A.F.FILIPPOV METHOD FOR OPTIMIZATION WITH NON-SMOOTH PENALTY FUNCTIONS

Vadim Utkin, Ohio State University

MOSCOW, MAY 13, 2015

Conventional Problem Statement

Find $\boldsymbol{min f(x)}$, $x \in R^n$, if

- $\boldsymbol{h_i(x) = 0 \quad i = 1, \ldots, m}$
- $h_i(x) \leq 0 \quad i = m+1, \ldots, m+l$

$F(x) = f(x) + \boldsymbol{\lambda H(x)}$, $\boldsymbol{H(x)}$ is a smooth function

$\boldsymbol{H(x)}$ – PENALTY FUNCTION

$\boldsymbol{H(x) = 0}$ ***in admissible domain***

$\boldsymbol{H(x) > 0}$ *in forbidden domain*

If $\boldsymbol{\lambda > 0}$ tends to infinity, then $\boldsymbol{x}$ tends to the solution $\boldsymbol{x}^*$

NON-SMOOTH PENALTY FUNCTION

$$\boldsymbol{H(x) = h^T(x)u},$$

$$h^T = (h_1, \dots, h_{m+l}),\ u^T = (u_1, \dots, u_{m+l}),$$

$$u_i = \begin{cases} \boldsymbol{\lambda_i} & \boldsymbol{if\ \ h_i > 0} \\ \boldsymbol{-\lambda_i} & \boldsymbol{if\ \ h_i < 0} \end{cases} \quad i = 1, \dots, m$$

$$u_i = \begin{cases} \boldsymbol{\lambda_i} & \boldsymbol{if\ \ h_i > 0} \\ 0 & if\ \ h_i < 0 \end{cases} \quad i = m+1, \dots, m+l$$

λ_i>0, λ_i-const

x^*: $\min F(x) = \min[f(x) + H(x)] = F(x^*)$

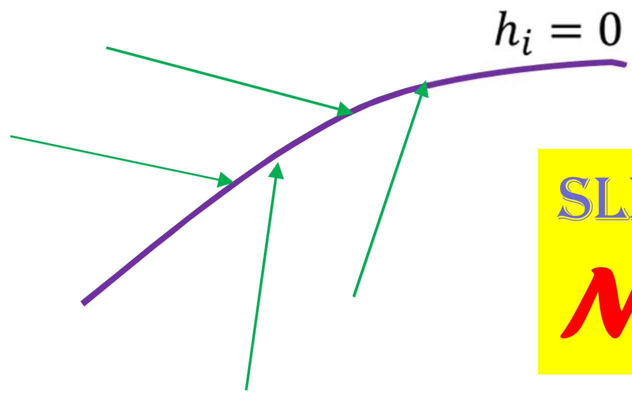
Theorem. *There exists λ_0, such that x^* is the solution to the original problem*, if all $\lambda_i > \lambda_0$.

Gradient search

$$\min F(x) = \min[f(x) + h^T(x)u]:$$

$$\dot{x} = -grad[f(x)] - G^T u, \; G = \{\partial h/\partial x\}$$

$$u_i = \begin{cases} \lambda_i & if \; h_i > 0 \\ -\lambda_i & if \; h_i < 0 \end{cases} \; i = 1, \dots, m, \quad u_i = \begin{cases} \lambda_i & if \; h_i > 0 \\ 0 & if \; h_i < 0 \end{cases} \; i = m+1, \dots, m+l$$

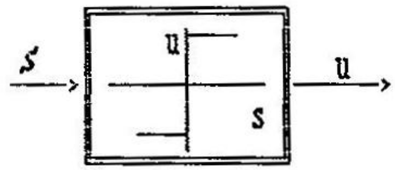


$$\lim_{h_i \to +0} \dot{h}_i < 0, \quad \lim_{h_i \to -0} \dot{h}_i > 0$$

SLIDING MODE

Motion Equations ???

RELAY ON - OFF CONTROL BANG - BANG



$$u = u_0 \operatorname{sign} s$$

I. Flügge - Lotz
1953

Ya. Tsypkin
1955

DISCONTINUOUS AUTOMATIC CONTROL

By IRMGARD FLÜGGE-LOTZ

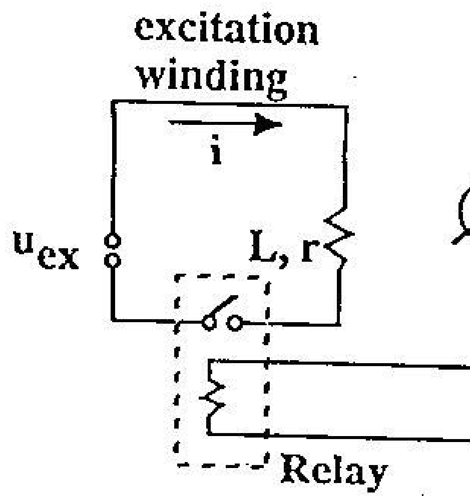
1953
PRINCETON UNIVERSITY PRESS
PRINCETON, NEW JERSEY

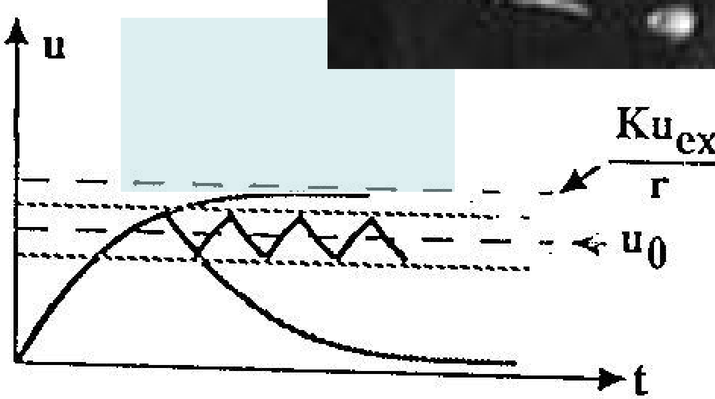
Я. З. ЦЫПКИН

ТЕОРИЯ
РЕЛЕЙНЫХ СИСТЕМ
АВТОМАТИЧЕСКОГО
РЕГУЛИРОВАНИЯ

ГОСУДАРСТВЕННОЕ ИЗДАТЕЛЬСТВО
ТЕХНИКО-ТЕОРЕТИЧЕСКОЙ ЛИТЕРАТУРЫ
МОСКВА 1955

1. VIBRATIONAL CON
AEROCRAFT D.C. GE
(KULEBAKIN V. 1932)



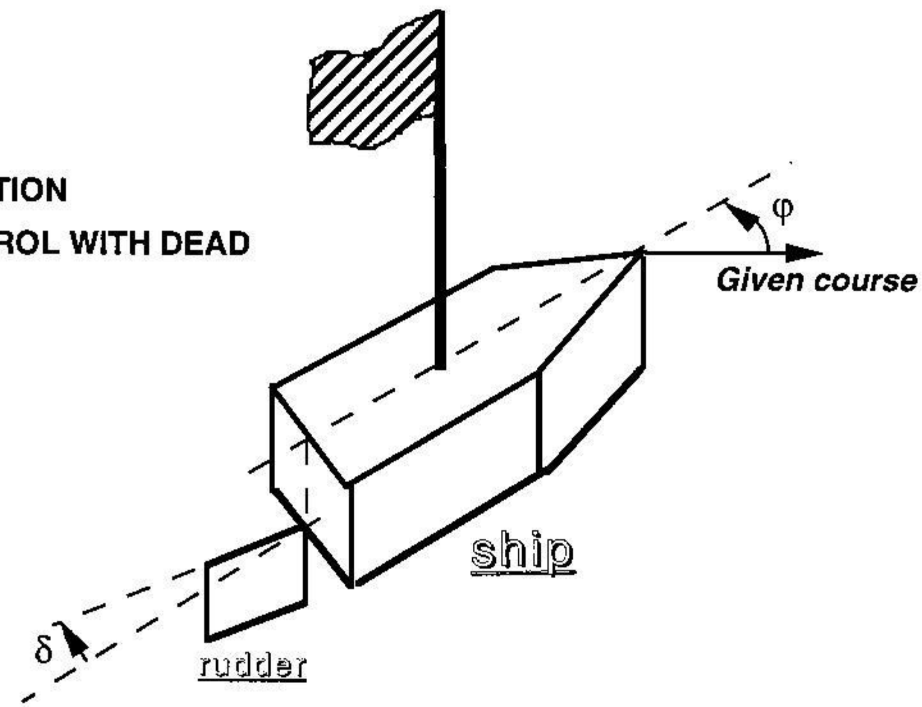


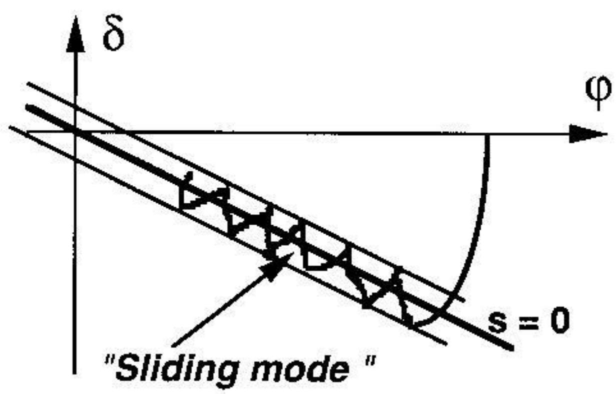
2. ON AUTOMATIC STABILITY OF A SHIP ON A GIVEN COURSE (NIKOLSKI G., 1934)

φ - SHIP COURSE

δ - RUDDER POSITION

u - ON-OFF CONTROL WITH DEAD ZONE





"PHASE PLANE"

"SWITCHING LINE "

