

Lyapunov Matrices for Time-Delay Systems

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System description

In this lecture we consider a system of the form

$$\frac{dx(t)}{dt} = A_0x(t) + A_1x(t - h). \quad (1)$$

Here A_0 and A_1 are real $n \times n$ matrices, and h is a non-negative time delay.

Initial value problem

The initial value problem for system (1) is stated as follows: Given an initial time instant $t = 0$, and an initial function

$$\varphi : [-h, 0] \rightarrow R^n,$$

find a solution of the system that satisfies the condition:

$$x(\theta) = \varphi(\theta), \quad \theta \in [-h, 0].$$

In this lecture we assume that initial functions belong to the space of piece-wise continuous functions

$$PC([-h, 0], R^n).$$

- The Euclidean/Hermitian norm is used for vectors
- The corresponding induced spectral norm is used for matrices

$$\|A\| = \max_{\|x\|=1} \|Ax\|$$

- The space

$$PC([-h, 0], R^n)$$

is equipped with the uniform norm,

$$\|\varphi\|_h = \sup_{\theta \in [-h, 0]} \|\varphi(\theta)\|.$$

The state of system (1) at a time instant t is defined as the restriction of the solution on the segment $[t - h, t]$:

$$x_t : \theta \rightarrow x(t + \theta), \quad \theta \in [-h, 0].$$

In the case when the initial function φ should be indicated explicitly we use notations $x(t, \varphi)$ and $x_t(\varphi)$.

Definition 1.2

It is said that $n \times n$ matrix $K(t)$ is the fundamental matrix of system (1) if it satisfies the matrix equation

$$\frac{d}{dt}K(t) = K(t)A_0 + K(t-h)A_1, \quad t \geq 0,$$

and $K(t) = 0_{n \times n}$, for $t < 0$, $K(0) = I$. Here I is the identity $n \times n$ matrix.

Theorem 1.2

Given an initial function $\varphi \in PC([-h, 0], R^n)$, then the following equality holds

$$x(t, \varphi) = K(t)\varphi(0) + \int_{-h}^0 K(t - \theta - h)A_1\varphi(\theta)d\theta, \quad t \geq 0.$$

This expression for $x(t, \varphi)$ is known as Cauchy formula.

Definition 2.2

System (1) is said to be exponentially stable if there exist $\gamma \geq 1$ and $\sigma > 0$, such that any solution $x(t, \varphi)$ of the system satisfies the inequality

$$\|x(t, \varphi)\| \leq \gamma e^{-\sigma t} \|\varphi\|_h, \quad t \geq 0.$$

Krasovskii Theorem

Theorem 2.2

System (1) is exponentially stable if there exists a functional

$$v : PC([-h, 0], R^n) \rightarrow R,$$

such that the following conditions hold

- *For some positive α_1, α_2 the functional admits a lower and an upper bounds of the form*

$$\alpha_1 \|\varphi(0)\|^2 \leq v(\varphi) \leq \alpha_2 \|\varphi\|_h^2, \quad \varphi \in PC([-h, 0], R^n).$$

- *For some $\beta > 0$ the inequality*

$$\frac{d}{dt}v(x_t) \leq -\beta \|x(t)\|^2, \quad t \geq 0,$$

holds along the solutions of the system.

Lyapunov-Krasovskii approach

We illustrate the scheme with the functional

$$v(\varphi) = \varphi^T(0)P\varphi(0) + \int_{-h}^0 \varphi^T(\theta)Q\varphi(\theta)d\theta.$$

If matrices P and Q are positive definite, then the first condition of the Krasovskii Theorem is satisfied with

$$\begin{aligned}\alpha_1 &= \lambda_{\min}(P), \\ \alpha_2 &= \lambda_{\max}(P) + h\lambda_{\max}(Q).\end{aligned}$$

The value of the functional along a solution $x(t)$ of system (1)

$$v(x_t) = x^T(t)Px(t) + \int_{-h}^0 x^T(t+\theta)Qx(t+\theta)d\theta.$$

Lyapunov-Krasovskii approach

The time derivative,

$$\begin{aligned} \frac{dv(x_t)}{dt} = & 2x^T(t)P[A_0x(t) + A_1x(t-h)] \\ & + x^T(t)Qx(t) - x^T(t-h)Qx(t-h), \end{aligned}$$

can be presented in the form

$$\frac{dv(x_t)}{dt} = (x^T(t), x^T(t-h)) \begin{pmatrix} A_0^T P + P A_0 + Q & P A_1 \\ A_1^T P & -Q \end{pmatrix} \begin{pmatrix} x(t) \\ x(t-h) \end{pmatrix}.$$

Lyapunov-Krasovskii approach

The second condition of Krasovskii Theorem is fulfilled when matrix

$$\begin{pmatrix} A_0^T P + P A_0 + Q & P A_1 \\ A_1^T P & -Q \end{pmatrix}$$

is negative definite.

