Stability and optimization of dual switching linear positive systems

P. Colaneri
DEIB, Politecnico di Milano - Italy

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Robustness in identification and control, Siena (Italy), 1998.

Они спорили о чем-то очень сложном и важном, причем ни один из них не мог победить другого. Они ни в чем не сходились друг с другом, и от этого их спор был особенно интересен и нескончаем.

Михаил Булгаков, “**Мастер и Маргарита**”

(They were arguing about something very complex and important, and none of them could defeat the other. They are in no way agreed with each other, and from that of their dispute was particularly interesting, and endless).

Summary

- **Motivations**
- **Positivity constraint**
- **Stability and Stabilization**
- **Performances and Optimization**
- **Examples**
- **Sufficiency of Pontryagin conditions via convexity**
- **Conjecture on Stochastic consensus**
- **Conclusions**

Thermal system

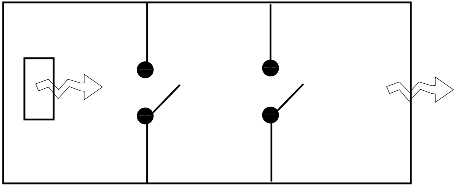

Figure 2: Switching thermal system

x_i, $i = 1,2,3$ temperatures in the three rooms. Two doors (open/closed) $\rightarrow \sigma \in \{1,2,3,4\}$.

$$\dot{x}(t) = A_{\sigma(t)} x(t) + Bu(t)$$

For instance: "worst" control problem $\max_\sigma x_3(T)$

Traffic system

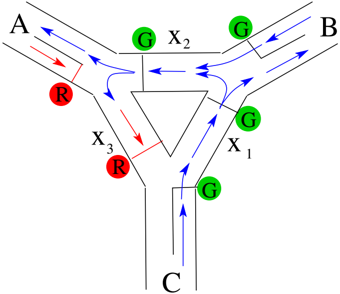

Figure 3: The traffic congestion control problem

Three main roads, 6 traffic lights (red/green), three buffer variables x_i, three symmetric configurations $\sigma \in \{1, 2, 3\}$.

$$\dot{x}(t) = A_{\sigma(t)} x(t) + b$$

For instance: given x_0 find $\sigma(t)$ so as to stabilize.

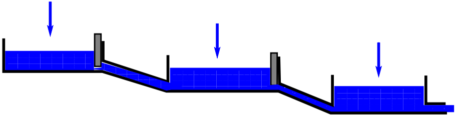

Figure 4: Flood control problem

Flow x_i, $i = 1, 2, 3$. Sluice gates with three positions (fully open, partially open, closed).

$$\dot{x}(t) = A_{\sigma(t)} x(t) + Bu(t)$$

For instance: emergency emptying strategy
$\min_\sigma \int_0^\infty (d_1 x_1(t) + d_2 x_2(t) + d_3 x_3(t)) dt$.

HIV system

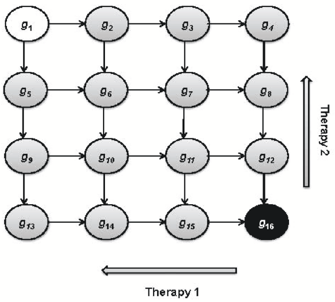

Figure 5: The network for 16 genotypes and two drug combinations. The direction of the arrows represents the strength of the therapy.

Viral loads of infected cells and machrophages x_i. Two therapies $\sigma = \{1,2\}$.

$$\dot{x}(t) = A_{\sigma(t)}x(t)$$

Chemical system

$$\dot{x}(t) = Sg(x,t) + g_0$$

with S stoichiometric matrix, $g(x,t)$ nonlinear reaction rate. Any solution is also a solution of a differential inclusion

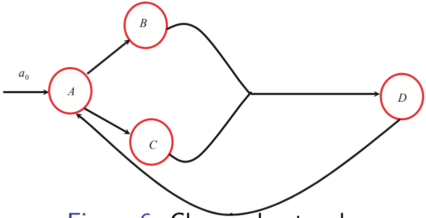

Figure 6: Chemical network

$$\dot{x}(t) \in A(t)x(t) + g_0$$

Boundedness via stability under arbitrary switching

Distributed power networks

Channel gains g_{ij}. Signal-to-Interference and Noise-Ratio γ_i. Variance of thermal noise at the receiver ν_i. Power level p_i.

$$\dot{p}_i = k_i\left(-p_i + \gamma_i \sum_{j\in N_j} g_{ij}p_j + \nu_i\right)$$

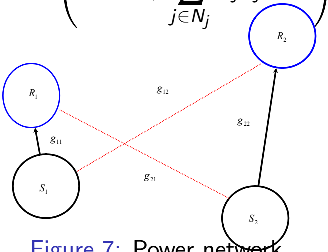

Figure 7: Power network

g_{ij} depend on the applied linear detector and coding properties but also incorporate path losses, shadowing and multi-path fading..... jumping parameters.