Introduction of Sliding Mode Control

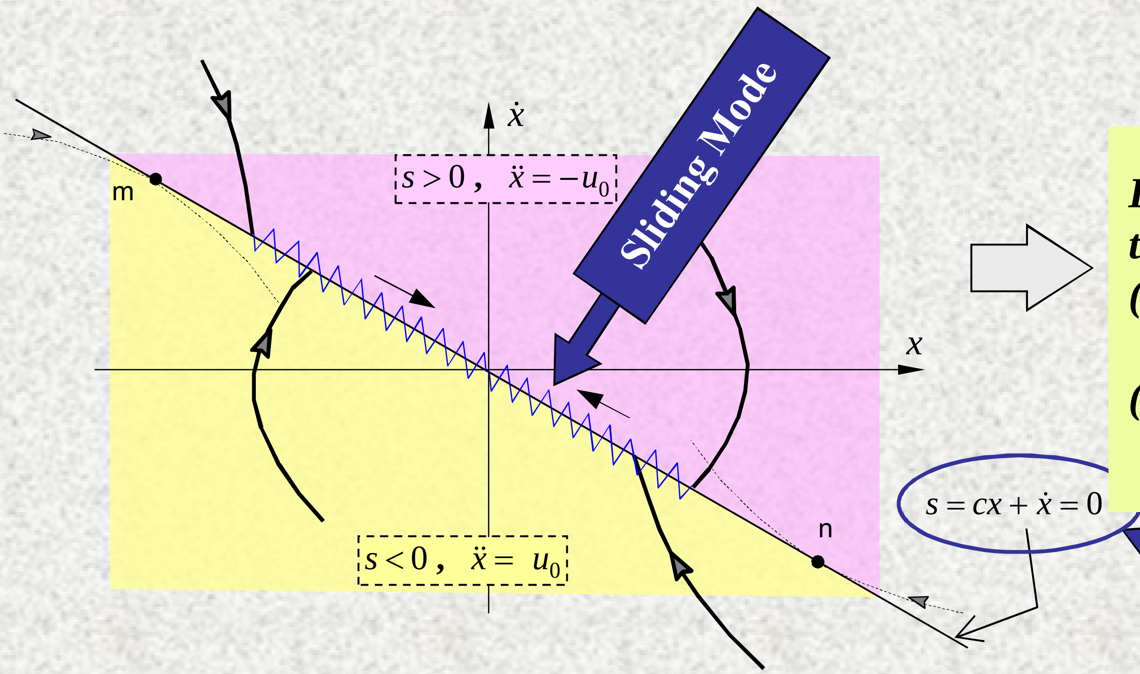
■ Concept of Sliding Mode (Second order relay system)

$$\ddot{x} = u,$$
$$u = -u_0 \operatorname{sgn}(s), \quad s = cx + \dot{x}, \quad u_0, c : \text{const}$$

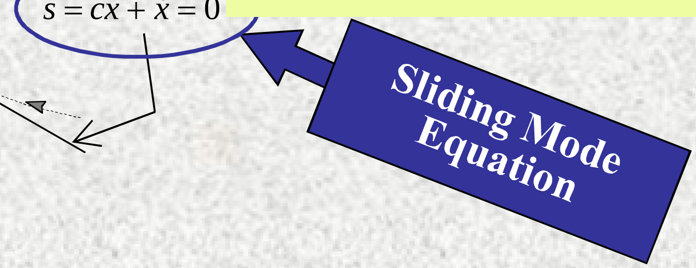
Upper semi-plane : $s > 0 \rightarrow u = -u_0 \rightarrow \ddot{x} = -u_0$

Lower semi-plane : $s < 0 \rightarrow u = u_0 \rightarrow \ddot{x} = u_0$

- State trajectories are towards the line switching line *s=0*
- State trajectories cannot leave and belong to the switching line *s=0* : sliding mode
- After sliding mode starts, further motion is governed by $s = cx + \dot{x} = 0$: sliding mode equation



In sliding mode,
the system motion is
(1) governed by 1st order equation (reduced order).
(2) depending only on 'c' not plant dynamics.



Canonical Space

$$x^{(n)} + a_n x^{(n-1)} + a_{n-1} x^{(n-2)} + \ldots + a_1 x = b\,u + f(t)$$

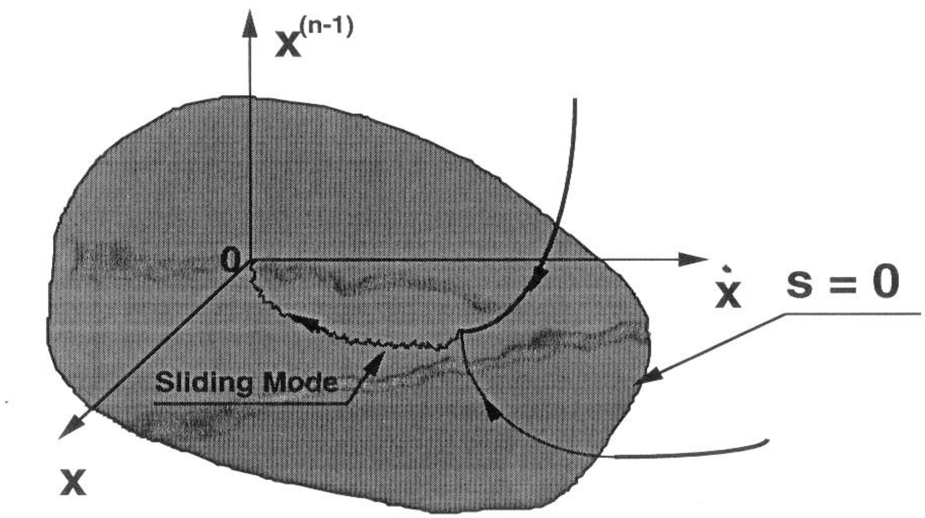
$a_i(t)$ and $b(t)$ are unknown parameters

$f(t)$ is disturbance

Control $u = \begin{cases} u^{+}(x,t) & \text{if } s > 0 \\ u^{-}(x,t) & \text{if } s < 0 \end{cases}$

$$u^{+}(x,t) \neq u^{-}(x,t)$$

$$s = x^{(n-1)} + c_{n-1} x^{(n-2)} + \ldots + c_1 x$$

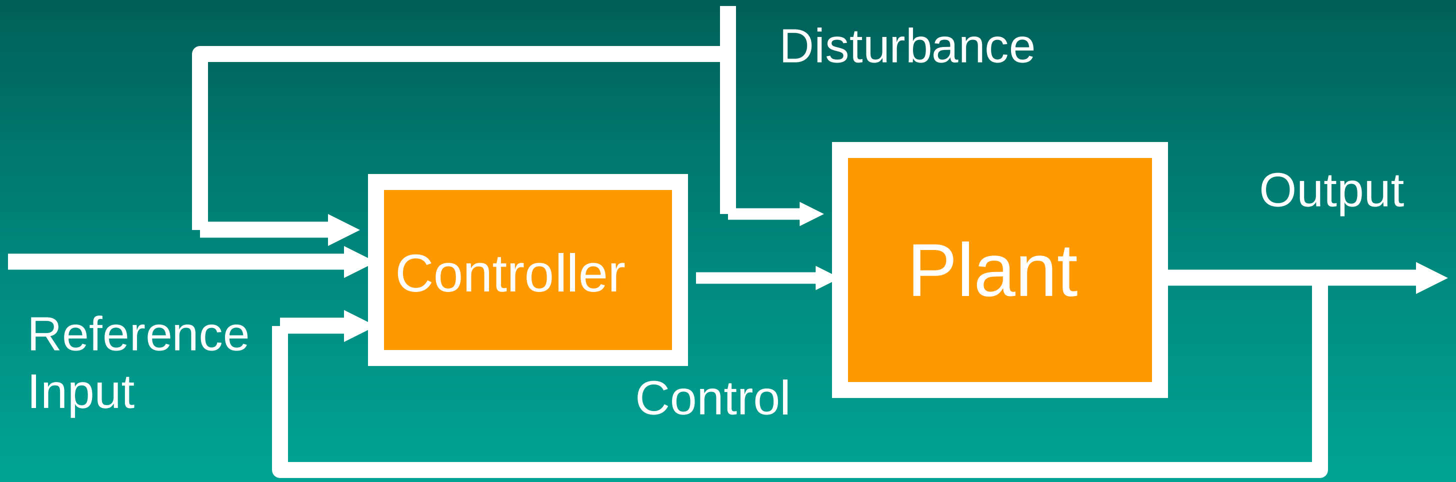


Sliding Mode Equation **does not depend** **on parameters and disturbances**

$$x^{(n-1)} + c_{n-1} x^{(n-2)} + \ldots + c_1 x = 0$$

SECOND STAGE

Finite Dimensional MIMO Systems



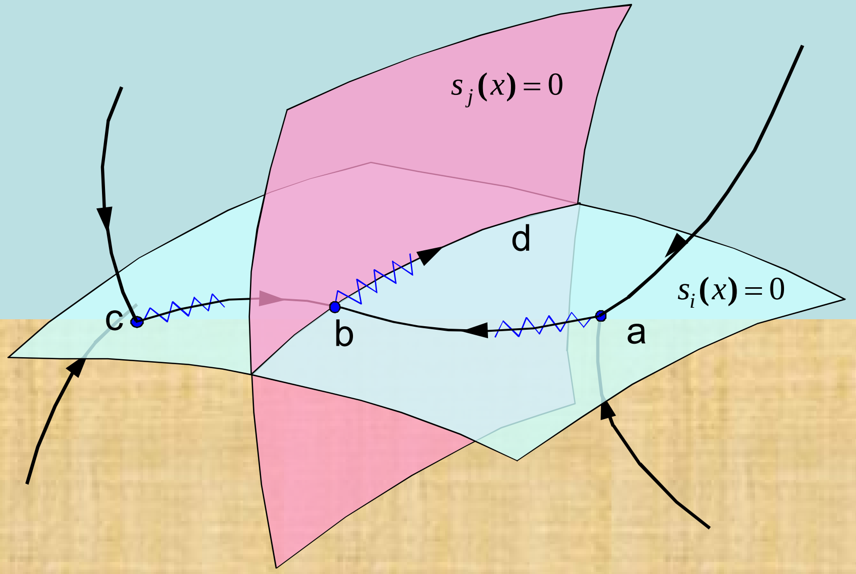
$$\dot{x} = f(x,t,u) \quad x \in R^n \qquad u \in R^m$$

