When the Expected Value is not Expected

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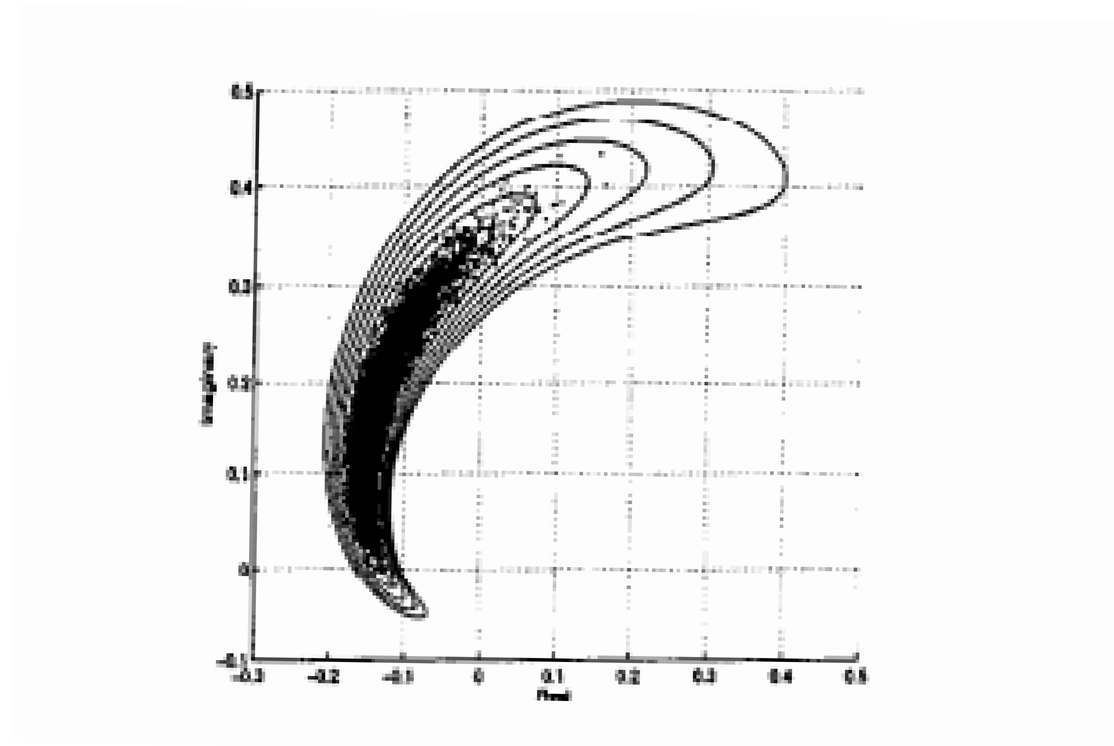
An Exciting Moment with Boris

We were working one day in Matlab (1995)

On our screen we saw

Polyak's Banana?

