text{otherwise.} \end{cases} \qquad (2)$$

Description of the inverse mappings

Theorem 1 [1-3]. Let a homeomorphism $\varphi : D \to D'$ have the following properties:

1) $\varphi \in W^1_{q,\mathrm{loc}}(D)$, $n-1 \leq q \leq \infty$,
2) φ has the finite codistortion: $\operatorname{adj} D\varphi(x) = 0$ a. e. on a set $Z = \{x \in D : J(x,\varphi)\}$,
3) $\mathcal{K}_{\varphi,p}(\cdot) \in L_\varrho(D)$ where $\frac{1}{\varrho} = \frac{n-1}{q} - \frac{n-1}{p}$, $n-1 \leq q \leq p \leq \infty$.

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Then the inverse homeomorphism has the following properties:

4) $\varphi^{-1} \in W^1_{p',\mathrm{loc}}(D')$, где $p' = \frac{p}{p-n+1}$,
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Moreover, $\|K_{\varphi^{-1},q'}(\cdot) \mid L_\varrho(D')\| = \|\mathcal{K}_{\varphi,p}(\cdot) \in L_\varrho(D)\|$.

New definition of quasiconformal mapping.

Definition. A homeomorphism $\varphi : D \to D'$ belonging to $W^1_{n,\mathrm{loc}}(D)$ and meeting $|D\varphi(x)|^n \leq K|J(x,\varphi)|$ for almost all $x \in D$, is called *quasiconformal*.

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If for some nonnegative number $M \in \mathbb{R}$ the inequality

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Thus, $\operatorname{adj} D\varphi(x) = 0$ on Z a.e. implies $D\varphi(x) = 0$ on Z a.e.

Some preliminaries

A mapping $f : \Omega \to \mathbb{R}^n$ of $W^1_{n,\mathrm{loc}}(\Omega)$, $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is called the mapping with bounded distortion if

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In 1967 Yuriy Reshetnyak [6] proved that

every non-constant mapping with bounded distortion is continuous open and discrete.

More general mappings. DEFINITION [7].

Let $\theta : \mathbb{R}^n \to [0, \infty]$ be measurable functions: $0 < \theta < \infty$ a. e.

- A continuous open and discrete mapping $f : \Omega \to \mathbb{R}^n$ is said to be a *mapping with bounded θ-weighted (p, q)-distortion,* $1 \leq q \leq p < \infty$, if:

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- the θ-weighted (q, p)-distortion function

$$\Omega \ni x \mapsto K^{\theta}_q(x, f) = \begin{cases} \dfrac{\theta^{\frac{1}{q}}(x)|Df|(x)}{J(x, f)^{\frac{1}{p}}}, & \text{if } J(x, f) \neq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

belongs to $L_{\kappa}(\Omega)$ where $\frac{1}{\kappa} = \frac{1}{q} - \frac{1}{p}$ ($\kappa = \infty$ if $p = q$).

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  belongs to $L_\kappa(\Omega)$ where $\frac{1}{\kappa} = \frac{1}{q} - \frac{1}{p}$ ($\kappa = \infty$ if $p = q$).
- If $\theta \equiv 1$, $q = p = n$, f is a m. b. d. If f is homeo then f is a qc mapping.

DEFINITION.

A weighted Sobolev space $L^1_q(\Omega,\omega)$, $1 \leq q < \infty$, consists of locally integrable functions $f : \Omega \to \mathbb{R}^n$

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All weighted functions are assumed to be locally integrable.

Analytic motivation

Theorem 3 [8]. A sense-preserving homeomorphism $f : (\Omega, \theta) \to \Omega'$ induces a bounded operator

$f^* : L^1_p(\Omega') \cap W^1_{\infty,\mathrm{loc}}(\Omega') \to L^1_q(\Omega, \theta)$, $1 \leq q \leq p < \infty$, of weighted Sobolev spaces by the rule $f^*(g) = g \circ f$ iff f has θ-weighted (p, q)-distortion.

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COROLLARY 1. Let $\theta\in A_q$. A sense-preserving homeomorphism

$f:(\Omega,\theta)\to\Omega'$ induces a b. operator $f^*: L^1_p(\Omega')\to L^1_q(\Omega,\theta)$, $1\le q\le p<\infty$, of weighted Sobolev spaces by the rule $f^*(g)=g\circ f$ iff f has θ-weighted (p,q)-distortion.

