Randomization and Gossiping in Social Networks

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Opinion Dynamics in Social Networks

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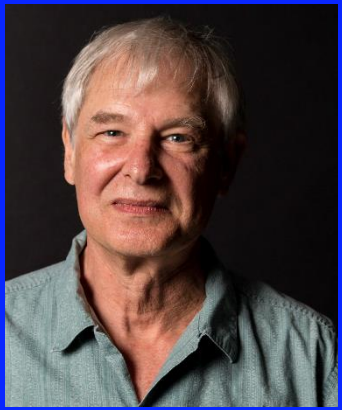
Joint work with



Sergey Parsegov Anton Proskurnikov Noah Friedkin

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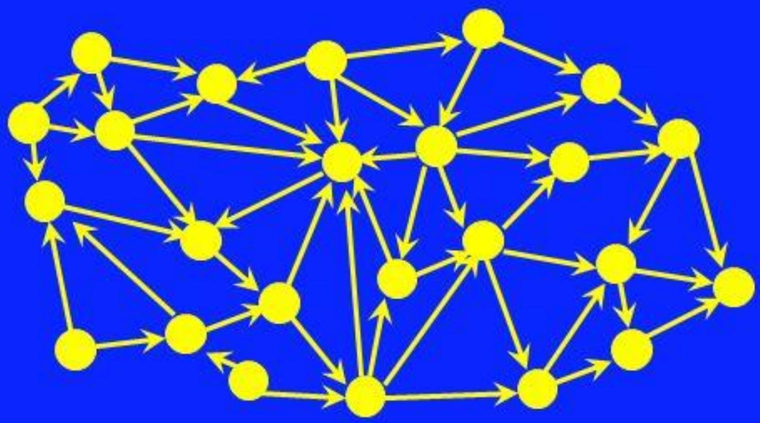
Networks of Interpersonal Influences

- Individuals discuss a certain topic based on interpersonal influences



- How opinions evolve over time?
- Opinions converge or fluctuate?
- Opinions aggregate into clusters?
- Opinion consensus or disagreement?

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Friedkin and Johnsen Model of Opinions - 1

- Discrete time model of opinions

$$x(k+1) = \Lambda W\, x(k) + (I-\Lambda)\, u \qquad x(0) = u$$

x is the belief or opinion $\qquad x \in \mathbf{R}^n$

n agents and each agent's opinion is a scalar value

u is the prejudices $\qquad u \in \mathbf{R}^n$

W interpersonal influences beetween agents

Λ (diag) sensitivity to opinion of other agents (weights)

N.E. Friedkin and E.C. Johnsen (1999)

Friedkin and Johnsen Model of Opinions - 2

- Discrete time model of opinions

$$x(k+1) = \Lambda W\, x(k) + (I-\Lambda)\, u \qquad x(0) = u$$

endogeneous *exogeneous*

W interpersonal influences beetween agents

Λ (diag) sensitivity to the opinion of other agents

- averaging: W is row stochastic ($W\mathbf{1} = \mathbf{1}$)

- diagonal sensitivity matrix: $0 \leq \Lambda \leq I$

sensitivity and interpersonal influences coupling: $\Lambda = I - \mathrm{diag}(W)$

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Example (Friedkin and Johnsen)

$u = [25\ 25\ 75\ 85]^T$ $u = x(0)$ prejudices

$$W = \begin{bmatrix} 0.220 & 0.120 & 0.360 & .300 \\ .147 & .215 & .344 & .294 \\ 0 & 0 & 1 & 0 \\ .090 & .178 & .446 & .286 \end{bmatrix}$$

W strength of interactions

Λ sensitivity

$$\Lambda = I - \text{diag}(W)$$

sensitivity and strength of interactions coupling

$$\Lambda = \text{diag}\begin{bmatrix} .780 & .785 & 0 & .714 \end{bmatrix}$$

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How the Opinions Evolve over Time?

- The opinion profile of agents is given by

$$x(k) = \left[(\Lambda W)^k + \sum_{j=0}^{k-1} (\Lambda W)^j \ (\mathrm{I}-\Lambda) \right] u$$

- Question: Do the opinions converge to a *stationary point* for $k \to \infty$?