$$u_i = \begin{cases} u_i^+(x,\mathrm{t}) & \text{if} \quad s_i(x) > 0 \\ u_i^-(x,t) & \text{if} \quad s_i(x) < 0 \end{cases}, \qquad i = 1,\ldots,m$$

$$u_i^+(x,\mathrm{t}) \neq u_i^-(x,t)$$

Introduction of Sliding Mode Control

Second Stage – Multidimensional Sliding Modes



Sliding mode in discontinuity surfaces and their intersection

Multi-dimensional Sliding Mode

- **In general case,**

$$\dot{x} = f(x,t) + B(x,t)u, \qquad x \in R^n, u \in R^m$$

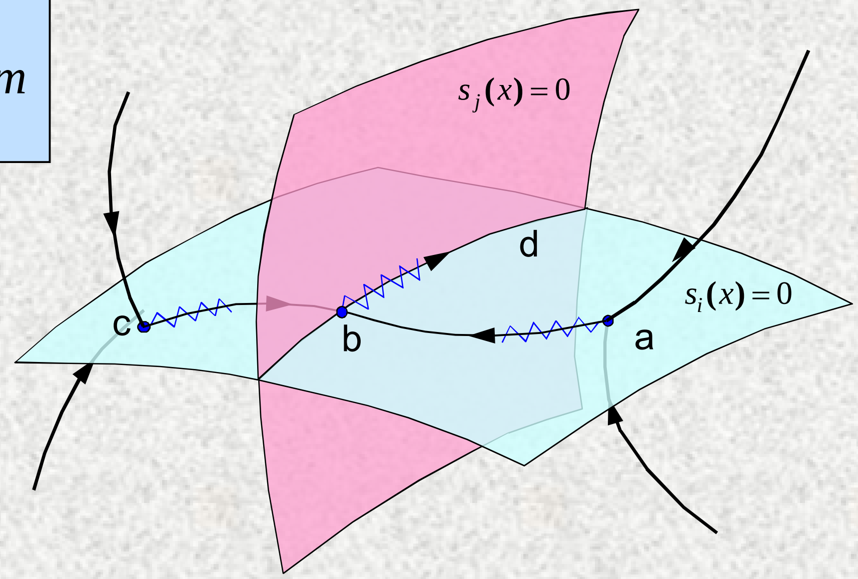
$$u_i = \begin{cases} u_i^+(x,t) & \text{if} \quad s_i(x) > 0 \\ u_i^-(x,t) & \text{if} \quad s_i(x) < 0 \end{cases} \qquad \text{i} = 1,\ldots,m$$

- ***Sliding mode occurs in the intersection of all m surfaces,***

$s_i(x) = 0 \quad (i = 1,..,m),$ *or*

in manifold $s(x) = 0,\ s^T = (s_1,\ldots,s_m).$

- ***The order of motion equation is by 'm' less than the order of original system***



Sliding mode in discontinuity surfaces and their intersection

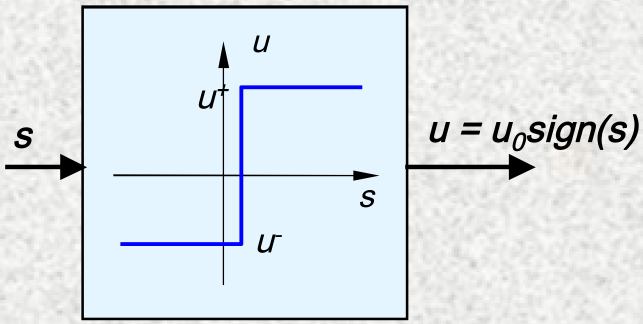
Outline of Sliding Mode Methodology

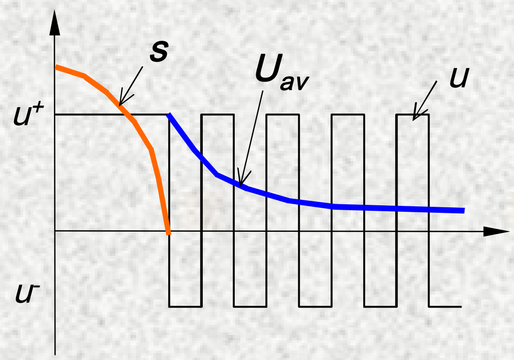
- **Order reduction of the system**

  ⇨ Enables a designer to simplify and decouple the design procedure.

- **Insensitivity with respect to parameter variations and disturbances**

  ⇨ Attained by finite control actions, unlike continuous high gain controls in conventional system.





$$K_{eq} = \frac{U_{av}}{s} \approx \infty \qquad (\because s \to 0)$$

Mathematical Aspects I

Sliding Mode Equations

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + b_1 u + d_1 f(t)$$
$$\dot{x}_2 = a_{12}x_1 + a_{22}x_2 + b_2 u + d_2 f(t)$$

$$u = -M sign(s), \; s = c_1 x_1 + c_2 x_2.$$

Sliding Mode Equation on s=0 ???

$$\ddot{x} = u, \; u = -u_0 sign(s), \; s = \dot{x} + cx \qquad \dot{x} + cx = 0$$ WHY ???

Yu. Kornilov (1950),

A. Popovski (1950)

$$b_1 = 0 \quad x_2 = -c_2^{-1} c_1 x_1$$

$$\dot{x}_1 = (a_{11} - a_{12}c_2^{-1}c_1)x_1 + d_1 f(t).$$

Ya. Dolgolenko (1955)

$$\dot{x} = Ax + bu, \qquad b^T = (0, 0, ..., 0, 1).$$

FILIPPOV METHOD 1958 →

MATHEMATICAL ASPECTS I
SLIDING MODE EQUATIONS (CONT).

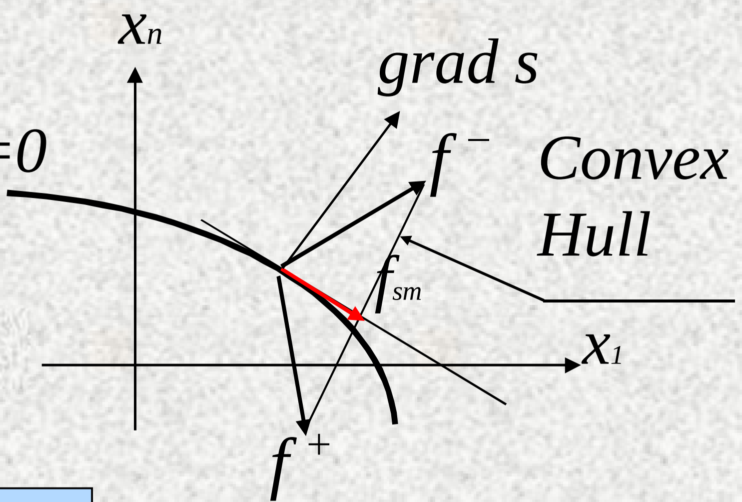
A.F. Filippov, Application of the theory of differential equations with discontinuous right-hand sides to non-linear problems of automatic control, *Proceedings of 1st IFAC Congress in Moscow, 1960,* Butterworths, London, 1961.

$$\dot{x} = f(x), \quad f(x) = \begin{cases} f^{+} & \text{if } s(x) > 0 \\ f^{-} & \text{if } s(x) < 0 \end{cases} \qquad s(x)=0$$

f_{sm} *belongs to convex hull*

$dx/dt = f_{sm}$

$$If\ \dot{x} = f(x,u,t), \qquad f_{sm} \in conv_u f(x,u,t)$$



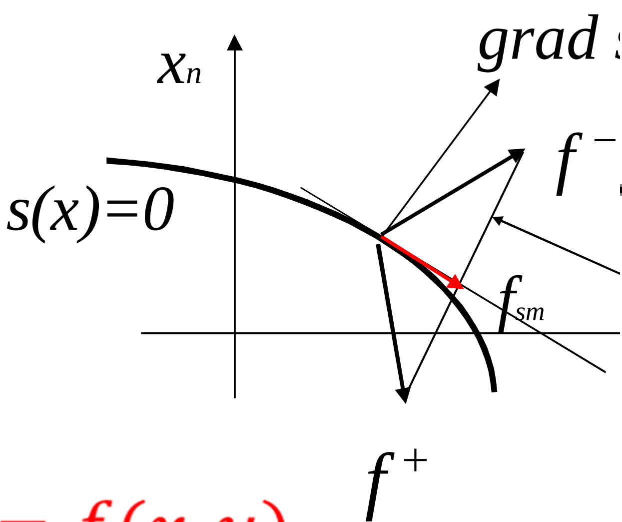
$$\ddot{x} = u,$$
$$u = -u_0 \operatorname{sgn}(s), \quad s = cx + \dot{x}, \quad u_0, c : \text{const}$$

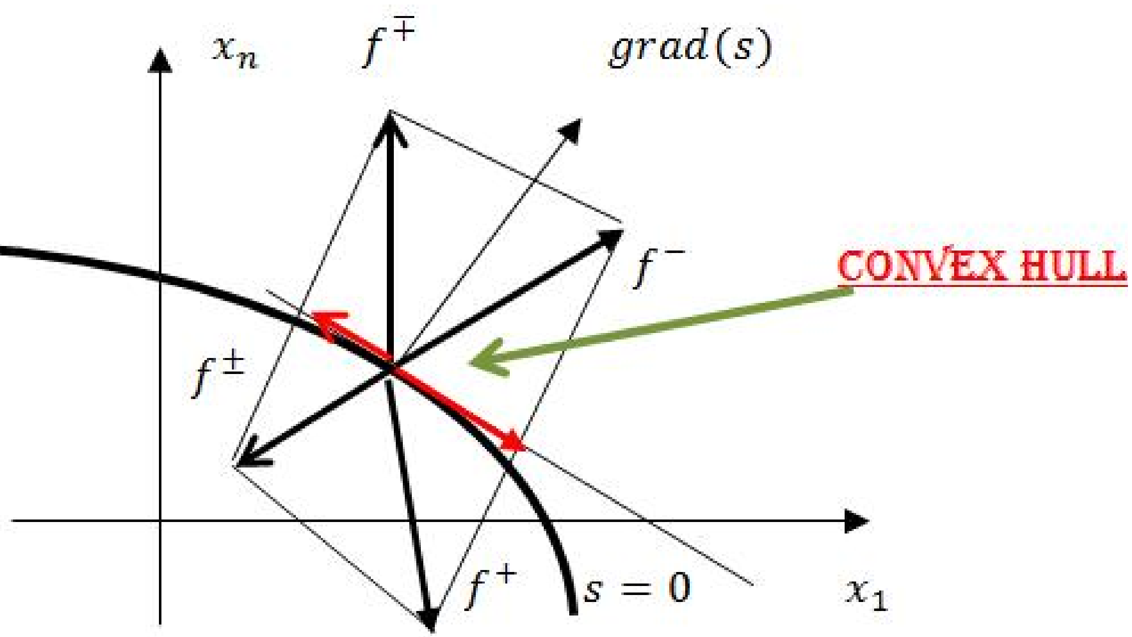
Sliding mode equation $\dot{x} + cx = 0$ results from **Filippov method**