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Moreover, $\alpha_{p,q} \big\| K^\theta_q(\cdot, f) \mid L_\kappa(\Omega') \big\| \le \|f^*\| \le \big\| K^\theta_q(\cdot, f) \mid L_\kappa(\Omega') \big\|$.

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Moreover, $\alpha_{p,q}\big\|K^\theta_q(\cdot,f)\mid L_\kappa(\Omega')\big\|\le\|f^*\|\le\big\|K^\theta_q(\cdot,f)\mid L_\kappa(\Omega')\big\|$.

REMARK: Under $p=q=n$, $\theta=1$, we obtain just qc mappings

Geometric motivation

A Riemannian manifold $\mathbb{M}$ has the bounded geometry if there exists $\varepsilon > 0$ such that every $B(x, \varepsilon) \subset \mathbb{M}$ is diffeomorphic to $B(0, 1) \subset \mathbb{R}^n$ with Lipschitz constant bounded uniformly.

Theorem 4 (Pierre Pansu proved the result when f is a diffeo.) Let $\mathbb{M}$ and $\mathbb{N}$ be Riemannian manifolds with bounded geometry, and assume that $\mathbb{N}$ satisfies an isoperimetric inequality of order $d > n$:

$$\mathrm{Area}(\partial\Omega)^{d/(d-1)} \geq \mathrm{const.}\ \mathrm{Vol}(\Omega)$$

for all smooth compact domain $\Omega \subset \mathbb{N}$ of volume ≥ 1.
If $d\frac{n-1}{d-1} < s < n$, then every homeomorphism $f \in W^{1,1}_{loc}(\mathbb{M}, \mathbb{N})$ with bounded (s, s)-distortion is a rough quasi-isometry.

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The last means $0 < \alpha^{-1} \leq \frac{d_{\mathbb{N}}(f(x), f(y))}{d_{\mathbb{M}}(x,y)} \leq \alpha$ for all $x, y \in \mathbb{M}$ with $d_{\mathbb{M}}(x, y) \geq 1$.

Poletskiy function and its properties

DEFINITION. Let $f : \Omega \to \mathbb{R}^n$ be a mapping with bounded θ-weighted (p, q)-distortion, $D \Subset \Omega$ be a normal domain: $f(\partial D) = \partial f(D)$. On $V = f(D)$ define the Poletskiy function

$$V \ni y \mapsto g_D(y) = \sum_{x \in f^{-1}(y) \cap D} i(x, f)x.$$

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Theorem 5 [9]. Let $q > n - 1$. The function g_D has the following properties:

1) $\operatorname{supp} g_D = f(\overline{D})$ is a compact set;
2) g_D is continuous;
3) $g_D \in \mathrm{ACL}(V)$;
4) $Dg_D(y) = 0$ a. e. on $Z = \{y \in V : \det Dg_D(y) = 0\}$;
5) $f(B_f \cap D) \subset Z$ where B_f is the branch set.

Push-forward functions

DEFINITION. Let $f:\Omega\to\mathbb{R}^n$ be a mapping with bounded θ-weighted (p,q)-distortion, $D\Subset\Omega$ be a normal domain, $f(\partial D)=\partial f(D)$, $u\in C_0^1(D)$. Define $v=f_*u:f(\Omega)\to\mathbb{R}$ as

$$f(\Omega)\ni y\mapsto v(y)=\begin{cases}\sum\limits_{x\in f^{-1}(y)} i(x,f)u(x), & y\in f(\operatorname{supp}u),\\ 0, & y\notin f(\operatorname{supp}u).\end{cases}$$

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Theorem 6 [7]. Let $q>n-1$. The function v has the following properties:

1) $\operatorname{supp} v=f(\operatorname{supp} u)$ is a compact set;
2) v is continuous, $v\in \mathrm{ACL}(f(D))$;
3) $Dv(y)=0$ a. e. on $Z=\{y\in f(\Omega):\det Dg_D(y)=0\}$;
4) $f(B_f\cap D)\subset Z$.

Push-forward functions and estimates for them

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Theorem 7 [7]. Let $q>n-1$. The function w has the following properties:

1) $\operatorname{supp} w = f(\operatorname{supp} u)$ is a compact set;
2) w is continuous, $w\in \mathrm{ACL}(f(D))$;
3) $Dw(y)=0$ a. e. on $Z=\{y\in f(\Omega):\det Dg_D(y)=0\}$;
4) $f(B_f\cap D)\subset Z$.