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Convergence of Friedkin and Johnsen (FJ) Model

- Definition [convergence]
- FJ model is convergent if the opinions have a limit for $k \to \infty$

$$x^{opd} = \lim_{k\to\infty} x(k) \quad => \quad x^{opd} = \Lambda W x^{opd} + (I-\Lambda)\, u$$

for any prejudices u

- *Sufficient condition*: if ΛW is Schur stable, then the opinions converge to the unique stationary point

$$x^{opd} = (I - \Lambda\, W)^{-1}\, (I-\Lambda)\, u$$

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Stubborness and Oblivion

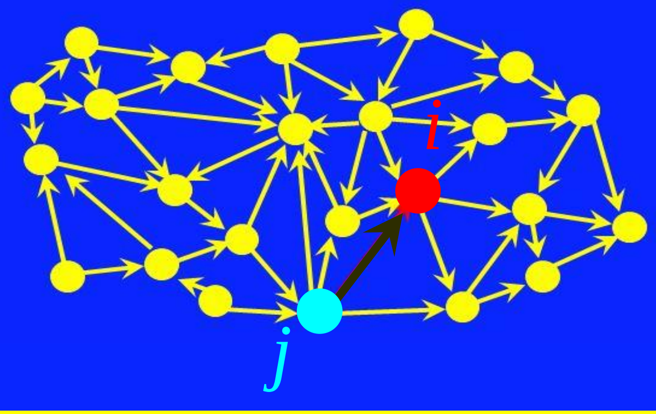
- Definition [stubbornness]

  - agent i is *stubborn* if $\lambda_{ii} < 1$

  - agent i is *totally stubborn* if $\lambda_{ii} = 0$

  - agent i is *implicitly stubborn* if it is stubborn or is influenced by at least a stubborn agent j

- Agents that are not implicitly stubborn are *oblivious* (nondegenerate case)



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Stability and Convergence of FJ Model

- Theorem [stability and convergence]

  - FJ model is Schur stable if and only if there are no oblivious agents

  - FJ model is convergent if and only if the matrix ΛW is regular

    ($\lim_{k\to\infty} (\Lambda W)^k$ exists)

Presence of oblivious agents is the only reason for instability

Regularity is the only condition needed for convergence

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DeGroot Opinion Pooling for Consensus - 1

- Uncoupled sensitivity and interpersonal influences
- FJ: *generalization* of DeGroot model

$$x(k+1) = W\, x(k)$$

(take $\Lambda = I$)

- This model is *never* stable
- Converges to a consensus x^{opd} if W is primitive

(W^m is positive for some $m > 0$)

- Convergence of DeGroot model is N&S for regularity and convergence of FJ model

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Example (Friedkin and Johnsen) - 1

$$u = [25\ 25\ 75\ 85]^T$$

$$W = \begin{bmatrix} 0.220 & 0.120 & 0.360 & .300 \\ .147 & .215 & .344 & .294 \\ 0 & 0 & 1 & 0 \\ .090 & .178 & .446 & .286 \end{bmatrix}$$