1st IFAC Congress.

dx/dt=Ax+bu+dv,
u=-sign(s), v=-sign(s),
s=cx

NONUNIQUENESS !?





$$\dot{x} = f(x,u)$$

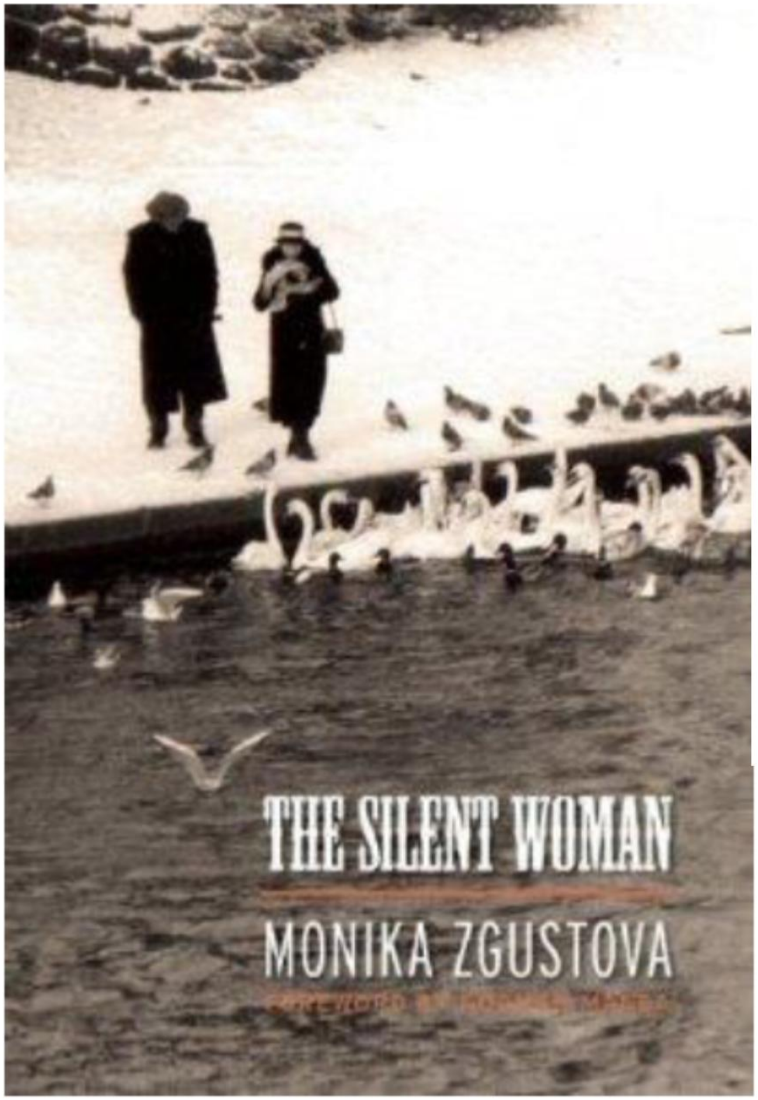
$f_i \to \Delta t_i, i = 1, \dots, k,$

$$\sum_1^k \Delta t_i = \Delta t, \qquad f_{sm} = lim_{\Delta t \to 0} \frac{\sum_1^k f_i \Delta t_i}{\Delta t} = \sum_1^k f_i \mu_i,$$

$$\mu_i = \frac{\Delta t_i}{\Delta t} \geq 0, \sum_1^k \mu_i = 1$$

“В статье Филиппова 1958 года рассматривались дифференцаильные уравнения с одной поверхностью разрыва правой части, а в вашей работе их несколько. Не уверен, что этот метод применим в вашем случае”.

$x = f\ sm,$



Fortner replied as if he were a schoolteacher, explaining something that everyone else already understood to a not very gifted student.

"Yes, I know Filippov's Theorem, of course," Fortner said, slowly, disguising his impatience, "but if you look at this piece of research from 1958, you will realise that Filippov developed his theorem with just one discontinuous surface, whereas you, in your article, employ three! This is why I have some objections, at least of a formal nature, to your article

Equivalent Control Method

Control system $\dot{x} = f(x,u,t)$

Discontinuous control u enforces sliding mode in manifold $s=(s_1,\ldots,s_m)=0.$ $G(x)=\{\partial s/\partial x\}.$

Then solution to $\dot{s} = Gf(x,t,u) = 0 \Rightarrow u_{eq} = u_{eq}(x,t)$

is substituted to system equation

$$\dot{x} = f(x,t,u_{eq})$$

$s(x)=0 \longrightarrow x_2=s_0(x_1)$ $x_2, s_0 \in \Re^m$ $x_1 \in \Re^{n-m}$:

