# Robust Stabilization with Time Constraints: Implicit Lyapunov Function Approach

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# Outline

1. Introduction

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# Mechanical Example (Deceleration of a Cart)

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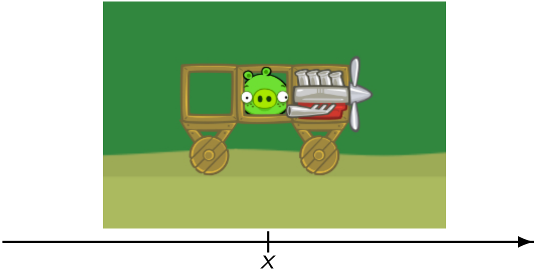

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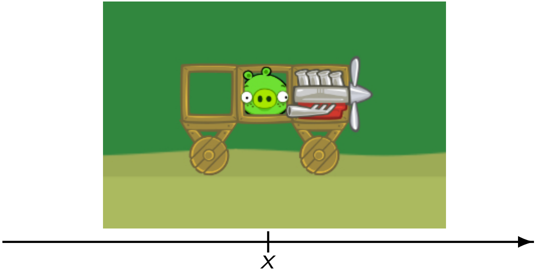


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$m$ - mass, $x$ - position and $k_d$, $k_v$ - coefficients of dry and viscous friction

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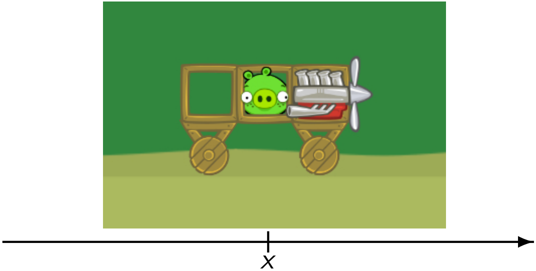


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$m$ - mass, $x$ - position and $k_d$, $k_v$ - coefficients of dry and viscous friction

$$\dot{x}(t) = 0 \;\text{ for all }\; t \geq T_{\max} = \frac{m\pi}{2\sqrt{k_d k_v}} \;\text{ and }\; \text{for any }\; (x_0, \dot{x}_0) \in \mathbb{R}^2$$

# Nonlinear Principles of Finite-Time Control

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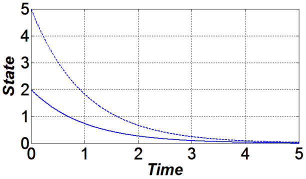
**Linear control**:

$$u(t) = -x(t)$$

$$x(t) = e^{-t}x_0 \to 0 \text{ if } t \to +\infty$$



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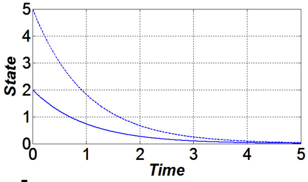


**Finite-time control:**

$$u(t) = -\,\mathrm{sign}[x(t)]$$

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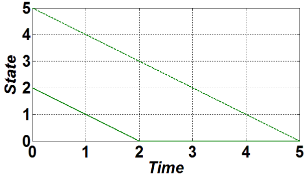

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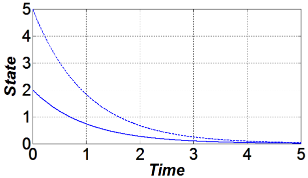


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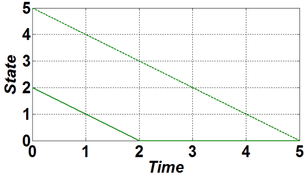
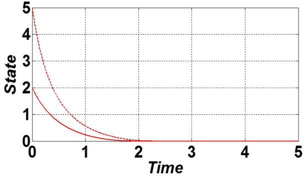



**Fixed-time control**:

$$u(t) = -\left(|x(t)|^{\frac{1}{2}} + |x(t)|^{\frac{3}{2}}\right)\mathrm{sign}[x(t)]$$

$$x(t) = 0 \text{ for } t \geq \pi$$

independently of $x_0$



**Homogeneus Systems**

# Definition of Weighted Homogeneity

$$r \in \mathbb{R}^n_+ \quad \text{and} \quad D_r(\lambda) = \begin{pmatrix} \lambda^{r_1} & 0 & \dots & 0 \\ 0 & \lambda^{r_2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \lambda^{r_n} \end{pmatrix}.$$