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t)$$

Positive dual switching systems

Consider the system

$$\begin{aligned}\dot{x}(t) &= A_{\sigma(t)}^{\gamma(t)}x(t)+B_{\sigma(t)}^{\gamma(t)}w(t), \quad x(0)=x_0>0\\ z(t) &= C_{\sigma(t)}^{\gamma(t)}x(t)+D_{\sigma(t)}^{\gamma(t)}w(t)\end{aligned}$$

- $x(t)$ is the state >0
- $w(t)$ is a deterministic entrywise positive disturbance >0
- $z(t)$ is the performance output >0
- $\gamma(t)\in\mathscr{M}=\{1,2,\dots,M\}$ is a control switching signal
- $\sigma(t)\in\mathscr{N}=\{1,2,\dots,N\}$ is a Markov chain with transition rate matrix $\Lambda=[\lambda_{ij}]$
- A_i^j Metzler matrices, B_i^j, C_i^j, D_i^j nonnegative, $\forall i,j$

Mean stability

$$\dot{x}(t) \quad = \quad A_{\sigma(t)}^{\gamma(t)} x(t)$$

The system is Mean-stable if

$$\lim_{t\to\infty} E[x(t)] = 0$$

for any $x_0 > 0$ and any initial probability distribution of $\sigma(0)$.
Analysis: Check M-stability for any $\gamma(t)$.
Design: Find $\gamma(t)$ M-stabilizing.
Information available to the designer: $\gamma(x,\sigma)$ (mode-dependent state-feedback), $\gamma(x)$ (mode-independent state-feedback), $\gamma(\sigma)$ (static map).

Mean stability

For $\gamma(t) = r$ constant mean stability is equivalent to Hurwitz stability of matrix

$$\begin{bmatrix} A_1^r + \lambda_{11} I_n & \lambda_{21} I_n & \cdots & \lambda_{N1} I_n \\ \lambda_{12} I_n & A_2^r + \lambda_{22} I_n & \cdots & \lambda_{N2} I_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{1N} I_n & \lambda_{2N} I_n & \cdots & A_N^r + \lambda_{NN} I_n \end{bmatrix}$$

L_1 induced performance

For $x_0 = 0$,

$$J_1(\gamma) = \sup_{w \in \mathscr{L}_1, w>0} \frac{E\left[\int_0^\infty \mathbf{1}_p^\top z(t)dt\right]}{E\left[\int_0^\infty \mathbf{1}_m^\top w(t)dt\right]}$$

L_∞ induced performance

For $x_0 = 0$,

$$J_\infty(\gamma) = \sup_{w \in \mathscr{L}_1, w>0} \frac{\sup_{t\geq 0,k} E[z(t)]_k}{\sup_{t\geq 0,k} E[w(t)]_k}$$

Analysis: Find the worst γ. Design: Find the best γ. Information available to the designer: $\gamma(x,\sigma)$ (mode-dependent state-feedback), $\gamma(x)$ (mode-independent state-feedback), $\gamma(\sigma)$ (static map).

From now on: w is deterministic, hence $E[w(t)] -- > w$

Performance indices

L_1 induced performance

For $x_0 = 0$, $\gamma(t) = r$ given and constant,

$$J_1(r) = \|G^r(0)\|_1, \quad J_\infty(r) = \|G^r(0)\|_\infty$$

Linear program to check $J_*(r) < \rho$. For example $J_1(r) < \rho$ IFF there exist $v_i^r \gg 0$ such that

$$\begin{aligned} (v_i^r)^\top A_i^r + \sum_{k=1}^{N} \lambda_{ik} (v_j^r)^\top + \mathbf{1}_p^\top C_i^r & \ll 0 \\ \sum_{i=1}^{N} \bar{\pi}_i \left((v_i^r)^\top B_i^r + \mathbf{1}_p D_i^r \right) & \ll \rho \mathbf{1}_m^\top \end{aligned}$$