$$\dot{x}_1 = f_1[x_1,t,s_0(x_1)],\ f_1 \in \Re^{n-m}.$$

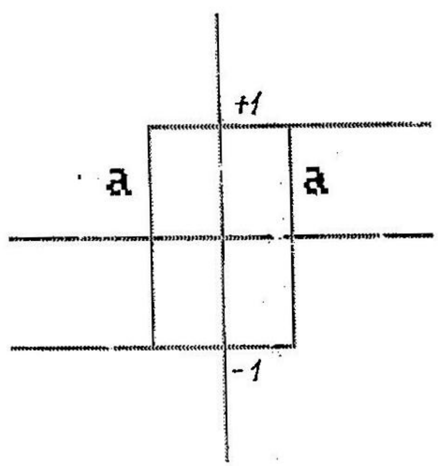
Uniqueness Problem

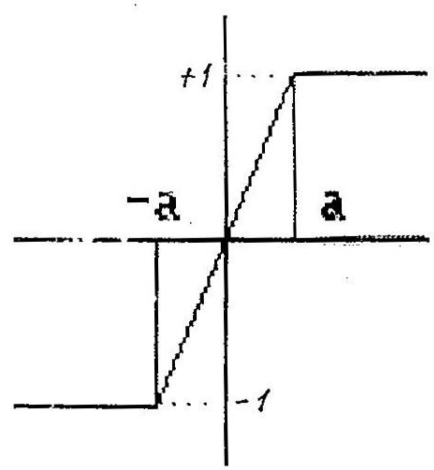
$$\dot{x}_1 = 0.3x_2 + ux_1$$

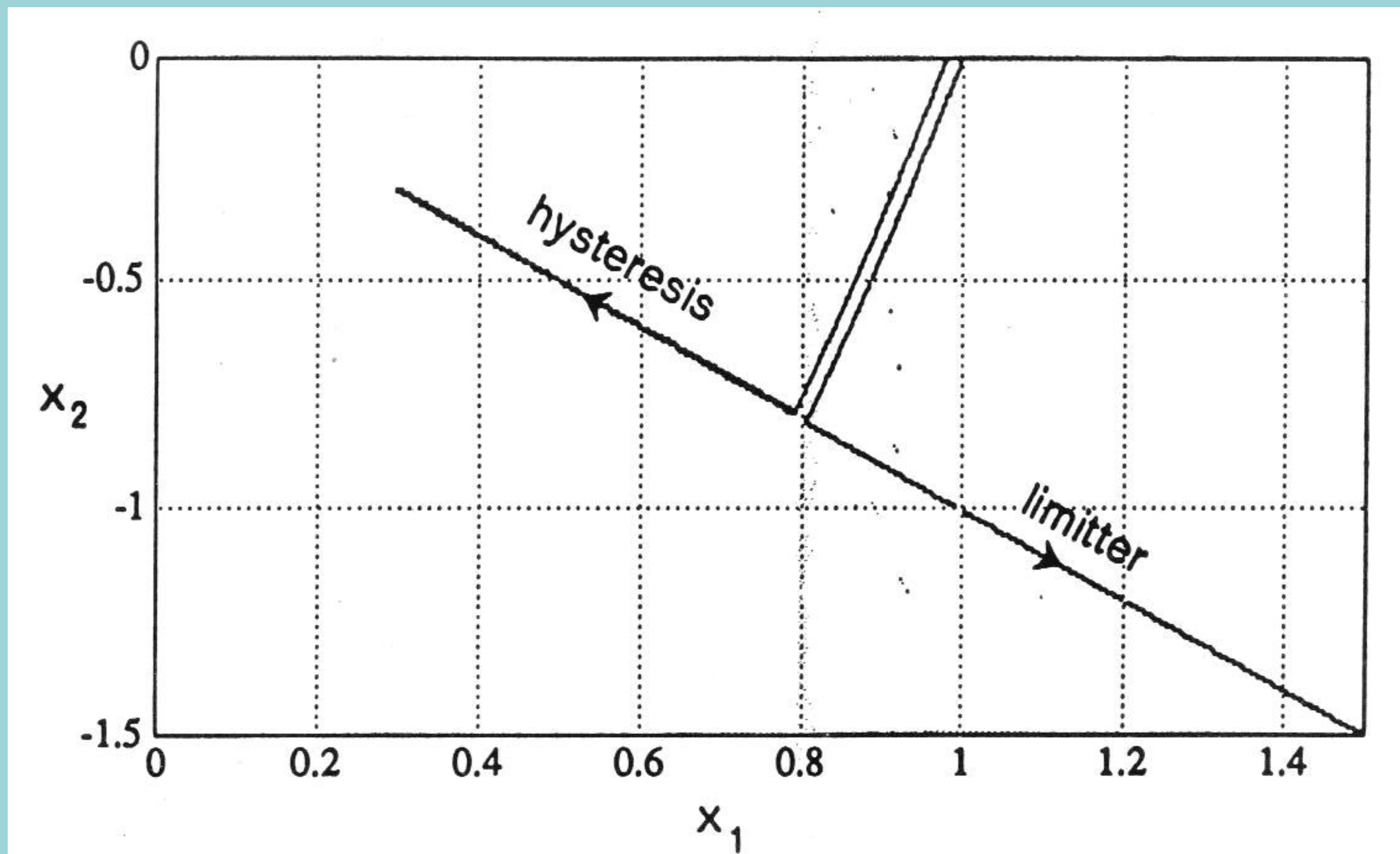
$$\dot{x}_2 = -0.7x_1 + 4u^3x_1,$$

$$u = -\mathrm{sign}(x_1 s),$$

$$s = x_1 + x_2.$$

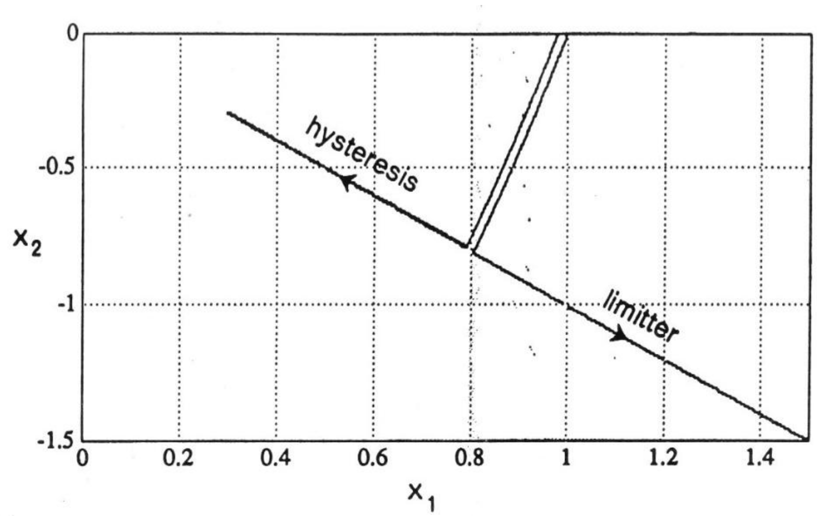


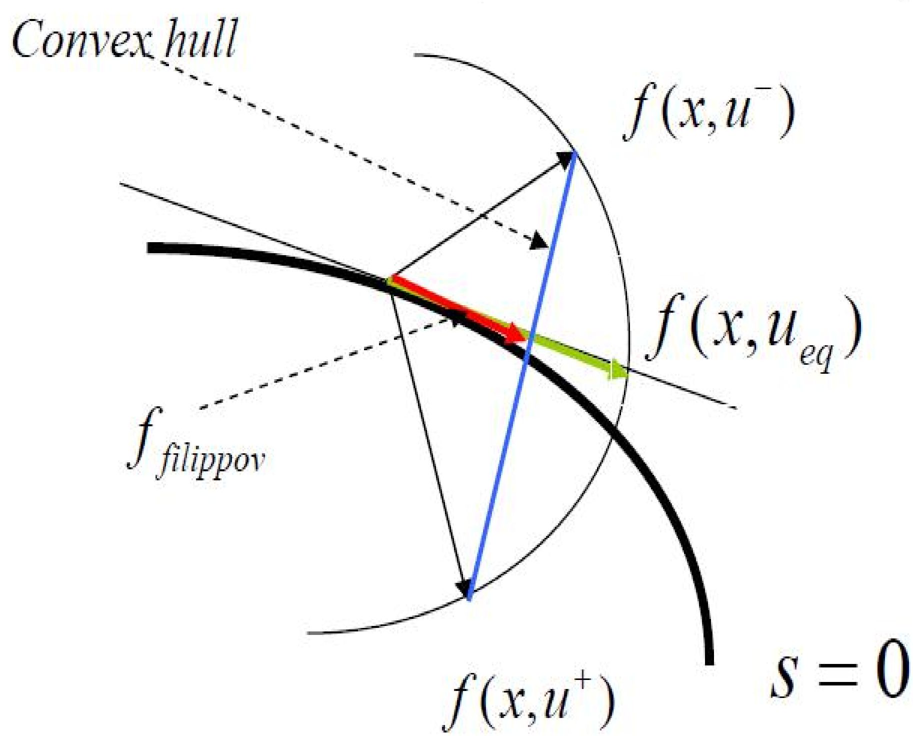


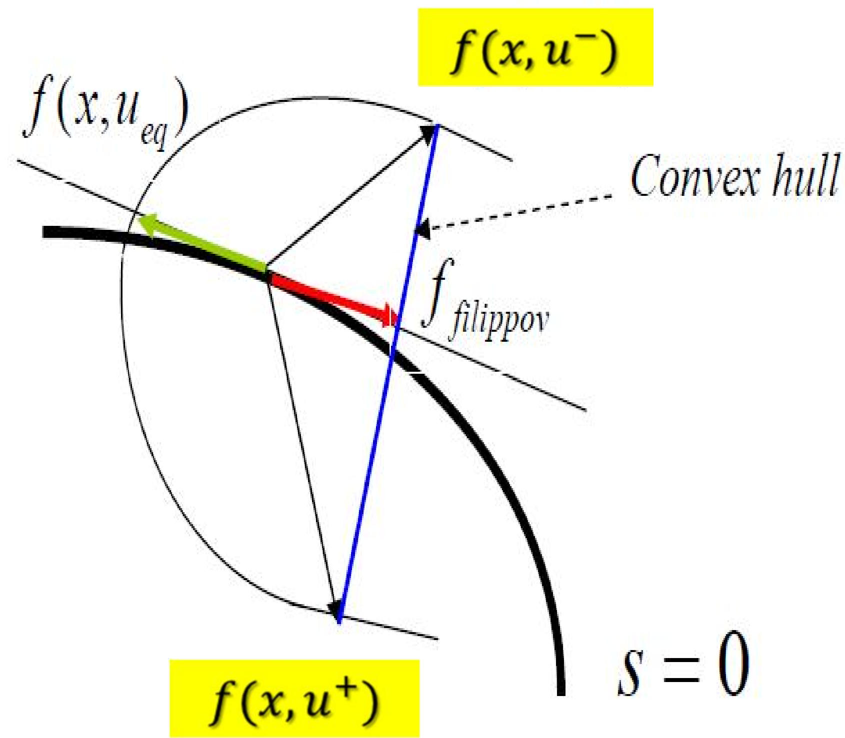


0
-0.5
-1
-1.5
x_2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
x_1
hysteresis
limitter

$$\dot{x} = f(x,u),\ u = \begin{cases} u^+ \text{ if } s(x) > 0 \\ u^- \text{ if } s(x) < 0 \end{cases}$$



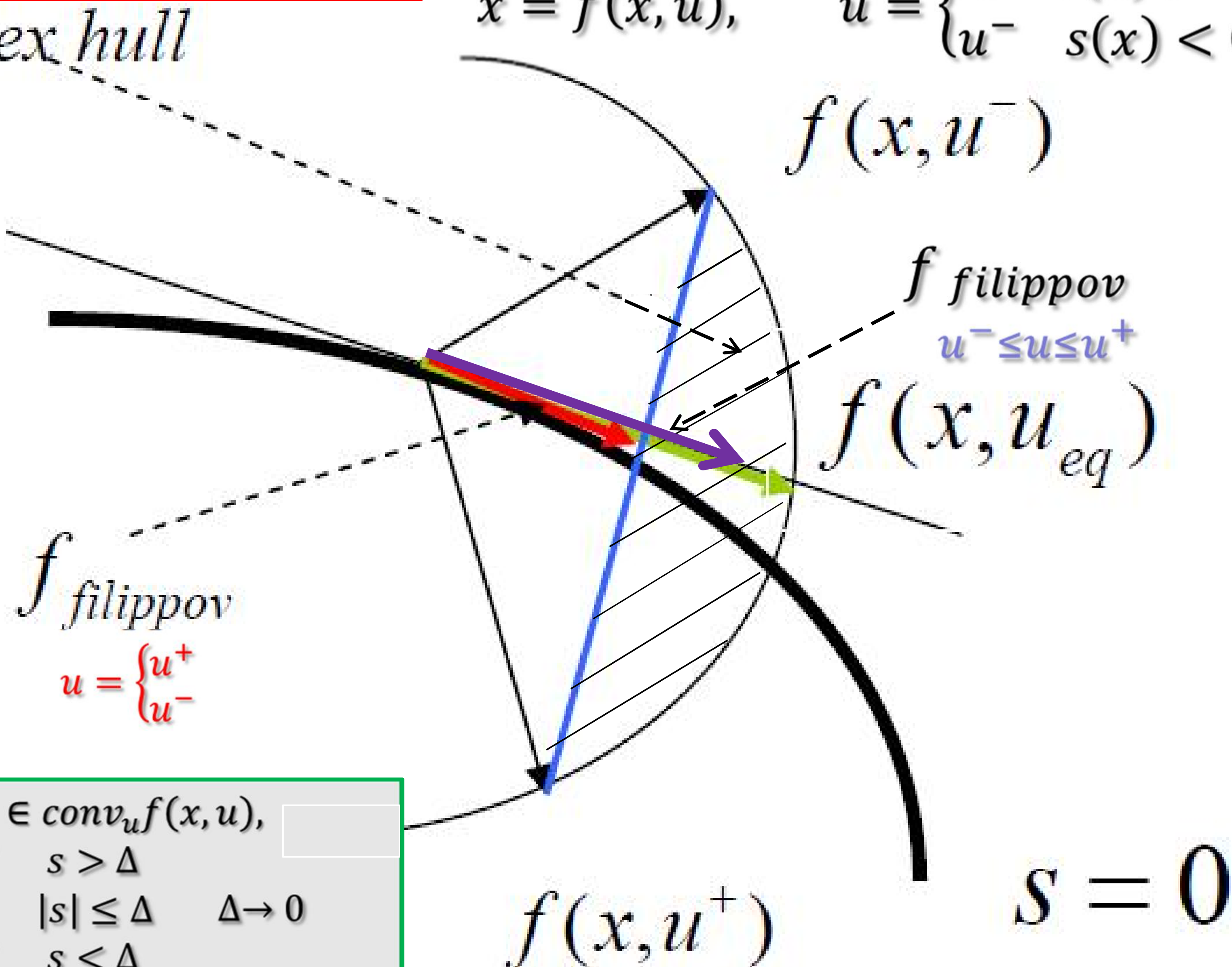




Мандельштам, 20-е годы

SLIDING MODE EQUATIONS

$$\dot{x} = f(x,u), \qquad u = \begin{cases} u^{+} & s(x) > 0 \\ u^{-} & s(x) < 0 \end{cases}$$