Theorem 3.2

System

$$\frac{dx}{dt} = Ax.$$

is exponentially stable if and only if there exists a Lyapunov function $v(x)$ that satisfies the following conditions

- *For some positive α_1, α_2 the function admits a lower and an upper bounds of the form*

$$\alpha_1 \|x\|^2 \leq v(x) \leq \alpha_2 \|x\|^2, \quad x \in R^n;$$

- *For some $\beta > 0$ the inequality*

$$\frac{d}{dt}v(x(t)) \leq -\beta \|x(t)\|^2, \quad t \geq 0,$$

holds along the solutions of the system.

Problem statement

Problem: Let system (1) be exponentially stable, given a quadratic form

$$w(x) = x^T W x,$$

find a functional

$$v_0(\varphi),$$

defined on $PC([-h, 0], R^n)$, such that along the solutions of the system the following equality holds

$$\frac{d}{dt}v_0(x_t) = -x^T(t)Wx(t), \quad t \geq 0.$$

Computation of $v_0(\varphi)$

Assume that system (1) is exponentially stable, then

$$v_0(\varphi) = \int_0^{\infty} x^T(t, \varphi) W x(t, \varphi) dt.$$

Functional $v_0(\varphi)$

Simple transformations allow to show that functional $v_0(\varphi)$ can be written as

$$\begin{aligned} v_0(\varphi) = & \varphi^T(0)U(0)\varphi(0) \\ & + 2\varphi^T(0) \int_{-h}^0 U(-h-\theta)A_1\varphi(\theta)d\theta \\ & + \int_{-h}^0 \varphi^T(\theta_1)A_1^T \left[\int_{-h}^0 U(\theta_1-\theta_2)A_1\varphi(\theta_2)d\theta_2 \right] d\theta_1. \end{aligned} \tag{2}$$

Definition 3.2

Matrix

$$U(\tau) = \int_0^{\infty} K^T(t) W K(t + \tau) dt, \quad (3)$$

is named Lyapunov matrix of system (1) associated with matrix W .

How to compute matrix $U(\tau)$?

Delay-free case

$$V = \int_0^{\infty} (e^{At})^T W e^{At} dt$$

$$A^T V + V A = -W$$

Time-delay case

$$U(\tau) = \int_0^{\infty} K^T(t) W K(t) dt$$

?

Lemma 1.2

Lyapunov matrix $U(\tau)$ satisfies the dynamic property

$$\frac{d}{d\tau}U(\tau) = U(\tau)A_0 + U(\tau - h)A_1, \quad \tau \geq 0.$$

Lemma 2.2

Lyapunov matrix satisfies the symmetry property

$$U(-\tau) = U^T(\tau), \quad \tau \geq 0.$$

Lemma 3.2

Lyapunov matrix satisfies the algebraic property

$$U(0)A_0 + U(-h)A_1 + A_0^T U(0) + A_1^T U(h) = -W.$$

Algebraic property

The symmetry property indicates that the first derivative of the Lyapunov matrix suffers discontinuity at point $\tau = 0$.

Lemma 4.2

The algebraic property can be written in the form

$$U'(+0) - U'(-0) = -W.$$

Here $U'(+0)$ and $U'(-0)$ stand for the right hand side, and the left hand side derivatives of matrix $U(\tau)$ at $\tau = 0$, respectively.

Lyapunov matrices: new definition

There are two serious limitations, associated with the definition of Lyapunov matrices by means of improper integral (3):

- The first one is that this definition is applicable to the exponentially stable systems, only.
- The second one is that the definition is of little help from the computational point of view. Indeed, it demands a preliminary computation of the fundamental matrix $K(t)$, for $t \in [0, \infty)$, that by itself is a difficult task, and the consequent evaluation of the integral (3) for different values of τ .

Lyapunov matrices: new definition

Definition 4.2

We say that matrix $U(\tau)$ is a Lyapunov matrix of system (1), associated with a symmetric matrix W , if it satisfies the dynamic property

$$\frac{d}{d\tau}U(\tau) = U(\tau)A_0 + U(\tau - h)A_1, \quad \tau \geq 0, \quad (4)$$

the symmetry property

$$U(-\tau) = U^T(\tau), \quad \tau \geq 0, \quad (5)$$

and the algebraic property

$$U(0)A_0 + U(-h)A_1 + A_0^T U(0) + A_1^T U(h) = -W. \quad (6)$$

Consistency Theorem

On the one hand, the new definition allows to overcome the first limitation of the original definition of the Lyapunov matrices - the exponential stability assumption. On the other hand, it poses the new question: Does Definition 4.2 define for the case of exponentially stable system (1) the same Lyapunov matrix as that defined by improper integral (3)?

Theorem 4.2

Let system (1) be exponentially stable. Then matrix (3) is the unique solution of equation (4) that satisfies properties (5), (6).