$$V(x,\sigma) = \min_r (v_\sigma^r)^\top x$$

Letting $\sigma(t) = s$, $x(t) = x$, $w(t) = w$, $\gamma(t) = g$, it is shown that the infinitesimal generator

$$\begin{aligned}
\mathscr{L}V(x,s) &= \lim_{h\to 0}\frac{1}{h}\left(E[V(x(t+h),\sigma(t+h))|(x,s,g,w)] - V(x,s)\right) \\
&\leq \left((v_s^g)^\top A_s^g + \sum_{j=1}^{N}\lambda_{sj} v_s^j + \mathbf{1}_p^\top C_s^g\right) x + \\
&\quad + \left((v_s^g)^\top B_s^g + \mathbf{1}_p^\top D_s^g\right) w - \mathbf{1}_p^\top z
\end{aligned}$$

Dynkin's formula ....

Mean-stabilization: bilinear version

Theorem 1

Let $w(t) = 0$ and assume that there exist $v_i^r \gg 0$, $i \in \mathcal{N}$, $r \in \mathcal{M}$ and a Metzler matrix $\Phi = [\varphi_{rs}] \in \mathcal{T}_M$ satisfying, $\forall i, r$, the inequalities

$$(v_i^j)^\top A_i^j + \sum_{k=1}^{N} \lambda_{ik} (v_k^j)^\top + \sum_{s=1}^{M} \varphi_{js} (v_i^s)^\top \ll 0$$

Then, the feedback switching law

$$\gamma^* = g(x, \sigma) = \mathrm{argmin}_j (v_\sigma^j)^\top x$$

makes the closed-loop system Mean-stable.

Theorem 2

Let $w(t) = 0$ and assume that there exist $v_i \gg 0$, $i \in \mathcal{N}$, and a map $\gamma = g(\sigma)$ satisfying, $\forall i, r$, the inequalities

$$v_i^\top A_i^{g(i)} + \sum_{k=1}^{N} \lambda_{ik} v_k^\top \ll 0$$

Let

$$(r_i^j)^\top := v_i^\top A_i^j + \sum_{k=1}^{N} \lambda_{ik} v_k^\top$$

Then, the feedback switching law

$$\gamma^* = g(x,\sigma) = \mathrm{argmin}_j (r_\sigma^j)^\top x$$

makes the closed-loop system Mean-stable.

Mean-stabilization: mode-independent

Theorem 3

Let $w(t) = 0$ and assume that there exist $v_i \gg 0$, $i \in \mathcal{N}$, and an $N \times M$ matrix Q satisfying, $\forall i, r$, the inequalities

$$v_i^\top A_i^j + \sum_{k=1}^{N} \lambda_{ik} v_k^\top - q_{ij} v_i^\top \ll 0, \quad Q\mathbf{1}_M \ll 0$$

Then, the feedback law

$$\mu^* = \arg \min_{\mu \in \Omega_M} \max_{\nu \in \Omega_M} \sum_{j=1}^{M} \sum_{i=1}^{N} \nu_i \mu_j q_{ij} v_i^\top x$$

makes the closed-loop system Mean-stable.

Notice:

$$\min_{\mu \in \Omega_M} \max_{\nu \in \Omega_M} \nu^\top H(x) \mu = \max_{\nu \in \Omega_N} \min_{\mu \in \Omega_N} \nu^\top H(x) \mu, \quad h_{ij}(x) = q_{ij} v_i^\top x$$

L_1 induced control: bilinear version

Theorem 4

Let $x_0 = 0$ and assume that there exist $v_i^r \gg 0$, $i \in \mathcal{N}$, $r \in \mathcal{M}$ and a Metzler matrix $\Phi = [\varphi_{rs}] \in \mathcal{T}_M$ satisfying $\forall i, r$, the inequalities

$$(v_i^r)^\top A_i^r + \sum_{j=1}^{N} \lambda_{ij} (v_j^r)^\top + \sum_{k=1}^{M} \varphi_{rk} (v_i^k)^\top + \mathbf{1}_p^\top C_i^r \ll 0$$