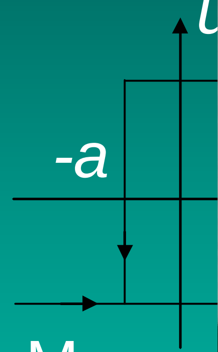


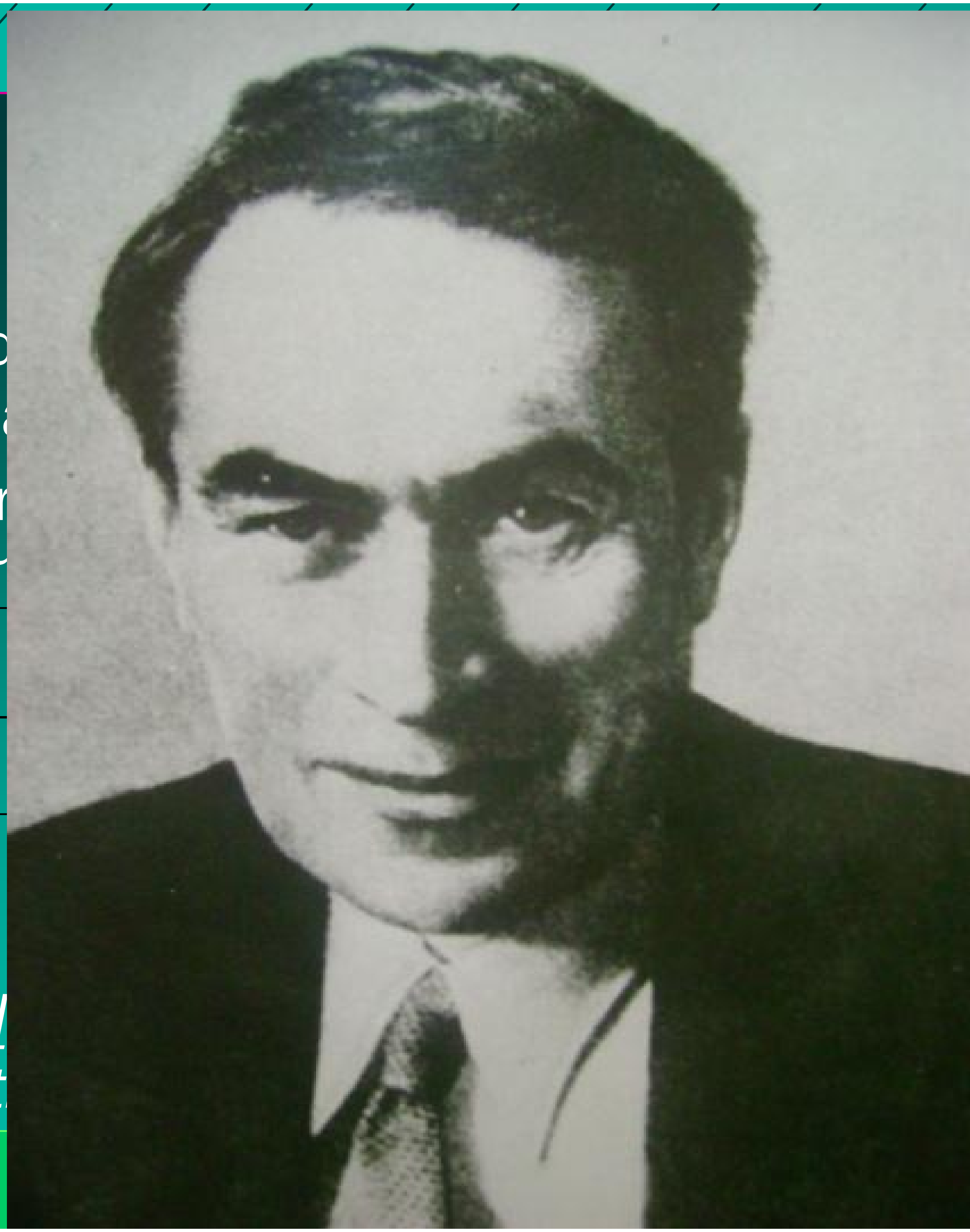
$$\dot{x} = f_{sm}, \; f_{sm} \in conv_u f(x,u),$$

$$u = \begin{cases} u^{+} & if \quad s > \Delta \\ \tilde{u} & if \quad |s| \le \Delta \qquad \Delta \to 0 \\ u^{-} & if \quad s < \Delta \end{cases}$$

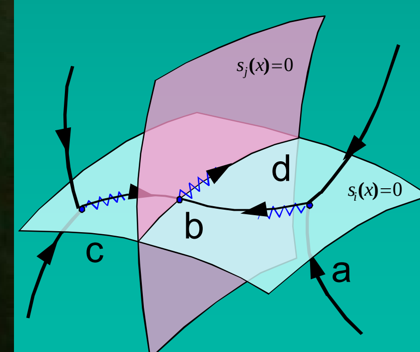
$\tilde{u} = ???$ 12th IFAC congress 1993

A.A. And by element
with dela ed to zero.
J. Andr nd hysteresis





Lim[
wit

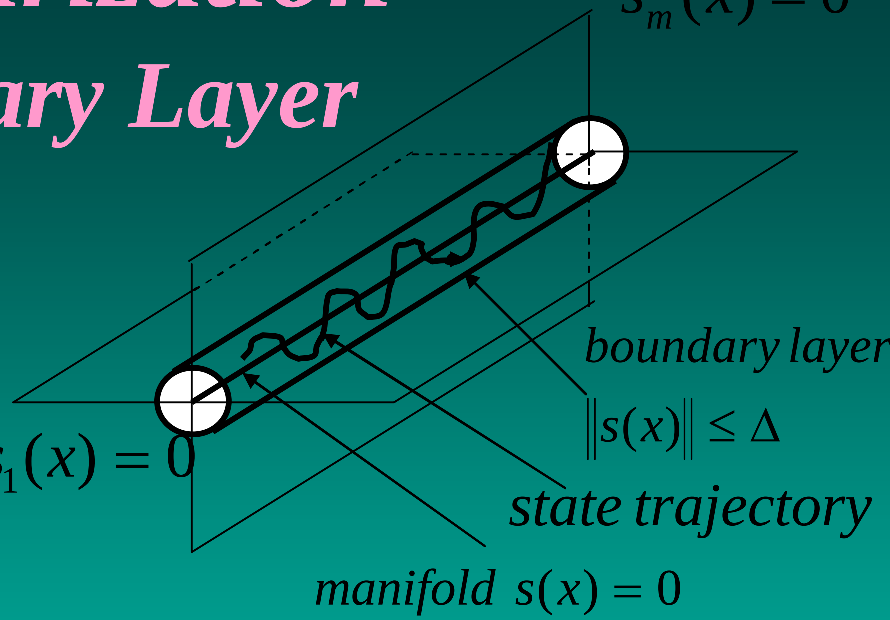


Regularization
Boundary Layer

$$\dot{x} = f(x,u),$$

$$x, f \in \Re^n, \; u(x) \in \Re^m,$$

$$u_i(x) = \begin{cases} u_i^+(x) & \text{if} \;\; s_i(x) > 0 \\ u_i^-(x) & \text{if} \;\; s_i(x) < 0 \end{cases}$$



$$\tilde{u}(x,\Delta) \Rightarrow \tilde{x}(t,\Delta), \quad \|s(x)\| \le \Delta, \quad \|s\|^2 = (s^T s).$$

$$\lim_{\Delta \to 0} \tilde{x}(t,\Delta) = x(t)$$

Equivalent Control Method

Affine system $\dot{x} = f(x,t) + B(x,t)u$

Discontinuous control u enforces sliding mode in manifold $s=(s_1,\ldots,s_m)=0.$ $\det(GB) \neq 0,\; G(x) = \{\partial s/\partial x\}.$