Auxiliary delay free boundary problem

Auxiliary matrices

Let $U(\tau)$ be a Lyapunov matrix associated with a symmetric matrix W . We define two auxiliary matrices

$$Y(\tau) = U(\tau), \quad Z(\tau) = U(\tau - h), \quad \tau \in [0, h].$$

Auxiliary delay free boundary problem

Lemma 5.2

Let $U(\tau)$ be a Lyapunov matrix associated with a symmetric matrix W , then the auxiliary matrices satisfy the following delay free system of matrix equations

$$\frac{d}{d\tau}Y(\tau) = Y(\tau)A_0 + Z(\tau)A_1, \quad \frac{d}{d\tau}Z(\tau) = -A_1^T Y(\tau) - A_0^T Z(\tau), \quad (7)$$

and the boundary value conditions

$$Y(0) = Z(h), \quad A_0^T Y(0) + Y(0)A_0 + A_1^T Y(h) + Z(0)A_1 = -W. \quad (8)$$

Lyapunov matrices: Existence issue

Now we show that conversely, any solution of the boundary value problem (7)-(8) generates a Lyapunov matrix associated with W .

Theorem 5.2

If a pair $(Y(\tau), Z(\tau))$ is a solution of the boundary value problem (7)-(8), then matrix

$$U(\tau) = \frac{1}{2} [Y(\tau) + Z^T(h - \tau)], \quad \tau \in [0, h],$$

is a Lyapunov matrix associated with W , if we extend it to $[-h, 0]$ by setting $U(-\tau) = U^T(\tau)$, for $\tau \in (0, h]$.

Corollary 1.2

If the boundary value problem (7)-(8) admits a unique solution $(Y(\tau), Z(\tau))$, then matrix

$$U(\tau) = Y(\tau), \quad \tau \in [0, h],$$

is a unique Lyapunov matrix associated with W .

Lyapunov condition

Definition 5.2

We say that system (1) satisfies the Lyapunov condition if the spectrum of the system,

$$\Lambda = \left\{ s \mid \det \left(sI - A_0 - e^{-sh} A_1 \right) = 0 \right\},$$

does not contain a point s_0 such that $-s_0$ also belongs to the spectrum, or say it in other way, there are no eigenvalues of the system disposed symmetrically with respect to the origin of the complex plane.

Remark 1.2

If system (1) satisfies the Lyapunov condition, then it has no eigenvalues on the imaginary axis of the complex plane.

Uniqueness Theorem

Theorem 6.2

System (1) admits a unique Lyapunov matrix associated with a given symmetric matrix W if and only if the system satisfies the Lyapunov condition.

Complete type functionals

Given three symmetric matrices W_j , $j = 0, 1, 2$, let us define the functional

$$w(\varphi) = \varphi^T(0)W_0\varphi(0) + \varphi^T(-h)W_1\varphi(-h) + \int_{-h}^0 \varphi^T(\theta)W_2\varphi(\theta)d\theta.$$

If there exists a Lyapunov matrix $U(\tau)$, associated with matrix $W = W_0 + W_1 + hW_2$, compute the functional

$$\begin{aligned} v_0(\varphi) &= \varphi^T(0)U(0)\varphi(0) \\ &\quad + 2\varphi^T(0) \int_{-h}^0 U(-h-\theta)A_1\varphi(\theta)d\theta \\ &\quad + \int_{-h}^0 \varphi^T(\theta_1)A_1^T \left[\int_{-h}^0 U(\theta_1-\theta_2)A_1\varphi(\theta_2)d\theta_2 \right] d\theta_1. \end{aligned}$$

Theorem 7.2

The time derivative of the modified functional

$$v(\varphi) = v_0(\varphi) + \int_{-h}^0 \varphi^T(\theta) [W_1 + (h + \theta)W_2] \varphi(\theta) d\theta \quad (9)$$

along the solutions of system (1) is such that the following equality holds

$$\frac{d}{dt}v(x_t) = -w(x_t), \quad t \geq 0.$$

Definition 6.2

We say that functional (9) is of the complete type if matrices W_j , $j = 0, 1, 2$, are positive definite.

Lemma 6.2

Let system (1) be exponentially stable. Given positive definite matrices W_j , $j = 0, 1, 2$, then there exists $\alpha_1 > 0$, such that the complete type functional (9) admits the following quadratic lower bound

$$\alpha_1 \|\varphi(0)\|^2 \leq v(\varphi), \quad \varphi \in PC([-h, 0], R^n).$$

Lemma 7.2

Let system (1) satisfy the Lyapunov condition, see Definition 5.2. Given symmetric matrices W_j , $j = 0, 1, 2$, then for some positive α_2 the functional (9) satisfies the inequality

$$v(\varphi) \leq \alpha_2 \|\varphi\|_h^2, \quad \varphi \in PC([-h, 0], R^n).$$

Krasovskii Theorem revisited

We return now to Krasovskii Theorem and show that the conditions of the theorem are necessary for the exponential stability of system (1).






Theorem 8.2

System (1) is exponentially stable if and only if there exists a functional $v : PC([-h, 0], R^n) \rightarrow R$ such that the following conditions are satisfied

- 1. $\alpha_1 \|\varphi(0)\|^2 \leq v(\varphi) \leq \alpha_2 \|\varphi\|_h^2$, for some positive α_1, α_2 .*
- 2. For some $\beta > 0$ the inequality*

$$\frac{d}{dt}v(x_t) \leq -\beta \|x(t)\|^2, \quad t \geq 0,$$

holds along the solutions of the system.

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