all agents are stubborn

agent 3 is totally stubborn

no oblivious agents

$\Rightarrow$ FJ model is Schur stable

$\Rightarrow$ it is convergent to a unique stationary point

$$\Lambda = \mathrm{I} - \mathrm{diag}(W)$$

$$\Lambda = \mathrm{diag}\left[.780 \quad .785 \quad 0 \quad .714\right]$$

$$x^{opd} = (\mathrm{I} - \Lambda\, W)^{-1}\, (\mathrm{I}-\Lambda)\, u$$

$$x^{opd} = \left[60 \quad 60 \quad 75 \quad 75\right]^T$$

Example (Friedkin and Johnsen) - 2

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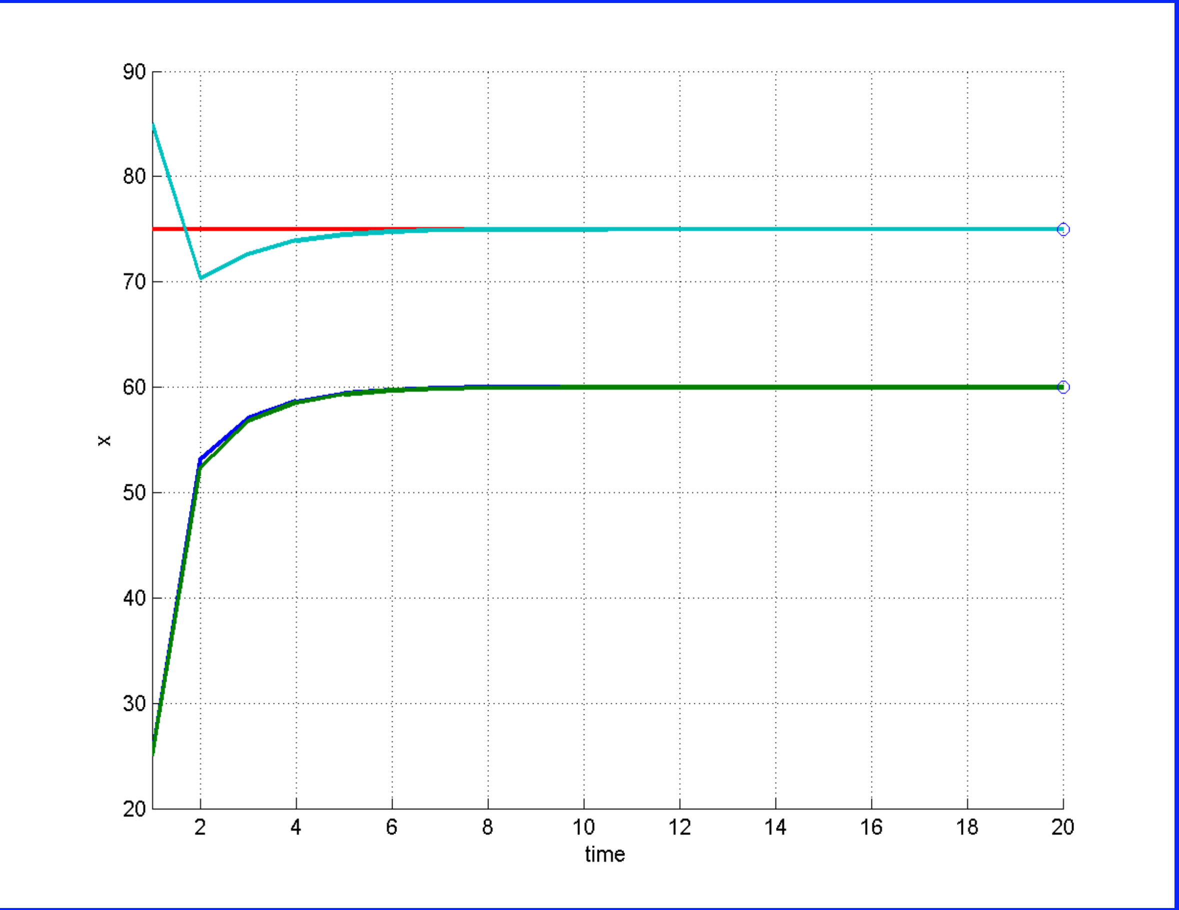
$u = [25\ 25\ 75\ 85]^T$

- Study opinion profile
- R (totally stubborn)
- B (stubborn)
- R and B reach a consensus
- Two distinct opinion clusters are formed
- Global consensus is not achieved



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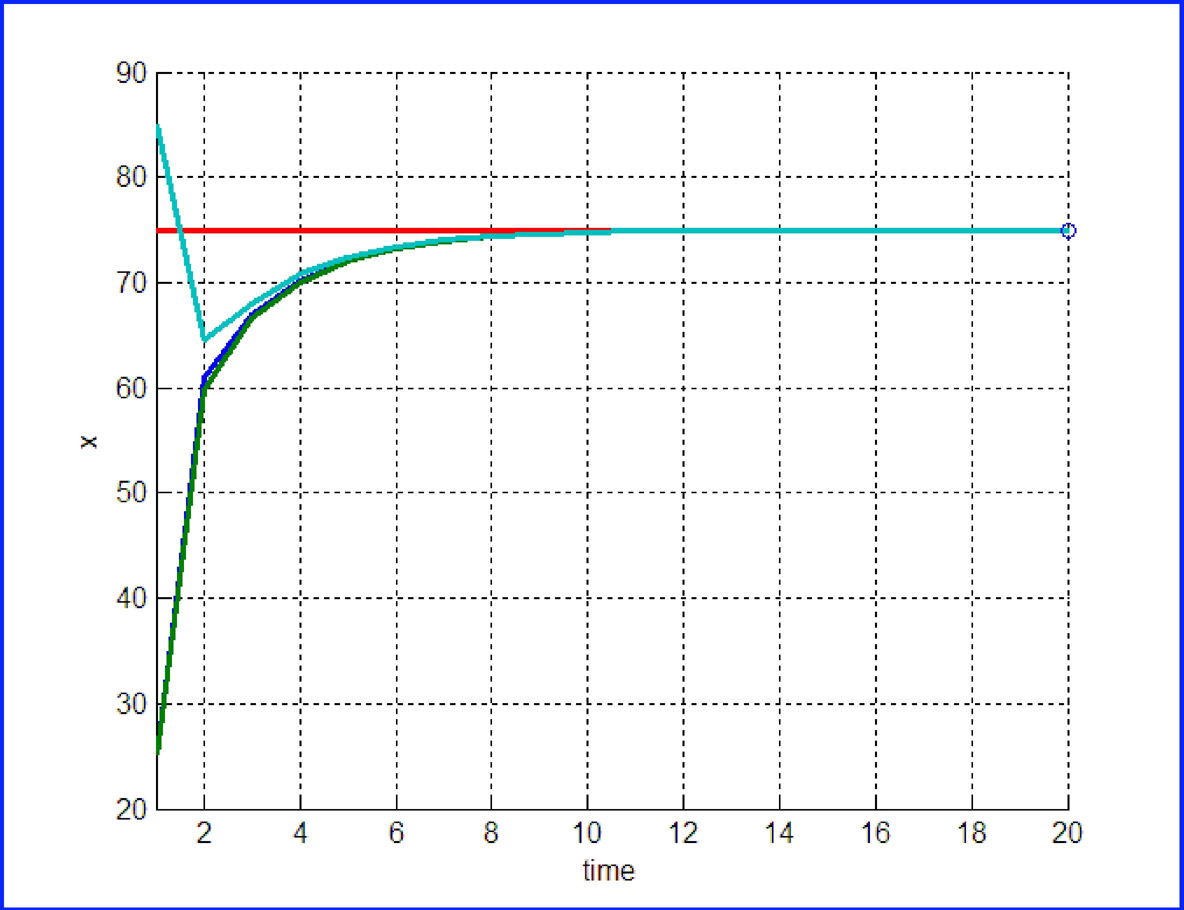
Example (DeGroot)