Theorem 8 [7]. Let $f:\Omega\to\mathbb{R}^n$ be a mapping with bounded θ-weighted (p,q)-distortion. For the push-forward function

1) $v=f_*u: f(\Omega)\to\mathbb{R}$ and a domain $D\subset\Omega$ such that $\operatorname{supp} u\Subset D$, the following estimate holds:

$$\|f_*u\mid L^1_s(f(D))\|\leq N(f,D)^{\frac{s-1}{s}}\big(K^{\theta}_{q,p}(f;D)\big)^{n-1}\|u\mid L^1_r(D,\omega)\|,$$

where $N(f,D)=\sup\limits_{y\in\mathbb{R}^n}\#\{f^{-1}(y)\cap D\}$;

Theorem 8 [7]. Let $f : \Omega \to \mathbb{R}^n$ be a mapping with bounded θ-weighted (p, q)-distortion. For the push-forward function

1) $v = f_* u : f(\Omega) \to \mathbb{R}$ and a domain $D \subset \Omega$ such that $\operatorname{supp} u \Subset D$, the following estimate holds:

$$\|f_* u \mid L^1_s(f(D))\| \leq N(f, D)^{\frac{s-1}{s}} \big(K^\theta_{q,p}(f; D)\big)^{n-1} \|u \mid L^1_r(D, \omega)\|,$$

where $N(f, D) = \sup\limits_{y \in \mathbb{R}^n} \#\{f^{-1}(y) \cap D\}$;

2) for $w : f(\Omega) \to \mathbb{R}$ and a domain $D \Subset \Omega$ such that $\operatorname{supp} u \Subset D$, the following estimate holds:

$$\|w \mid L^1_s(f(D))\| \leq K^\theta_{q,p}(f; \Omega)^{n-1} \|u \mid L^1_r(D, \omega)\|$$

holds where $s = \frac{p}{p-(n-1)}$, $r = \frac{q}{q-(n-1)}$, $\omega(x) = \theta^{-\frac{n-1}{q-(n-1)}}(x) \in L_1$.

Estimates for capacity:

Definition of capacity. Let $F_0, F_1 \subset \overline{D}$ be closed sets where $D \subset \mathbb{R}^n$ be an open set. The value

$$\operatorname{cap}_p^\omega(E) = \operatorname{cap}_p^\omega(F_0, F_1; D) = \inf \int\limits_D |\nabla u|^p(x)\omega(x)\,dx$$

where infimum is taken over all functions $u \in C(\overline{D}) \cap W^1_{\infty,\mathrm{loc}}(D) \cap L^1_p(D,\omega)$ such that $u \geq 1$ on F_1, $u = 0$ on F_0, is called *ω-weighted p-capacity of the condenser* $E = (F_0, F_1; D)$.

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If U is an open set, $C \subset U$ is a compact, then the condenser $E = (\partial U, C; U)$ will be denoted by a symbol $E = (U, C)$.

Estimates for capacity

Corollary 2. Let $f:\Omega\to\mathbb{R}^n$ be a mapping with bounded $(\theta,1)$-weighted (p,q)-distortion, $n-1<q\le p<\infty$. If $E=(A,C)$ is a condenser in Ω such that $\overline{A}\subset\Omega$, C is a compact in A and $N(f,A)<\infty$, then

$$\left(\operatorname{cap}_s f(E)\right)^{1/s}\le\frac{(K^{\theta,1}_{q,p}(f;\Omega))^{n-1}(N(f,A))^{(s-1)/s}}{M(f,C)}\left(\operatorname{cap}^{\omega}_r E\right)^{1/r},$$

where $s=\frac{p}{p-(n-1)}$, $r=\frac{q}{q-(n-1)}$ and
$M(f,C)=\inf\limits_{x\in f(C)}\sum\limits_{z\in f^{-1}(x)\cap C} i(z,f)$, $\omega(x)=\theta^{-\frac{n-1}{q-(n-1)}}(x)$.

Estimates for capacity

Corollary 3. Let $f : \Omega \to \mathbb{R}^n$ be a mapping with bounded θ-weighted (p,q)-distortion, $n-1 < q \le p < \infty$. If $E = (A, C)$ is a condenser in Ω such that $\overline{A} \subset \Omega$, C is a compact in A, then

$$\left(\operatorname{cap}_s f(E)\right)^{1/s} \le K^{\theta}_{p,q}(f;\Omega)^{n-1}\left(\operatorname{cap}^{\omega}_r E\right)^{1/r}$$

where $r = \frac{q}{q-(n-1)}$, $s = \frac{p}{p-(n-1)}$ and $\omega(x) = \theta^{-\frac{n-1}{q-(n-1)}}(x)$.