## Definition (Zubov 1958)

The vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is called $r$-homogeneous of degree $\nu \in \mathbb{R}$ if

$$f(D_r(\lambda)x) = \lambda^\nu D_r(\lambda) f(x), \qquad \forall \lambda > 0, \forall x \in \mathbb{R}^n.$$

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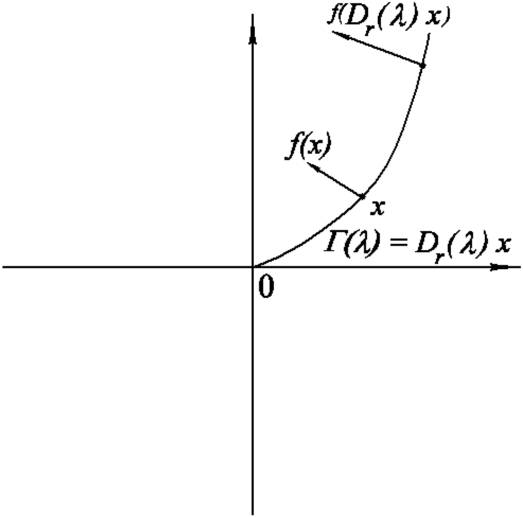
## Example

The system $\dot{x}(t) = f(x)$, $x = (x_1, x_2)^T \in \mathbb{R}^2$ with

$$f(x) := \begin{pmatrix} x_2 \\ -k_1|x_1|^{\frac{\alpha}{2-\alpha}} \operatorname{sign}[x_1] - k_2|x_2|^\alpha \operatorname{sign}[x_2] \end{pmatrix}$$

is $r$-homogeneous of degree $\nu = \alpha - 1$ with the weights $r = (2-\alpha, 1)$, $\alpha \geq 0$.

# Geometric Illustration of Weighted Homogeneity




$$f(D_r(\lambda)x) = \lambda^{\nu} D_r(\lambda) f(x)$$

# Properties

Linear system is homogeneous of degree 0 with $D(\lambda) = \lambda I$

| | **Linear System** $\dot{x} = Ax$ | **Homogeneous System** $\dot{x} = f(x)$ |
|---|---|---|
| **Trajectory Scaling** | $y(t, \lambda x_0) = \lambda x(t, x_0)$ | $y\left(t, D(\lambda) x_0\right) = D(\lambda) x(\lambda^\nu t, x_0)$ |
| **Local $\Leftrightarrow$ Global** | ✓ | ✓ |
| **Invariance $\Leftrightarrow$ Stability** | ✓ | ✓ |
| **Robustness** (Input-to-State Stability) | $\dot{x} = Ax + Dw$ <br> $w \in \mathbb{L}^\infty$ | $\dot{x} = f(x, w)$ <br> $w \in \mathbb{L}^\infty$ |
| **Control Design Tool** for $\dot{x} = Ax + Bu$ | Linear Matrix Inequalities <br> $AX + XA^T + BY + (YB)^T < 0$ | Linear Matrix Inequalities <br> (via Implicit Lyapunov Function) |

**Implicit Lypunov Function Method**

# Implicit Lyapunov Function (ILF)

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$$\exists V : \mathbb{R}^n \to \mathbb{R} \quad V(x) > 0 \text{ and } \dot{V} = \frac{\partial V}{\partial x} f(x) \leq 0 \; (< 0) \;\; \text{for all } x \in \mathbb{R}^n \neq 0$$

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$\exists Q : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that the equation $Q(V, x) = 0$ implicitly defines the Lyapunov function $V(x)$, i.e. $\exists V(x) > 0 : Q(V(x), x) = 0$.

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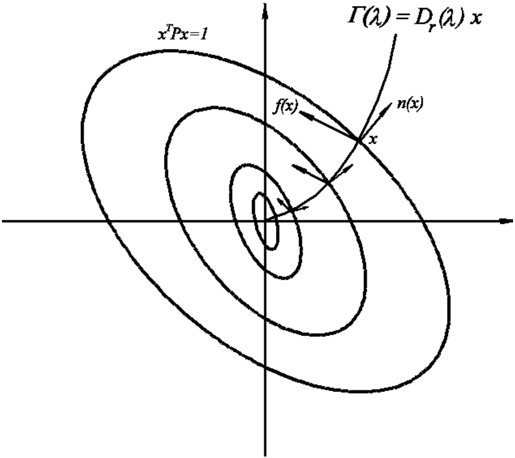
$$\frac{\partial V}{\partial x} = -\left[\frac{\partial Q}{\partial V}\right]^{-1} \frac{\partial Q}{\partial x}$$