What we actually saw

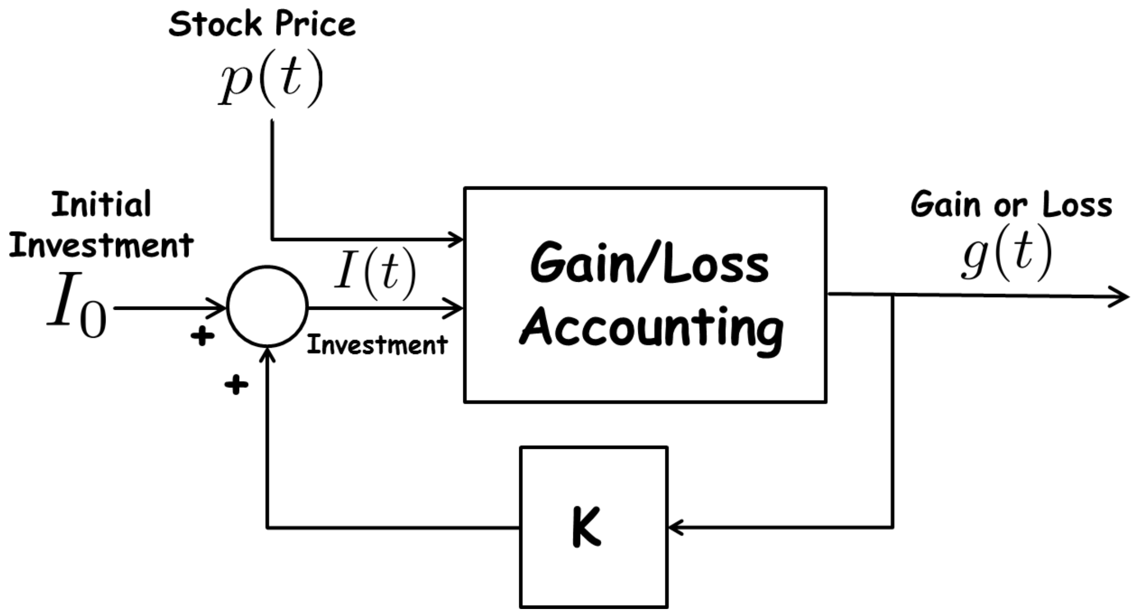


This Talk Describes

A New Definition: The "Conservative Expected Value"

My Personal Motivation

Stock Trading Via Feedback Control



$$I(t) = I_0 + Kg(t)$$

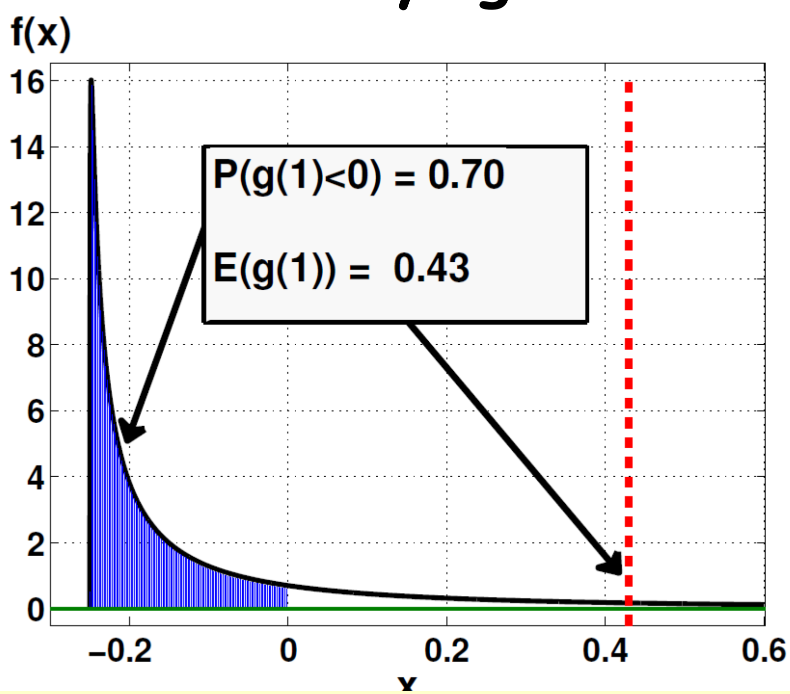
Probability density function for g(t) can be highly skewed

My Personal Motivation (2)

$$\mu = 0.25, \ \sigma = 0.5, \ K = 4, \ t = 1$$

$$f(x) = \frac{1}{\sqrt{\frac{\pi}{2}}(1+4x)} e^{-\frac{\left(\log(1+4x)+1\right)^2}{8}}$$ **for x >-0.25.**

- **Large Right-Sided Skewed Return**
- **Distrust in Underlying Distribution**



A Discounted Alternative to Expected Value?

Mindset for CEV Definition

- The Larger X, the Better
- High Risk Aversion "Conservative"
- Mission-Critical Considerations
- Robustness Considerations wrt PDF
- Distrust in Probability Distribution
- Worst-Case X is Bounded "Support"

We first spend some time motivating:

The Conservative Expected Value: $\mathbb{CEV}(X)$

Saying the Obvious

Conservatism Dictates

$$\mathbb{CEV}(X) \leq \mathbb{E}(X)$$

The theory also includes an associated risk measure called Conservative Semi-Variance (CSV)

$$\mathbb{CSV}(X)$$

leading to New Reward-Risk Pair (CEV,CSV)

Towards Definition of Conservative Expected Value (CEV)