Properties of mappings with θ-weighted (p,q)-distortion, $n-1<q\le p<\infty$.

Theorem 9 (Liouville type theorem) [7]. Let $f:\Omega\to\mathbb{R}^n$ be a mapping with bounded θ-weighted (p,q)-distortion,

$n-1<q\le p<\infty$. If $E_k=(B_k,C)$ is a condenser in $\mathbb{R}^n$ such that $\operatorname{cap}_r^{\omega}(E_k)>0$ and $\operatorname{cap}_r^{\omega}(E_k)\to 0$ as $k\to\infty$ then

$$\operatorname{cap}_s\big(\mathbb{R}^n\setminus f(\mathbb{R}^n);W_s^1(\mathbb{R}^n)\big)=0$$

where $r=\frac{q}{q-(n-1)}$, $s=\frac{p}{p-(n-1)}$ and $\omega(x)=\theta^{-\frac{n-1}{q-(n-1)}}(x)$.

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Corollary 4.

Under conditions of previous theorem $f(\mathbb{R}^n)=\mathbb{R}^n$ if $n-1<q\le p<n$.

Properties of mappings with θ-weighted (p,q)-distortion, $n-1<q\leq p<\infty$.

Theorem 10 (Removability) [7]. Let $f:\Omega\setminus F\to\mathbb{R}^n$ be a mapping with bounded θ-weighted (p,q)-distortion, $n\leq q\leq p$,

and $F\subset\Omega$ be a closed set such that $\operatorname{cap}^{\omega}(F;W^1_r(\mathbb{R}^n))=0$, $s=\frac{p}{p-(n-1)}$, $r=\frac{q}{q-(n-1)}$, $\omega(x)=\theta^{-\frac{n-1}{q-(n-1)}}(x)$, $\omega\in A_r$. Then
1) at $n<q\leq p<\frac{(n-1)^2}{(n-2)}$ the mapping f can be extended to a continuous mapping $\widetilde{f}:\Omega\to\mathbb{R}^n$;
2) at $n=q\leq p<\frac{(n-1)^2}{(n-2)}$ and $\operatorname{cap}(\mathbb{R}^n\setminus f(\Omega\setminus F);W^1_n(\mathbb{R}^n))>0$ the mapping f is extended to a continuous mapping $\widetilde{f}:\Omega\to\overline{\mathbb{R}}^n$.

Lavrentiev — Zorich Theorem for mapping with bounded θ-weighted (n, n)-distortion.

Now $f : \Omega \to \mathbb{R}^n$ belongs to $W^1_{n,\mathrm{loc}}(\Omega)$, has the finite distortion and meets

$$\theta(x)|Df|^n(x) \le J(x, f) \quad \text{for almost all } x \in \Omega. \tag{4}$$

Theorem 10. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a locally homeomorphic mapping meeting condition (4).

If $E_k = (B_k, C)$ is a condenser in $\mathbb{R}^n$ such that $\operatorname{cap}_r^{\omega}(E_k) > 0$ and $\operatorname{cap}_r^{\omega}(E_k) \to 0$ as $k \to \infty$ where $\omega(x) = \theta^{1-n}(x)$,
then $f : \mathbb{R}^n \to \mathbb{R}^n$ is an homeomorphism on $\mathbb{R}^n$.

Applications: Classification of manifolds

Let $\mathbb{M}$ and $\mathbb{N}$ be Riemannian manifolds of topological dimension $n \geq 2$, θ be a weighted function on $\mathbb{M}$ and $n-1 < q \leq p < \infty$.

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A Riemannian manifold $\mathbb{M}$ is said to be (ω, r)-parabolic

if $\operatorname{cap}_{\omega,r}(D, \mathbb{M}) = 0$ for any compact set $D \subset \mathbb{M}$ where $\omega(x) = \theta^{-\frac{n-1}{q-(n-1)}}(x)$, $r = \frac{q}{q-(n-1)}$, $q > n-1$.

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Theorem 11 [7]. *Let $f : \mathbb{M} \to \mathbb{N}$ be a mapping with bounded θ-weighted (p, q)-distortion, $n-1 < q \leq p < \infty$.*

Then if $\mathbb{M}$ is (ω, r)-parabolic then N is s-parabolic where $r = \frac{q}{q-(n-1)}$, $s = \frac{p}{p-(n-1)}$, $\omega(x) = \theta^{-\frac{n-1}{q-(n-1)}}(x)$.