Let $\hat{\gamma}(\sigma) = \arg\min_j \max_k \left((v_\sigma^j)^\top B_\sigma^j + \mathbf{1}_p^\top D_\sigma^j \right) e_k$. Then, the feedback switching law

$$\gamma^* = g(x, \sigma) = \begin{cases} \arg\min_r (v_\sigma^r)^\top x, & x > 0 \\ \hat{\gamma}(\sigma), & x = 0 \end{cases}$$

makes the closed-loop system mean-stable and guarantees

$$J_1(\gamma^*) < \max_k \sum_{i=1}^{N} \bar{\pi}_i \left((v_i^{\hat{\gamma}(i)})^\top B_i^{\hat{\gamma}(i)} + \mathbf{1}_p^\top D_i^{\hat{\gamma}(i)} \right) \mathbf{e}_k$$

L_1 induced control: linear version

Theorem 5

Let $x_0 = 0$ *and assume that there exist* $v_i \gg 0$, $i \in \mathcal{N}$, *and a map* $\gamma = g(\sigma)$ *satisfying,* $\forall i, r$, *the inequalities*

$$(v_i)^\top A_i^{g(i)} + \sum_{k=1}^{N} \lambda_{ik} v_k^\top + \mathbf{1}_p^\top C_i^{g(i)} \ll 0$$

Let $(r_i^j)^\top := v_i^\top A_i^j + \sum_{k=1}^{N} \lambda_{ik} v_k^\top + \mathbf{1}_p^\top C_i^j$ *and* $\hat{\gamma}(\sigma) = \arg\min_j \max_k \left((v_\sigma^\top B_\sigma^j + \mathbf{1}_p^\top D_\sigma^j \right) e_k$. *Then, the feedback law*

$$\gamma^* = g(x, \sigma) = \begin{cases} \arg\min_j (r_\sigma^j)^\top x, & x > 0 \\ \hat{\gamma}(\sigma), & x = 0 \end{cases}$$

makes the closed-loop system mean-stable and guarantees

$$J_1(\gamma^*) < \max_k \sum_{i=1}^{N} \bar{\pi}_i \left((v_i^\top B_i^{\hat{\gamma}(i)} + \mathbf{1}_p^\top D_i^{\hat{\gamma}(i)} \right) e_k$$

L_1 induced control: mode-independent

Theorem 6

Let $w(t) = 0$ *and assume that there exist* $v_i \gg 0$, $i \in \mathcal{N}$, *and an* $N \times M$ *matrix* Q *satisfying,* $\forall i, r$, *the inequalities*

$$v_i^\top A_i^j + \sum_{k=1}^{N} \lambda_{ik} v_k^\top + \mathbf{1}_p^\top C_i^j - q_{ij} v_i^\top \ll 0, \quad Q\mathbf{1}_M \ll 0$$

Let $\hat{\gamma} = \arg\min_j \max_k E\left[\left(\left(v_\sigma^\top B_\sigma^j + \mathbf{1}_p^\top D_\sigma^j\right) e_k\right]\right.$. *Then, the feedback law*

$$\begin{array}{ll} \arg \min\limits_{\mu \in \Omega_M} \max\limits_{\nu \in \Omega_M} \sum\limits_{j=1}^{M} \sum\limits_{i=1}^{N} \nu_i \mu_j q_{ij} v_i^\top x, & x > 0 \\ \hat{\gamma}, & x = 0 \end{array}$$

makes the closed-loop system Mean-stable and guarantees

$$J_1(\mu^*) < \min_j \max_k \sum_{i=1}^{N} \bar{\pi}_i \left(\left(v_i^\top B_i^j + \mathbf{1}_p^\top D_i^j\right) e_k\right.$$

Example

$$
\begin{aligned}
A_1^1 &= \begin{bmatrix} -1 & 0 \\ 1 & 0.1 \end{bmatrix}, \quad A_2^1 = \begin{bmatrix} 0.1 & 1 \\ 0 & -2 \end{bmatrix} \\
A_1^2 &= \begin{bmatrix} -1.25 & 0 \\ 1 & 0.2 \end{bmatrix}, \quad A_2^2 = \begin{bmatrix} 0.2 & 1 \\ 0 & -2.1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \\
B_1^1 &= \begin{bmatrix} 1 & 0.5 \\ 0 & 0.5 \end{bmatrix}, \quad B_2^1 = \begin{bmatrix} 0 & 0.5 \\ 1 & 0.5 \end{bmatrix}, B_1^2 = \begin{bmatrix} 0 & 0.5 \\ 1 & 0.5 \end{bmatrix}, B_2^2 = \begin{bmatrix} 1 & 0.5 \\ 0 & 0.5 \end{bmatrix} \\
C_1^1 &= C_2^1 = C_1^2 = C_2^2 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad D_1^1 = D_2^1 = D_1^2 = D_2^2 = \begin{bmatrix} 0 & 0 \end{bmatrix}
\end{aligned}
$$

