Domain Specific Languages for Convex Optimization

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International Conference on Optimization
and Applications in Control and Data Science

May 13 2015, Moscow, Russia

On the occasion of Boris Polyak's 80th birthday

Outline

Constructive convex analysis

Cone representation

Canonicalization

Modeling frameworks

Conclusions

How do you solve a convex problem?

- use someone else's ('standard') solver (LP, QP, SOCP, . . .)
  - easy, but your problem **must** be in a standard form
  - cost of solver development amortized across many users
- write your own (custom) solver
  - lots of work, but can take advantage of special structure
- transform your problem into a standard form, and use a standard solver
  - extends reach of problems solvable by standard solvers
- **this talk:** methods to formalize and automate last approach

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How can you tell if a problem is convex?

approaches:

- use basic definition, first or second order conditions, *e.g.*, $\nabla^2 f(x) \succeq 0$
- via convex calculus: construct f using
  - library of basic functions that are convex
  - calculus rules or transformations that preserve convexity

Convex functions: Basic examples

- x^p ($p \geq 1$ or $p \leq 0$), $-x^p$ ($0 \leq p \leq 1$)
- e^x, $-\log x$, $x \log x$
- $a^T x + b$
- $x^T P x$ ($P \succeq 0$)
- $\|x\|$ (any norm)
- $\max(x_1, \ldots, x_n)$

Convex functions: Less basic examples

- $x^T x/y$ $(y > 0)$, $x^T Y^{-1} x$ $(Y \succ 0)$
- $\log(e^{x_1} + \cdots + e^{x_n})$
- $-\log \Phi(x)$ (Φ is Gaussian CDF)
- $\log \det X^{-1}$ $(X \succ 0)$
- $\lambda_{\max}(X)$ $(X = X^T)$
- $f(x) = x_{[1]} + \cdots + x_{[k]}$ (sum of largest k entries)

Calculus rules

- **nonnegative scaling**: f convex, $\alpha \geq 0 \implies \alpha f$ convex
- **sum**: f, h convex $\implies f + g$ convex
- **affine composition**: f convex $\longrightarrow f(Ax + b)$ convex
- **pointwise maximum**: $f_1, \ldots, f_m$ convex $\implies \max_i f_i(x)$ convex
- **partial minimization**: $f(x, y)$ convex $\implies \inf_y f(x, y)$ convex
- **composition**: h convex increasing, f convex $\implies h(f(x))$ convex

A general composition rule

$h(f_1(x), \ldots, f_k(x))$ is convex when h is convex and for each i

- h is increasing in argument i, and f_i is convex, or
- h is decreasing in argument i, and f_i is concave, or
- f_i is affine

- there's a similar rule for concave compositions
- this one rule subsumes most of the others
- in turn, it can be derived from the partial minimization rule

Constructive convexity verification

- start with function given as **expression**
- build parse tree for expression
  - leaves are variables or constants/parameters
  - nodes are functions of children, following general rule
- tag each subexpression as convex, concave, affine, constant
  - variation: tag subexpression signs, use for monotonicity
    e.g., $(\cdot)^2$ is increasing if its argument is nonnegative
- sufficient (but not necessary) for convexity

Example

for $x < 1$, $y < 1$

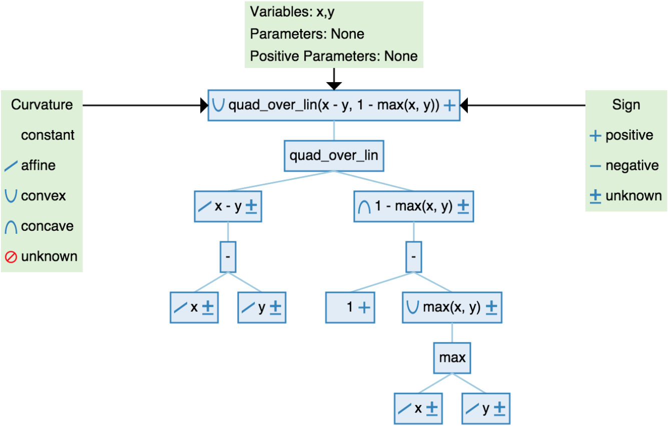
$$\frac{(x-y)^2}{1-\max(x,y)}$$

is convex

- (leaves) x, y, and 1 are affine expressions
- $\max(x,y)$ is convex; $x-y$ is affine
- $1-\max(x,y)$ is concave
- function u^2/v is convex, monotone decreasing in v for $v>0$
  hence, convex with $u=x-y$, $v=1-\max(x,y)$

Example

analyzed by `dcp.stanford.edu` (*Diamond 2014*)



Disciplined convex programming (DCP)

- framework for describing convex optimization problems
- based on constructive convex analysis
- sufficient but not necessary for convexity
- basis for several domain specific languages and tools for convex optimization

Disciplined convex program: Structure

a DCP has

- zero or one **objective**, with form
  - minimize {scalar convex expression} or
  - maximize {scalar concave expression}

- zero or more **constraints**, with form
  - {convex expression} <= {concave expression} or
  - {concave expression} >= {convex expression} or
  - {affine expression} == {affine expression}

Disciplined convex program: Expressions

- expressions formed from
  - **variables**,
  - **constants/parameters**,
  - and **functions** from a library
- library functions have known convexity, monotonicity, and sign properties
- all subexpressions match general composition rule

Disciplined convex program

- a valid DCP is
  - convex-by-construction (cf. posterior convexity analysis)
  - 'syntactically' convex (can be checked 'locally')
- convexity depends only on attributes of library functions, and not their meanings
  - *e.g.*, could swap $\sqrt{\cdot}$ and $\sqrt[4]{\cdot}$, or $\exp \cdot$ and $(\cdot)_+$, since their attributes match