Non-linear elasticity theory, J. Ball approach [10]:

For hyperelastic materials a typical boundary-value problem takes the form of finding a deformation $\psi : \Omega \to \mathbb{R}^n$ making the integral

$$I(\varphi) = \int_\Omega W(x, D\psi(x))\, dx$$

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$\mathcal{A}_B = \{\varphi \in W^1_p(\Omega),\ I(\varphi) < \infty,\ \varphi|_\Gamma = \overline{\varphi}|_\Gamma$ a. e. on $\Gamma = \partial\Omega$,
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$\mathcal{A}_B = \{\varphi \in W_p^1(\Omega),\ I(\varphi) < \infty,\ \varphi|_\Gamma = \overline{\varphi}|_\Gamma$ a. e. on $\Gamma = \partial\Omega$, $J(x, \varphi) > 0$ a. e. in $\Omega\}$

The integrand $W(x, F)$ is the stored-energy function. It is polyconvex and meets a coercitive estimate: if $\det F > 0$ then $W(x, F) \geq \alpha(||F||^p + ||\operatorname{adj} F||^q + (\det F)^r + (\det F)^{-s}) + g(x)$ for some $\alpha > 0$, $p > n$, $q > n$, $r > 1$, $s > \frac{(n-1)q}{q-n}$ and $g \in L_1(\Omega)$.

A new class of admissible deformations

Let $\overline{\varphi} : \Omega \to \Omega'$, $\overline{\varphi} \in W^1_n(\Omega)$, $M \in L_1(\Omega)$. Consider a new class of admissible deformations:

$$\mathcal{A} = \{\varphi \in W^1_n(\Omega) \text{ has finite distortion, } J(x,\varphi) \in L_r(\Omega), \\ \frac{|D\varphi(x)|}{J(x,\varphi)^{1/n}} < M(x) \in L_{ns}(\Omega), \\ s > n-1,\ \varphi|_\Gamma = \overline{\varphi}|_\Gamma \text{ a. e. on } \Gamma,\ J(x,\varphi) \geq 0 \text{ a. e. in } \Omega\}$$

Polyconvexity and coercitivity

$\mathbb{M}^n_{\geq 0}$ is a set of $(n \times n)$-matrixes with nonnegative determinant.

Polyconvexity: A function $W : \Omega \times \mathbb{M}^n \to \mathbb{R}$ is called

polyconvex if there exists a convex function $G(x, \cdot) : \mathbb{M}^n \times \mathbb{M}^n \times \mathbb{R}_+ \to \mathbb{R}$ such that for all $F \in \mathbb{M}^n_{\geq 0}$ the equality

$$G(x, F, \operatorname{adj} F, \det F) = W(x, F) \quad \text{holds.}$$

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Coercitivity: There exist constants $\alpha > 0$, $r > 1$ and

a function $g \in L_1(\Omega)$ such that

$$W(x, F) \geq \alpha(\|F\|^n + (\det F)^r) + g(x)$$

for almost all $x \in \Omega$ and for all $F \in \mathbb{M}^n_{\geq 0}$.

The main theorem

Theorem 12 [11]. Let Ω and Ω' be bounded domains with Lipschitz boundaries and

1) the stored-energy function $W(x, F)$ be polyconvex and coercive,
2) $\overline{\varphi} : \overline{\Omega} \to \overline{\Omega'}$ be a homeomorphism,
3) the set $\mathcal{A}$ be nonempty and $\inf\limits_{\psi \in \mathcal{A}} I(\psi) < \infty$.

Then there exists a homeomorphism $\varphi \in \mathcal{A}$ such that

$$I(\varphi) = \inf_{\psi \in \mathcal{A}} I(\psi).$$

Example:

$F \in \mathbb{M}^3_{\geq 0}$

$$W(F) = \alpha\,\mathrm{trace}(F^T F)^{\frac{3}{2}} + c(\det F)^r, \quad r > 1.$$

$W(F)$ meets the coercitive condition

$$W(F) \geq \alpha\,\|F\|^3 + c(\det F)^r$$

but it does not meet the coercitive condition by Ball

$$W(x, F) \geq \alpha\,(\|F\|^p + \|\,\mathrm{adj}\,F\|^q + (\det F)^r) + g(x)$$

for some $p, q > 3$ and $r > 1$.

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