Both PMJLS are M-stable and their respective L_1-induced gains are

$$\|(A_\sigma^1, B_\sigma^1, C_\sigma^1, D_\sigma^1, \Lambda\|_{\mathscr{L}_1} = 11.2936, \quad \|(A_\sigma^2, B_\sigma^2, C_\sigma^2, D_\sigma^2, \Lambda\|_{\mathscr{L}_1} = 11.4802$$

The aim is to design a switching strategy so as to improve the $\mathscr{L}_1$-induced gain.

Example - cont.

It is interesting to observe that by using the static map

$$\gamma(t) = \begin{cases} 2 & \sigma(t) = 1 \\ 1 & \sigma(t) = 2 \end{cases}$$

the true gain is reduced to 8.2724. Next, the open-loop control via Markov switching has been designed. After tuning the parameter Φ, the associated guaranteed gain is 10.8119. Some realizations of the controlled system under the worst case disturbance (an impulse applied to the second input channel) are shown

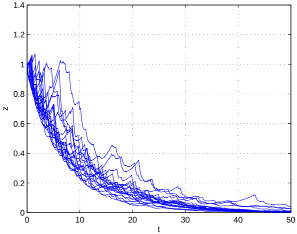

Figure 8: MonteCarlo simulations

Example - cont.

Finally, the state-feedback strategy has been designed. It turns out that the switching signal $\gamma(t)$ commutes according to the following rule:

$$\gamma = g(\mathbf{x},\sigma) = \begin{cases} \sigma, & x_2 \geq 3.1361 x_1 \\ 2, & 1.6504 x_1 < x_2 < 3.1361 x_1 \\ 3-\sigma, & x_2 \leq 1.6504 x_1 \end{cases}$$

The guaranteed gain is 10.8085. The sample average of the cost function over these realizations is 7.8479.

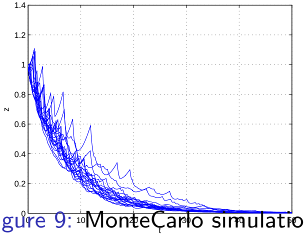

Figure 9: MonteCarlo simulations

Convexity for diagonally switched systems

Recent research (Automatica, 2013, 2014).

$$\dot{x} = A_{\sigma(t)}^{\gamma(t)} x(t) \longrightarrow \dot{x} = \left(\sum_{i=1}^{N} u_i(t) A_{\sigma(t)}^{i} \right) x(t)$$

but γ affects only the diagonal entries of A_σ^γ. The cost

$$J(x_0) = E[c^\top x(t_f)]$$

Then $J(x_0)$ is convex with respect to u. Therefore the optimal control (open loop) can be found using gradient.

Probabilistic consensus in Markovian networks

Recent research (ECC 2014). N agents, M discrete states for each agent

$$\dot{\pi}^{[r]} = \left(\frac{1}{N-1} \sum_{i=1}^{M} \sum_{j \neq r}^{N} \lambda_i A_i^j \pi_i^{[j]} \right) \pi^{[r]}, \quad r = 1, 2, \ldots, N$$

with $\mathbf{1}_n^\top A_i^j = 0$. An agent, in state i, transmits this information to the neighbors at rate λ_i. An agent is induced to increase its own transition rate towards the state of another agent of a fixed amount (induction mechanism). For a complete (or star) graph:

$$\lim_{t \to \infty} \left(\pi^{[r]} - \pi^{[s]} \right) = 0$$

Then

$$\dot{\pi} = \left(\sum_{i=1}^{M} \lambda_i \bar{A}_i \pi_i \right) \pi, \quad \mathbf{1}_n^\top \bar{A}_i = 0$$

Stability of the fixed point $\bar{\pi}$?
Other graph topologies?
Control problem $\pi^o \longrightarrow \lambda_i$

Conclusions

- Switching strategies have been devised for stabilization and suboptimal performance of dual switching linear positive systems
- The methods are for applications in epidemiological, biochemical systems and queuing networks.
- Possible extensions concern mitigation of chattering, co-design of controller/scheduler, and switching with partial information
- Possible extensions to probabilistic consensus in Markovian networks (aware and perception of Facebook News Feed curation algorithms).