The Conservative Expected Value (CEV) is defined for random variables with finite leftmost point support, e.g., the worst-case loss in trading via feedback, the lifetime of a component in a system

Given a random variable X with the CDF $F_X(x)$, **with leftmost support point**

$$\alpha_X \doteq \inf\{x : x \in \mathbb{R} \text{ such that } F_X(x) > 0\}$$

Examples:

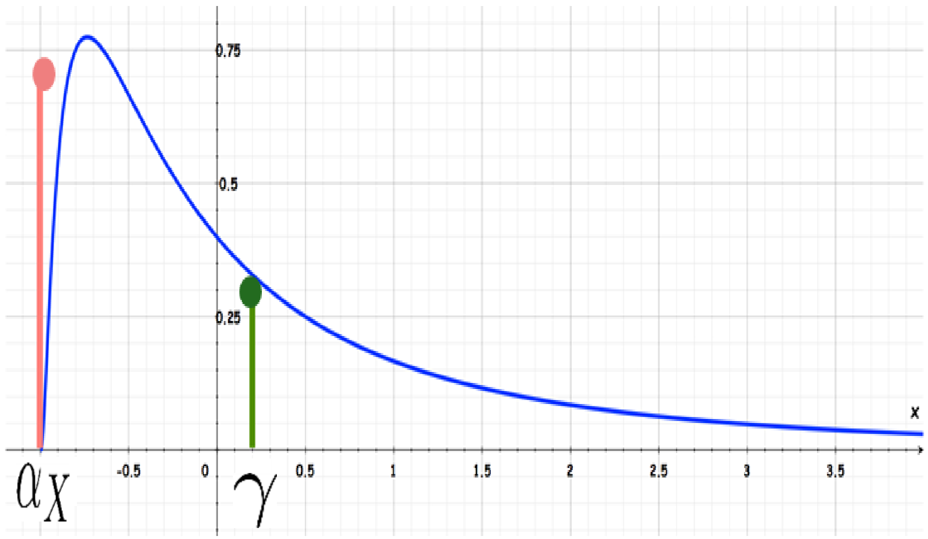
$$X \sim \text{Uniform}[a, b] \rightarrow \alpha_X = a$$

$$X \sim \exp(\lambda) \rightarrow \alpha_X = 0$$

Bernoullizing Procedure: Editing

Consider Target: γ

Mass-Shifted Random Variable: X_γ



This is similar to what happens in the "Editing Phase" in Prospect Theory. Kahnemann and Tverski (1979)

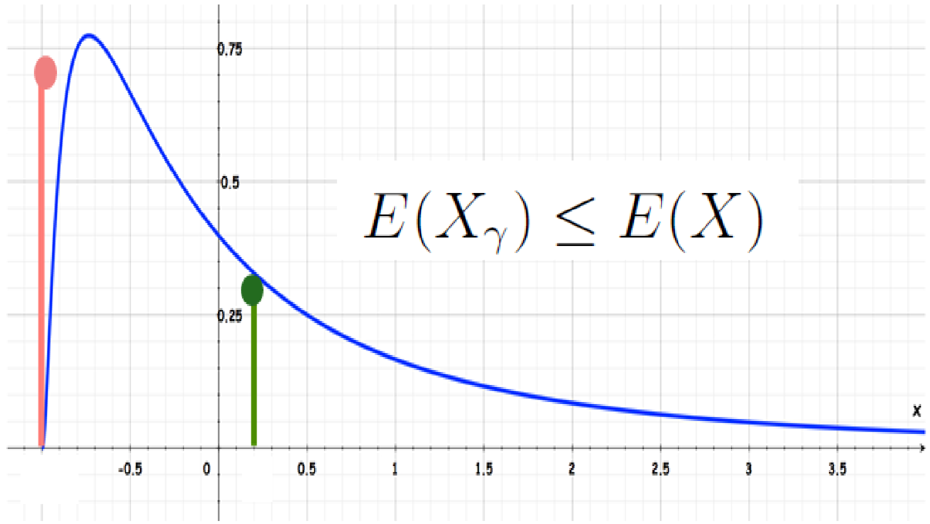
$$E(X_\gamma) \leq E(X)$$

What value of γ to use?

How conservative is too conservative?