# Implicit Lyapunov Function for Homogeneous System (I)




$$x^T D_r(\lambda) P D_r(\lambda) x = 1, \lambda > 0$$

# Implicit Lyapunov Function for Homogeneous System (II)

Let us consider the following ILF candidate

$$Q(V,x) := x^T D_r(V^{-1}) P D_r(V^{-1}) x - 1, \tag{1}$$

$V \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, $D_r(\cdot)$ is a dilation matrix, $P \in \mathbb{R}^{n\times n} : P = P^T > 0$ .

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## Theorem (Polyakov et al 2014)

*If $f:\mathbb{R}^n \to \mathbb{R}^n$ is r-homogeneous of degree $\mu$ with $r=(r_1,r_2,\ldots,r_n)^T \in \mathbb{R}^n_+$*

$$\mathrm{diag}(r)P + P\,\mathrm{diag}(r) > 0, \quad P > 0$$

$$z^T P f(z) + f^T(z) P z \le -z^T(\mathrm{diag}(r)P + P\,\mathrm{diag}(r))z, \quad z \in \mathbb{R}^n : z^T P z = 1$$

*then the function (1) is ILF of the system $\dot{x} = f(x)$ and its origin is*

- *asymptotically stable for $\mu > 0$;*
- *exponentially stable for $\mu = 0$;*
- **finite-time stable** *for $\mu < 0$.*

## Implicit Feedback Control

# Robust Stabilitzation with Time-Constraints[1]

$$\dot{x}(t) = Ax(t) + Bu(t) + d(t, x(t)),$$

$$x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \quad d : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n.$$

## Implicit Feedback Control

$$u = KD_r(V^{-1})Gx \quad \text{with} \quad Q(V, x) = 0$$

where $D_r$ is a proper dilation matrix, $K = YX^{-1}$,

$$\begin{cases} \tilde{A}X + X\tilde{A}^T + \tilde{B}Y + Y^T\tilde{B}^T + \mathrm{diag}(r)X + X\,\mathrm{diag}(r) \leq 0, \\ X\,\mathrm{diag}(r) + \mathrm{diag}(r)X > 0, \quad X > 0, \quad X \in \mathbb{R}^{n\times n}, \quad Y \in \mathbb{R}^{m\times n}. \end{cases}$$

Settling time (for $d = 0$) is estimated as follows

$$T(x_0) \leq V_0 \quad \text{with} \quad Q(V_0, x_0) = 0$$

[1] *Polyakov et al*, Automatica, 2015; *Polyakov et al*, Int. J. Rob. Nonlin. Control, 2015

# Robust Stabilitzation with Time-Constraints[2]

$$\dot{x}(t) = Ax(t) + Bu(t) + d(t, x(t)),$$

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$$u = KD_{r_i}(V^{-1})Gx \quad \text{with} \quad Q(V, x) = 0$$

where $D_{r_i}$, $i = 1, 2$ is a proper dilation matrix, $K = YX^{-1}$,

$$\begin{cases} \tilde{A}X + X\tilde{A}^T + \tilde{B}Y + Y^T\tilde{B}^T + \mathrm{diag}(r_i)X + X\,\mathrm{diag}(r_i) \leq 0, \\ X\,\mathrm{diag}(r_i) + \mathrm{diag}(r_i)X > 0, \quad X > 0, \quad X \in \mathbb{R}^{n\times n}, \quad Y \in \mathbb{R}^{m\times n}. \end{cases}$$

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[2] *Polyakov et al*, Automatica, 2015; *Polyakov et al*, Int. J. Rob. Nonlin. Control, 2015

## Example: Time Optimal Implicit Feedback Control

# Problem Statement

$$\dot{x}(t) = Ax(t) + bu(t), \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}$ is the control input,

$$A = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ \dots \\ 0 \\ 1 \end{pmatrix},$$

## Minimum Time Control Problem

$$T \to \min \tag{3}$$

subject to

$$\begin{cases} \dot{x}(t) = Ax(t) + bu(t), \\ x(0) = x_0, \quad x(T) = 0, \end{cases} \tag{4}$$

and the control $u$ is a bounded homogeneous feedback, $|u(x)| \leq u_0 \in \mathbb{R}^+$.