Then solution to $\dot{s} = Gf + GBu = 0, \quad u_{eq} = -(GB)^{-1}Gf.$

is substituted to system equation

$$\dot{x} = f - B(GB)^{-1}Gf$$

$s(x)=0 \longrightarrow x_2=s_0(x_1)$ $x_2, s_0 \in \Re^m \quad x_1 \in \Re^{n-m}:$

$$\dot{x}_1 = f_1[x_1, t, s_0(x_1)],\; f_1 \in \Re^{n-m}.$$

E.C.M. is substantiated by boundary layer regularization

E.C.M ≡ *Filippov Equation for Affine Systems*

Multi-dimensional Sliding Mode

- **In general case,**

$$\dot{x} = f(x,t) + B(x,t)u, \qquad x \in R^n, u \in R^m$$

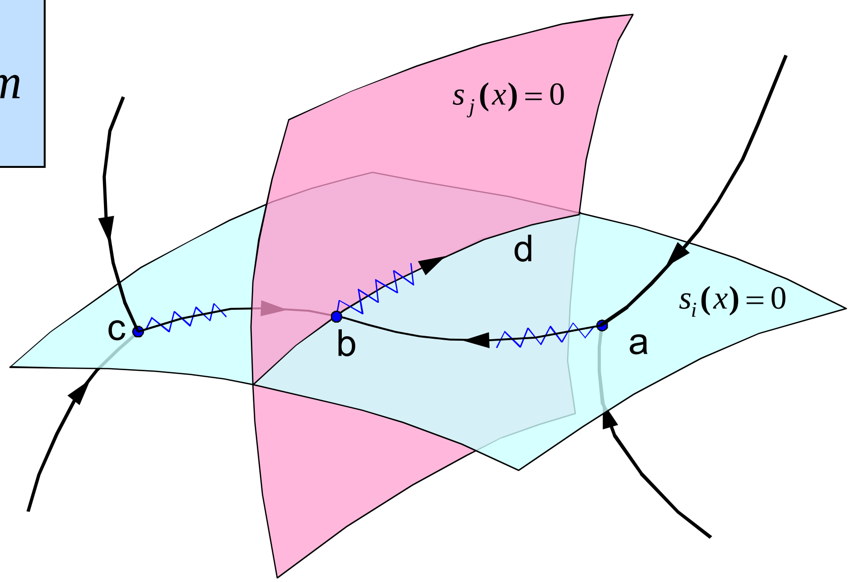
$$u_i = \begin{cases} u_i^+(x,t) & \text{if} \quad s_i(x) > 0 \\ u_i^-(x,t) & \text{if} \quad s_i(x) < 0 \end{cases} \qquad i = 1,\ldots,m$$

- ***Sliding mode occurs in the intersection of all m surfaces,***

  $s_i(x) = 0 \quad (i = 1,..,m)$, *or*

  in manifold $s(x) = 0,\ s^T = (s_1,\ldots,s_m)$.

- ***The order of motion equation is by 'm' less than the order of original system***



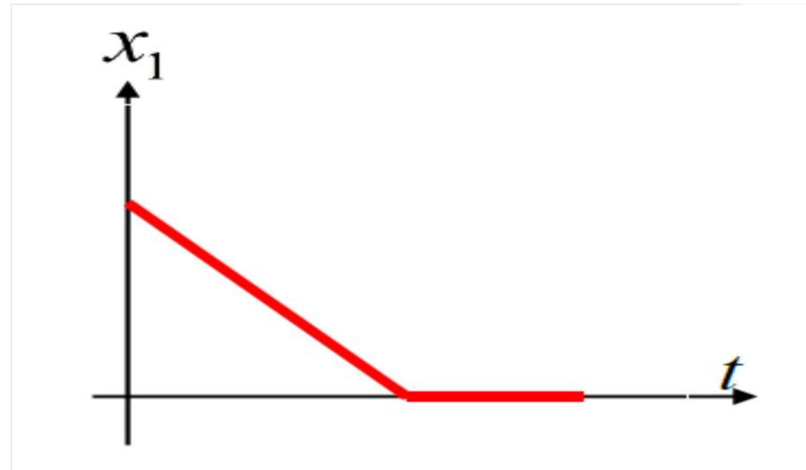
Sliding mode in discontinuity surfaces and their intersection