Conservative Expected Value: Defined

What Value of γ to Use?



$$\mathbb{E}[X_\gamma] = \alpha_X P(X < \gamma) + \gamma\big(1 - P(X \geq \gamma)\big)$$

$$\begin{aligned}\mathbb{CEV}(X) &\doteq \sup_\gamma \mathbb{E}(X_\gamma) \\ &= \sup_\gamma \alpha_X F_X(\gamma) + \gamma\big(1 - F_X(\gamma)\big)\end{aligned}$$

Why supremum above? Interpretation?

Understanding the CEV

Domination Property

$$\mathbb{CEV}(X) \doteq \sup_{\gamma} \mathbb{E}(X_\gamma)$$
$$= \sup_{\gamma} \alpha_X F_X(\gamma) + \gamma\big(1 - F_X(\gamma)\big)$$

Suppose supremum above attained by γ^*. That is,

$$\mathbb{CEV}(X) = \mathbb{E}(X_{\gamma^*})$$

Then for $\gamma < \gamma^*$, the pair $(\gamma, \mathbb{E}(X_\gamma))$ is dominated by $(\gamma^*, \mathbb{CEV}(X))$

For $\gamma > \gamma^*$, there is a tradeoff.

Understanding the CEV (2)

Geometric Interpretation

Consider the complementary CDF

$$\bar{F}_X(x) = 1 - F_X(x)$$

For a non-negative random variable, we use fact that

$$\mathbb{E}(X) = \int_0^\infty \bar{F}_X(x)dx$$

Now for X having zero as left support point, we have

$$\mathbb{CEV}(X) = \sup_\gamma \gamma P(X > \gamma) = \sup_\gamma \gamma \bar{F}_X(\gamma)$$

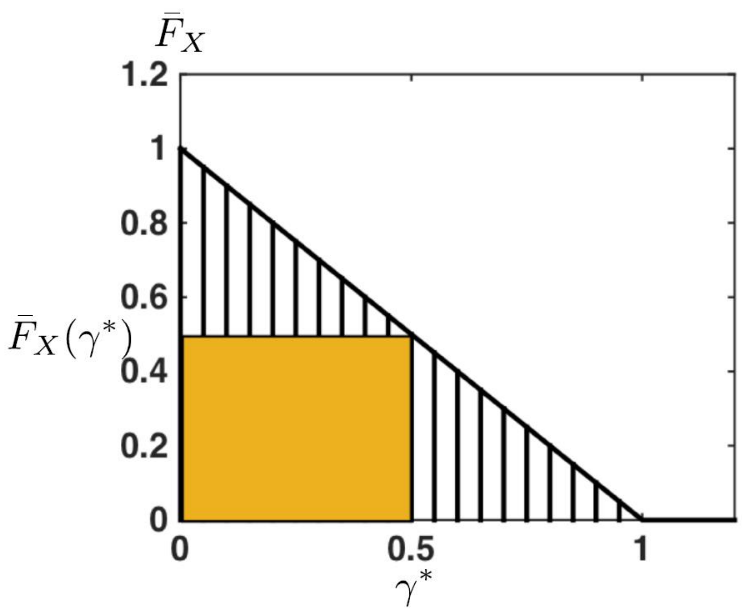
Let's consider the geometry associated with these facts:

Understanding the CEV (3)