# On Optimal Stabilizing Feedbacks [3]

## Optimal Stabilization Problem

$J(x(\cdot), u(\cdot)) = \int_0^{+\infty} w(x(\tau), u(\tau))d\tau \to \min$

subject to

$$\dot{x}(t) = f(x(t), u(t)), \quad t > 0 \quad x \in \mathbb{R}^n, \quad u \in \mathbb{U} \subset \mathbb{R}^m,$$

where $w(\cdot, \cdot)$ is positive definite, $f(0, 0) = 0$ and $u(\cdot) := u(x(\cdot))$ is continuous feedback such that $\dot{x} = f(x, u(x))$ is asymptotically stable.

[3] see, for example, Letov 1962, Krasovski 1968, et al.

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## Sufficient Condition of Optimality

The feedback $u = u(x)$ is optimal, if there exists a Lyapunov function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ such that

- $\dot{V} = -w(x, u(x))$
- $\dot{V} > -w(x, u)$ for $u \in \mathbb{U}$ such that $u \neq u(x)$.

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$$J(x(\cdot),u(\cdot)) = \int_0^{+\infty} w(x(\tau),u(\tau))d\tau \to \min \qquad \left( w = \begin{cases} 1 & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases} \right)$$

subject to

$$\dot{x}(t) = f(x(t),u(t)), \quad t > 0 \quad x \in \mathbb{R}^n, \quad u \in \mathbb{U} \subset \mathbb{R}^m,$$

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- $\dot{V} = -w(x,u(x)) = \begin{cases} -1 & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases};$
- $\dot{V} > -w(x,u)$ for $u \in \mathbb{U}$ such that $u \neq u(x)$.

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# ILF Candidate for Finite-time Stability Analysis

For a chain of integrators (2) consider the ILF candidate

$$Q(V, x) := x^T D_r(V^{-1}) P D_r(V^{-1}) x - 1 \tag{5}$$

where $D_r(\lambda)$ is the dilation matrix with $r = (n,\ n-1,\ \ldots,\ 1)^T$

$$D_r(\lambda) = \begin{pmatrix} \lambda^n & 0 & \ldots & 0 \\ 0 & \lambda^{n-1} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \lambda \end{pmatrix}.$$

# Finite-time Stabilization of a Chain of Integrators

## Proposition (A. Polyakov and J.-P. Richard 2014)

*If the system of matrix inequalities*

$$\begin{cases} AX + XA^T + by + y^T b^T + X\,\mathrm{diag}(r) + \mathrm{diag}(r)X = 0, \\ X\,\mathrm{diag}(r) + \mathrm{diag}(r)X > 0, \quad X > 0 \end{cases} \tag{6}$$

*is feasible for some* $X \in \mathbb{R}^{n\times n}$ *and* $y \in \mathbb{R}^{1\times n}$ *then*

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*2) the closed-loop system* (2) *with the feedback law*

$$u(x) = kD_r(V^{-1}(x))x, \tag{7}$$

*and* $k = yX^{-1}$ *is globally finite-time stable;*

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*3) the settling time function has the form* $T(x_0) = V(x_0)$ *and*
$\dot{V}(x) = -1$ *for* $x \neq 0$*.*

# Properties of ILF-based Feedback

- The control is smooth outside the origin and discontinuous at the origin.
- $u(t) = u(V(t), x(t))$, where $V(t) = V_0 - t$ for $t \in [0, V_0)$.

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**Corollary**

*(1) If for some $T \in \mathbb{R}_+$ the LMI system (6) is feasible together with*

$$\begin{pmatrix} 1 & x_0^\top \\ x_0 & D_r(T) X D_r(T) \end{pmatrix} \geq 0, \tag{8}$$

*then*

$$T(x_0) \leq T,$$

*where $x_0 \in \mathbb{R}^n$ is a given initial state.*

# Suboptimal Feedback Design

## Corollary (2)

*Let* $T \in \mathbb{R}_+$ *and* $u_0 \in \mathbb{R}_+$ *the LMI system* (6), (8) *is feasible together with*