TWO DIMENSIONAL SLIDING MODE

$$\dot{x}_1 = u_1 \qquad u_1 = -sign(x_1)$$

$$\dot{x}_2 = u_2 \qquad u_2 = -sign(x_2)$$

$$\dot{x}_3 = u_1 u_2$$

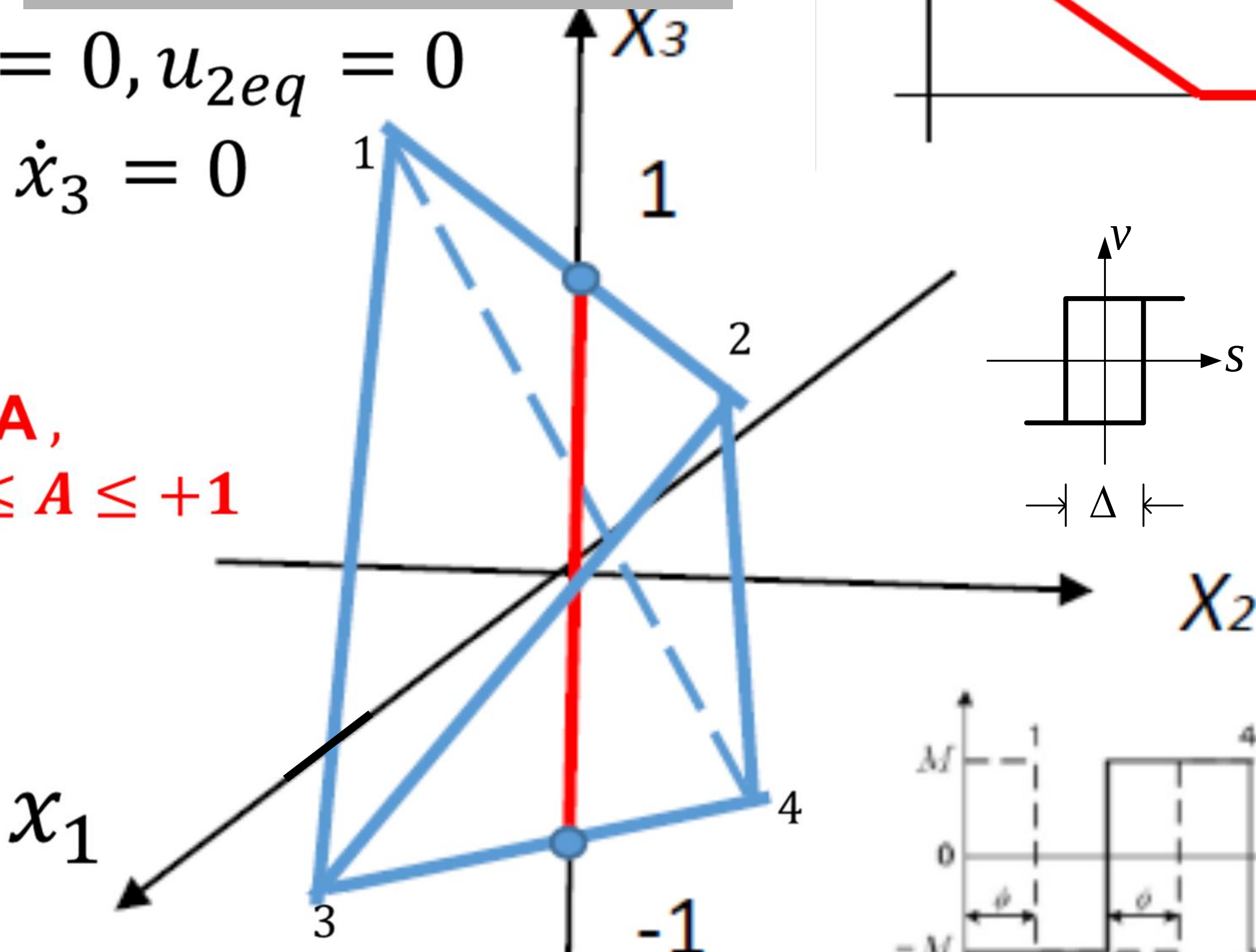


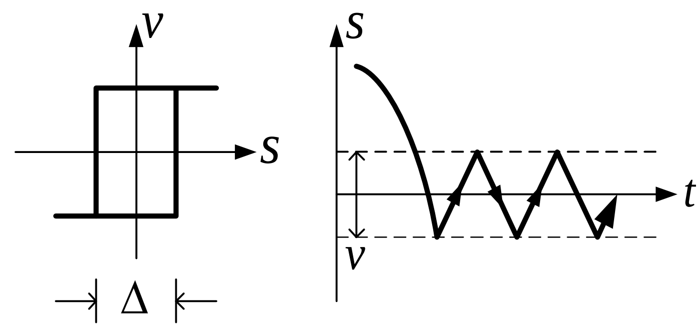
$$u_{1eq} = 0, u_{2eq} = 0$$

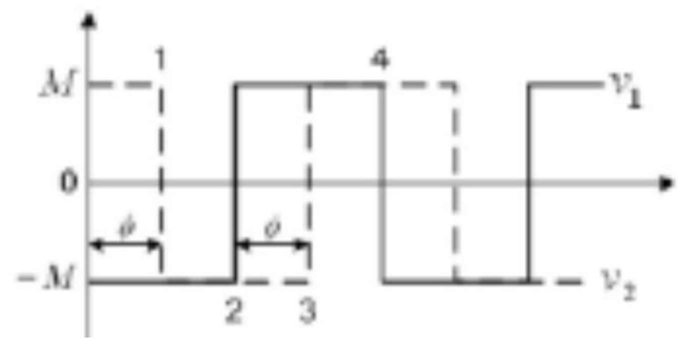
$$\dot{x}_3 = 0$$

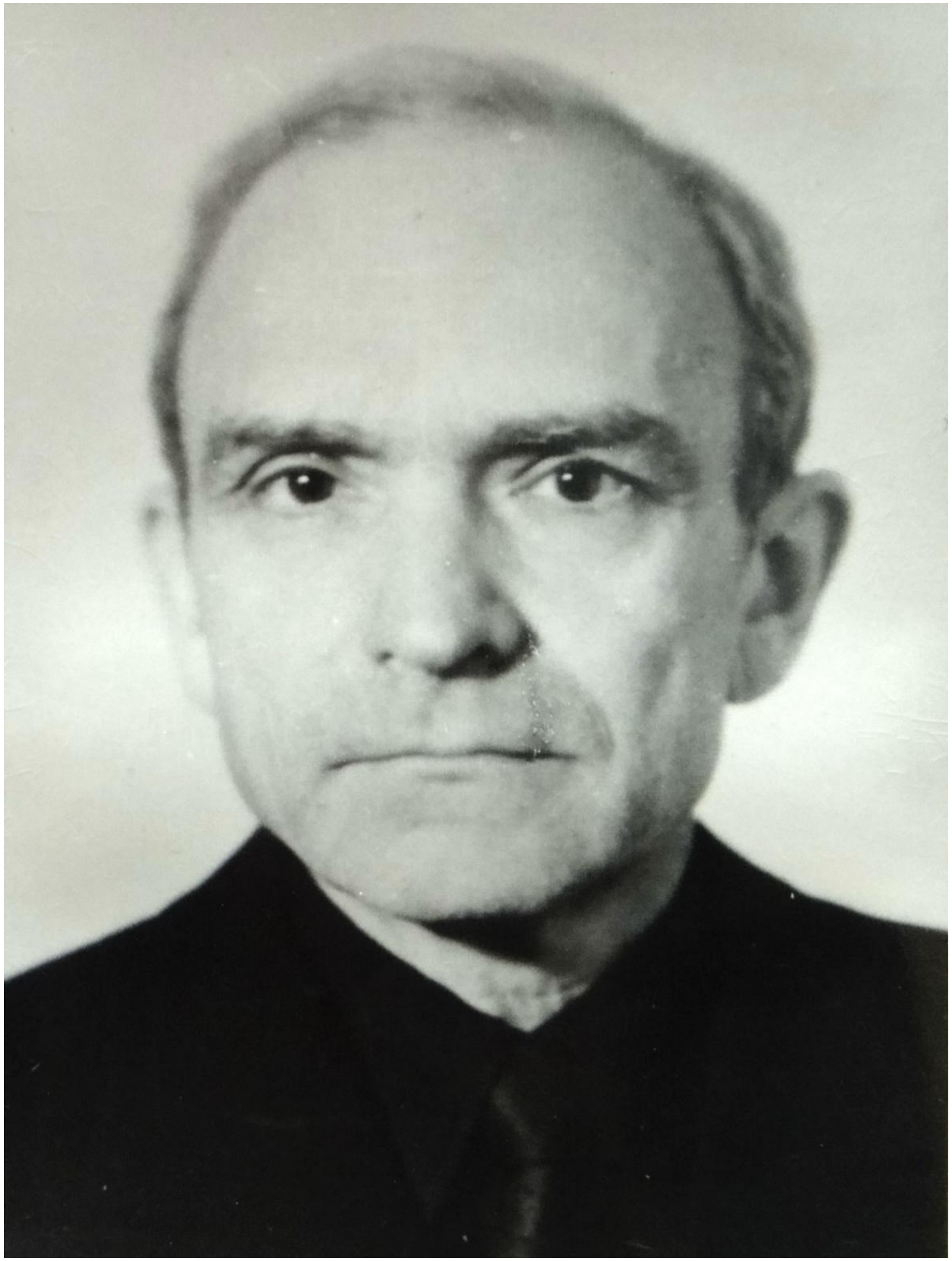
$\dot{x}_3$=A ,

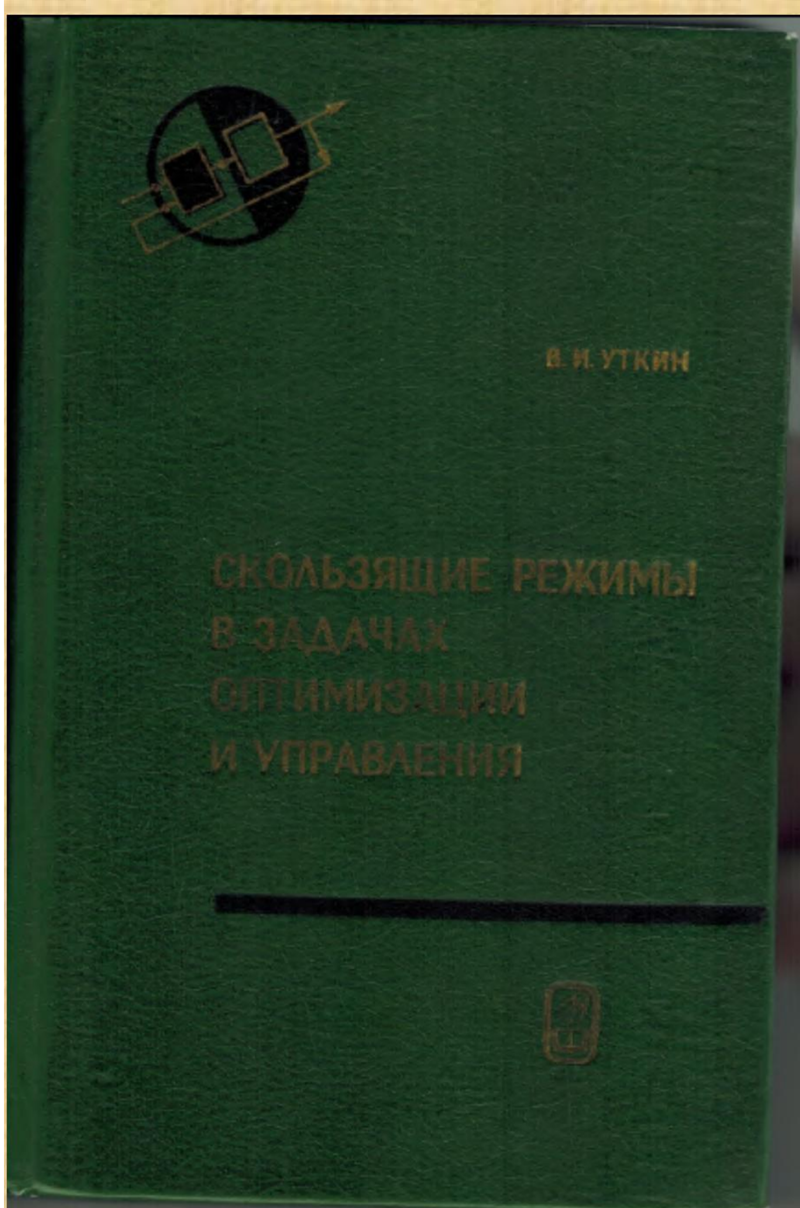
$-1 \le A \le +1$











Результаты § 3 главы IV явились «прямым следствием» одной из дискуссий с проф. А. Ф. Филипповым по проблеме существования многомерных скользящих режимов.

Несомненной удачей для автора оказалось решение издательства направить книгу на рецензирование проф. А. Ф. Филиппову, который проанализировал ее с присущей математикам строгостью и дал ряд полезных советов по улучшению рукописи.

Глубокоуважаемому
Вадиму Ивановичу
Уткину
от автора
с благодарностью
за постановку
и обсуждение
этих задач.
Филиппов

МАТЕМАТИЧЕСКИЕ ЗАМЕТКИ

т. 27, № 2 (1980)

СИСТЕМА ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С НЕСКОЛЬКИМИ РАЗРЫВНЫМИ ФУНКЦИЯМИ

А. Ф. Филиппов

1. Исследование устойчивости положения равновесия системы с разрывными правыми частями в ряде случаев сводится к исследованию системы

$$\dot{x}_i = -\sum_{j=1}^{n} a_{ij}\,\mathrm{sgn}\,x_j, \qquad i = 1, \ldots, n. \tag{1}$$

Необходимые и достаточные условия асимптотической устойчивости нулевого решения системы (1) в случае $n = 1$ очевидны ($a_{11} > 0$), в случае $n = 2$ имеют вид [1]

$$a_{11}a_{22} - a_{12}a_{21} > 0, \; a_{11}\,|\,a_{21}\,| + a_{22}\,|\,a_{12}\,| > 0$$

(при $a_{12} = a_{21} = 0$ эти неравенства заменяются на $a_{11} > 0$, $a_{22} > 0$). В случае $n = 3$ необходимые и достаточные условия асимптотической устойчивости получены в [2]. Они имеют различный вид в разных случаях в зависимости от знаков $\dot{x}_i$ в координатных октантах и наличия скользящих режимов.

Для исследования систем с коэффициентами, зависящими от параметров, удобнее пользоваться достаточными условиями асимптотической устойчивости, имеющими один и тот же вид во всех случаях, независимо от наличия скользящих режимов. Такие условия известны для системы (1) при любом n, например следующие (каждое из них является достаточным, что доказывается с помощью функции Ляпунова $V = |\,x_1\,| + \ldots + |\,x_n\,|$):

А) $a_{ii} \geq \sum_{j \neq i} |\,a_{ij}\,|, \quad i = 1, \ldots, n.$

Б) $a_{jj} > \sum_{i \neq j} |\,a_{ij}\,|, \quad j = 1, \ldots, n.$

OPTIMIZATION PROBLEM

Gradient search

$$\min F(x) = \min[f(x) + h^T(x)u]:$$

$$\dot{x} = -grad[f(x)] - G^T u, \; G = \{\partial h/\partial x\}$$

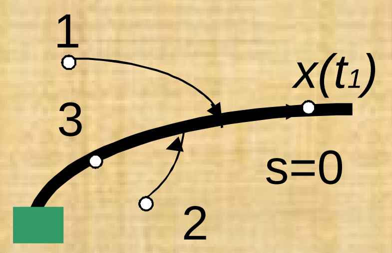
EQUIVALENT CONTROL METHOD IS APPLICABLE !!!

All conventional theorems on gradient search work

A Little Fantasy

The evolution of state in *dynamic systems x(t)* is represented by shift operator **$x(t)=F[x(0),t]$** with unique values for any time t .

The author of Minimum Principle **Pontryagin** mentioned in 1953 that the systems with “glued trajectories” formally can not be called dynamic systems.



Indeed the point $x(t_1)$ can be reached from outside $s=0$ (point 1 and 2) and then in sliding mode, or in sliding mode only (point 3). It means that family of shift operator at the point x_1 does not have inverse. Then the family is called ***semigroup.***

New Definition: the point of the system $x(t) = F[x,t]$ is called ***“sliding mode point”*** if the set $F[x,t]$ constitutes a ***semigroup.***

CHALLENGE: **to create general sliding mode and sliding mode control theory in terms of shift operators embracing all types of dynamic systems.**