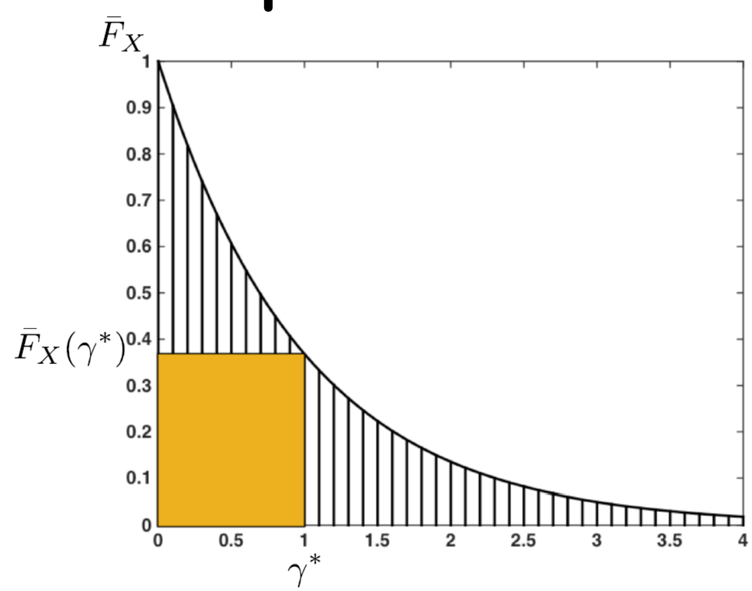
Geometric Interpretation

$$\mathbb{E}(X) = \int_0^\infty \bar{F}_X(x)dx \qquad \mathbb{CEV}(X) = \sup_\gamma \gamma P(X > \gamma) = \sup_\gamma \gamma \bar{F}_X(\gamma)$$

Uniform on [0,1]



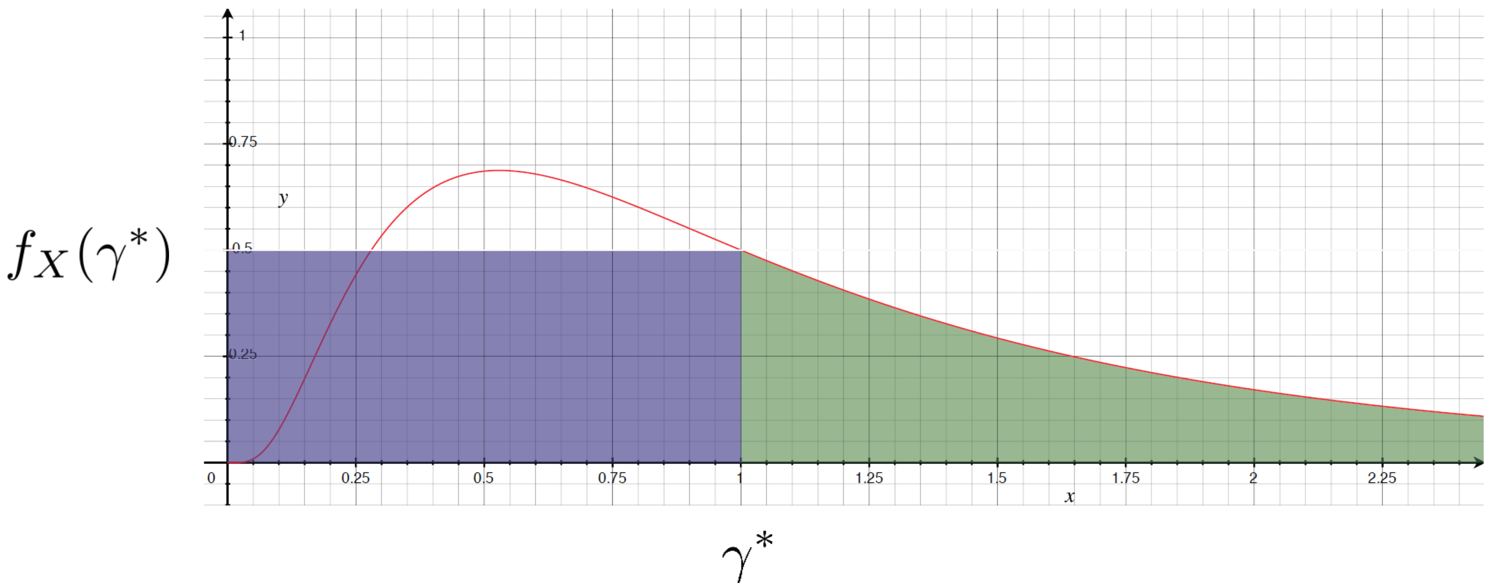
Exponential PDF



CEV = Area of Maximal Inscribed Rectangle

Understanding the CEV (3)

Equivalent Geometric Interpretation



Two Equal Areas Under the PDF

CEV of Famous Distributions

Random Variable	Expected Value	Conservative Expected Value
Uniform [a,b]	(a+b)/2	(3a+b)/4
Exponential (λ)	1/λ	exp(-1)/λ
Bernoulli (p)	(1-p)	(1-p)

Nicolas Bernoulli

1729 Portrait

Example: St. Petersburg Paradox

$$X = 2^k \quad \text{with probability} \quad \left(\frac{1}{2}\right)^{k+1}, \quad k = 0, 1, 2, 3, \ldots$$

$$\mathbb{E}(X) = \infty$$

How much would you pay to play this game?