Outline

Cone representation

($\textit{Nesterov, Nemirovsky}$)

cone representation of (convex) function f:

- $f(x)$ is optimal value of cone program

$$
\begin{array}{ll}
\text{minimize} & c^T x + d^T y + e \\
\text{subject to} & A \begin{bmatrix} x \\ y \end{bmatrix} = b, \quad \begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{K}
\end{array}
$$

  - cone program in (x, y), we but minimize only over y
- *i.e.*, we define f by partial minimization of cone program

Examples

- $f(x) = -(xy)^{1/2}$ is optimal value of SDP

$$\begin{array}{ll} \text{minimize} & -t \\ \text{subject to} & \begin{bmatrix} x & t \\ t & y \end{bmatrix} \succeq 0 \end{array}$$

  with variable t

- $f(x) = x_{[1]} + \cdots + x_{[k]}$ is optimal value of LP

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T \lambda - k\nu \\ \text{subject to} & x + \nu \mathbf{1} = \lambda - \mu \\ & \lambda \succeq 0, \quad \mu \succeq 0 \end{array}$$

  with variables λ, μ, ν

SDP representations

Nesterov, Nemirovsky, and others have worked out SDP representations for many functions, *e.g.*,

- x^p, $p \geq 1$ rational
- $-(\det X)^{1/n}$
- $\sum_{i=1}^k \lambda_i(X)$ $(X = X^T)$
- $\|X\| = \sigma_1(X)$ $(X \in \mathbf{R}^{m \times n})$
- $\|X\|_* = \sum_i \sigma_i(X)$ $(X \in \mathbf{R}^{m \times n})$

some of these representations are not obvious . . .

Outline

Canonicalization

- start with problem in DCP form, with cone representable library functions
- **automatically** transform to equivalent cone program

Canonicalization: How it's done

- for each (non-affine) library function $f(x)$ appearing in parse tree, with cone representation

$$\begin{array}{ll} \text{minimize} & c^T x + d^T y + e \\ \text{subject to} & A \begin{bmatrix} x \\ y \end{bmatrix} = b, \quad \begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{K} \end{array}$$

  - add new variable y, and constraints above
  - replace $f(x)$ with affine expression $c^T x + d^T y + e$

- yields problem with linear equality and cone constaints
- DCP ensures equivalence of resulting cone program

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Example

- constrained least-squares problem with ℓ_1 regularization

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_2^2 + \gamma\|x\|_1 \\ \text{subject to} & \|x\|_\infty \leq 1 \end{array}$$

  - variable $x \in \mathbf{R}^n$
  - constants/parameters A, b, $\gamma > 0$

CVX

- developed by M. Grant
- embedded in Matlab; targets multiple cone solvers
- CVX specification for example problem:

```
cvx_begin
  variable x(n)   % declare vector variable
  minimize sum(square(A*x-b,2)) + gamma*norm(x,1)
  subject to norm(x,inf) <= 1
cvx_end
```

- here A, b, γ are **constants**

Some functions in the CVX library

function	meaning	attributes
`norm(x, p)`	$\|x\|_p$, $p \geq 1$	cvx
`square(x)`	x^2	cvx
`square_pos(x)`	$(x_+)^2$	cvx, nondecr
`pos(x)`	x_+	cvx, nondecr
`sum_largest(x,k)`	$x_{[1]} + \cdots + x_{[k]}$	cvx, nondecr
`sqrt(x)`	$\sqrt{x}$, $x \geq 0$	ccv, nondecr
`inv_pos(x)`	$1/x$, $x > 0$	cvx, nonincr
`max(x)`	$\max\{x_1, \ldots, x_n\}$	cvx, nondecr
`quad_over_lin(x,y)`	x^2/y, $y > 0$	cvx, nonincr in y
`lambda_max(X)`	$\lambda_{\max}(X)$, $X = X^T$	cvx
`huber(x)`	$\begin{cases} x^2, & \|x\| \leq 1 \\ 2\|x\| - 1, & \|x\| > 1 \end{cases}$	cvx

CVXPY

- developed by S. Diamond
- embedded in Python; targets multiple cone solvers

- CVXPY specification for example problem:

```
from cvxpy import *
x = Variable(n)
cost = sum_squares(A*x-b) + gamma*norm(x,1)
obj = Minimize(cost)
constr = [norm(x,"inf") <= 1]
prob = Problem(obj,constr)
opt_val = prob.solve()
solution = x.value
```

Parameters in CVXPY

- symbolic representations of constants
- can specify sign
- change value of constant without re-parsing problem

- computing a trade-off curve for example problem:

```
x_values = []
for val in numpy.logspace(-4, 2, num=100):
    gamma.value = val
    prob.solve()
    x_values.append(x.value)
```

Signed DCP in CVXPY

function	meaning	attributes
`norm(x, p)`	$\|x\|_p,\ p \geq 1$	cvx, nondecr for $x \geq 0$, nonincr for $x \leq 0$
`square(x)`	x^2	cvx, nondecr for $x \geq 0$, nonincr for $x \leq 0$
`huber(x)`	$\begin{cases} x^2, & \|x\| \leq 1 \\ 2\|x\| - 1, & \|x\| > 1 \end{cases}$	cvx, nondecr for $x \geq 0$, nonincr for $x \leq 0$

Outline

Conclusions

- DCP is a formalization of constructive convex analysis
  - simple method to certify problem as convex
  - basis of several domain specific languages for convex optimization

- modeling frameworks make rapid prototyping easy

Happy Birthday Boris!