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*where* $X \in \mathbb{R}^{n \times n}$ *and* $y \in \mathbb{R}^{1 \times n}$*. Then the ILF control* (7) *is bounded by* $u_0$*, i.e.*

$$|u(x)| \leq u_0 \quad \forall x \in \mathbb{R}^n.$$

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$$\begin{pmatrix} X & y^\top \\ y & u_0^2 \end{pmatrix} \geq 0, \tag{9}$$

*where $X \in \mathbb{R}^{n\times n}$ and $y \in \mathbb{R}^{1\times n}$. Then the ILF control* (7) *is bounded by $u_0$, i.e.*

$$|u(x)| \leq u_0 \quad \forall x \in \mathbb{R}^n.$$

## SDP formulation

$$T \to \min_{X\in\mathbb{R}^n, y\in\mathbb{R}^{1\times n}} \quad \text{subject to } (6), (8), (9). \tag{10}$$

# On Robustness of Sampled-Time Implementation

$$u(V,x) = kD_r(V^{-1})x, \text{ where } Q(V,x) = 0.$$

## Corollary (3)

*Let* $\{t_i\}_{i=0}^{\infty}$ *be an arbitrary strictly increasing sequence of time instants,* $0 = t_0 < t_1 < t_2 < \ldots$.
*Let the function* $u(V,x)$ *be defined according to Proposition 1 then the origin of the system* (2) *with the switching control*

$$u(x) := u(V_i, x) \quad \textit{for} \quad t \in [t_i, t_{i+1}) \tag{11}$$

*where* $V_i > 0 : Q(V_i, x(t_i)) = 0$ *is globally asymptotically stable.*

## Numerical Example

# Sub-optimal ILF Control vs Time Optimal Control

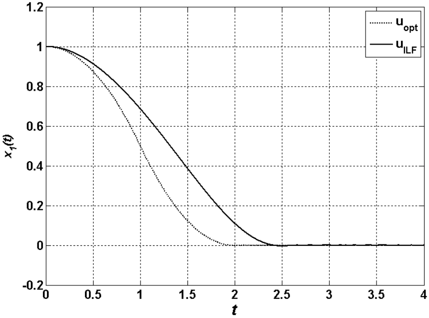
- Let $n = 2$ and $|u| \le 1$. The optimal feedback is

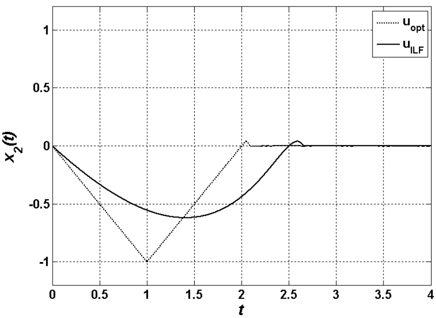
$$u_{opt}(x_1, x_2) = -\operatorname{sign}\left(x_2 + \sqrt{2|x_1|}\operatorname{sign}[x_1]\right).$$

- The ILF-based sub-optimal feedback is designed by the optimization procedure (10) for $\mu = 1$ and $x_0 = (1, 0)$ has the parameters

$$P = \begin{pmatrix} 49.6139 & 18.8965 \\ 18.8965 & 9.4482 \end{pmatrix}, \quad k = (-5.2511 \quad -3.0000).$$

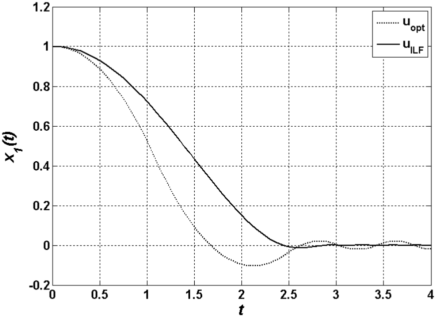
# Numerical Simulation I ("sampling-period=0.001")

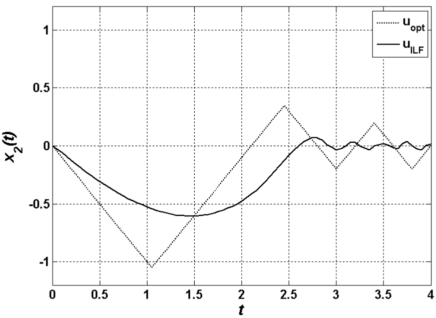



Simulations results for fast sampling

# Numerical Simulation II ("sampling-period=0.05")




Simulations results for slow samplings

**Thank you for your attention**