For $\gamma \in [2^k, 2^{k+1})$ **, we obtain** $\mathbb{E}(X_\gamma) = 1 + \frac{\gamma - 1}{2^{k+1}}$

Then a straightforward calculation leads to

$$\mathbb{CEV}(X) = 2$$

Properties of the CEV

Bounds on CEV:

$$\frac{\text{median}(X) + \alpha_X}{2} \leq \mathbb{CEV}(X) \leq \mathbb{E}(X)$$

Affine Linearity:

$$\mathbb{CEV}(aX + b) = a\mathbb{CEV}(X) + b$$

Finiteness Condition:

$$\mathbb{CEV}(X) < \infty \Longleftrightarrow \limsup_{\gamma \to \infty} \gamma\big(1 - F_X(\gamma)\big) < \infty.$$

Infinite Possible:

$$f_X(x) = \frac{1}{2x^{3/2}} \quad x \geq 1.$$

Properties of CEV (2)

Mean of Large IID Sums

Theorem: *For positive integers k, let X_k be a sequence of i.i.d. random variables with finite mean μ, finite variance σ^2 and finite leftmost support point. Then, with sequence of means*

$$M_n \doteq \frac{1}{n}\sum_{k=1}^{n} X_k,$$

it follows that

$$\lim_{n\to\infty} \mathbb{CEV}(M_n) = \mu$$

and

$$\lim_{n\to\infty} \mathbb{CSV}(M_n) = 0.$$

Properties of CEV (3)

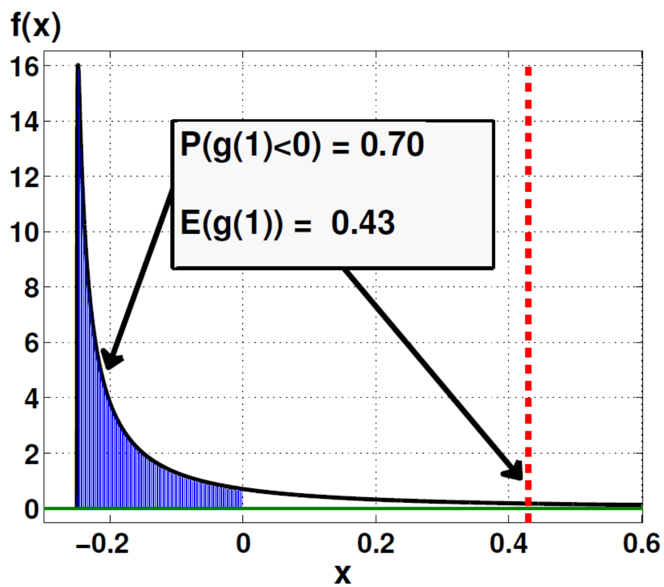
Convexity Property

Lemma: *Let the probability density function f_X of the random variable X be the convex combination above of the probability density functions f_{X_i} of the n random variables, $X_1, X_2, \ldots, X_n$. Then X has a conservative expected value satisfying*

$$\mathbb{CEV}(X) \leq \sum_{i=1}^{n} \lambda_i \mathbb{CEV}(X_i).$$

Revisit of Motivating Example

Trading Stock With Linear Feedback



Recall PDF for gains-losses g(t):

$$f(x) = \frac{1}{\sqrt{\frac{\pi}{2}}(1+4x)} e^{-\frac{\left(\log(1+4x)+1\right)^2}{8}}$$

for x >-0.25.

E(g(1)) = 0.43 VERSUS CEV(g(1)) ≈ -0.12

The CEV more accurately reflects attractiveness of trade without introducing any additional moments.

Concluding Remarks

- Model Distrust, Large Skewness
- CEV: Bernoullizing and Optimization
- Avoidance of Higher Order Moments
- Really (CEV,CSV) Combination Used
- Some Basic Properties: Rich Theory

Boris Polyak

- Fantastic Field-Changing Results
- Inspiration, Endurance, Enthusiasm
- Depth of Technical Thought
- Collegiality and Friendship

Happy Birthday Boris