$u = [25\ 25\ 75\ 85]^T$

- Study opinion profile
- R (totally stubborn)
- Opinions converge to the totally stubborn agent
- We reach consensus



Multidimensional Model of Opinions

Multidimensional Model of Opinions - 1

- Motivations: FJ model is univariate (n agents discuss one issue)
- Multidimensional extension: *n agents discuss m issues*

$$x_1(k), \dots , x_n(k) \quad x_i(k) \in \mathbf{R}^m$$

- Multi-issue dependence

$$y_i(k) = C\, x_i(k) \qquad y_i \text{ is the } \textit{impact}$$

Matrix C: multi-issues dependence structure (MiDS)

- C is row stochastic

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Example (Multidimensional Model) - 1

$$u = [25 \;\; 25 \;\; 25 \;\; 15 \;\; 75 \;\; -50 \;\; 85 \;\; 5]^T$$

Two interdependent issues: fish and salmon

$$W = \begin{bmatrix} 0.220 & 0.120 & 0.360 & .300 \\ .147 & .215 & .344 & .294 \\ 0 & 0 & 1 & 0 \\ .090 & .178 & .446 & .286 \end{bmatrix}$$

agent 3 is totally stubborn

strong liking for fish

disliking salmon

$$\Lambda = \mathrm{diag}\left[.780 \quad .785 \quad 0 \quad .714\right]$$

MiDS matrix C of multi-issue dependence

$$C = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}$$

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Multidimensional Model of Opinions - 2

- Vectors of opinions and prejudices

$$x(k) = [x_1(k)^{\mathrm{T}} \ \cdots \ x_n(k)^{\mathrm{T}}]^{\mathrm{T}}$$

$$u = [u_1^{\mathrm{T}} \ \cdots \ u_n^{\mathrm{T}}]^{\mathrm{T}}$$

- Multidimensional model of opinions

$$x(k+1) = (\Lambda W \otimes C)\, x(k) + ((\mathrm{I}-\Lambda) \otimes \mathrm{I})\, u \qquad x(0)=u$$

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Multidimensional Model of Opinions - 3

- Questions: Do multidimensional opinions converge to a *stationary point* ?

- Theorem 1: If C is row stochastic, stability is equivalent to the stability of FJ model

(no oblivious agents)

- Theorem 2: If some agents are oblivious need to assume also regularity of C for convergence

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Design of the MiDS Matrix

- Multidimensional model of opinions

$$x(k+1) = (\Lambda W \otimes C)\, x(k) + ((\mathrm{I}-\Lambda) \otimes \mathrm{I})\, u \qquad x(0)=u$$

- Assume that we know strength of interaction W and sensitivity Λ, target final opinion x_∞ and prejudices u
- Question: can we estimate MiDS matrix C ?
- Given Λ, W, x_∞, u solve

$$\min \| (\mathrm{I}_{mn} - (\Lambda W \otimes C)\,)\, x_\infty - ((\mathrm{I}-\Lambda) \otimes \mathrm{I})\, u \|$$

s.t. C row stochastic

- For l_2 convex quadratic program, for l_1, l_∞ linear program

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Example (Multidimensional Model) - 1

x_∞ = [35 11 35 10 75 -50 53 5]^T u = [25 25 25 15 75 -50 85 5]^T

$$W = \begin{bmatrix} 0.220 & 0.120 & 0.360 & .300 \\ .147 & .215 & .344 & .294 \\ 0 & 0 & 1 & 0 \\ .090 & .178 & .446 & .286 \end{bmatrix}$$

$$\Lambda = \mathrm{diag}\begin{bmatrix} .780 & .785 & 0 & .714 \end{bmatrix}$$

compute MiDS matrix C of multi-issue dependence with QP

$$C = \begin{bmatrix} .7562 & .2438 \\ .3032 & .6968 \end{bmatrix}$$

Opinion Dynamics: Multidimensional FJ Model

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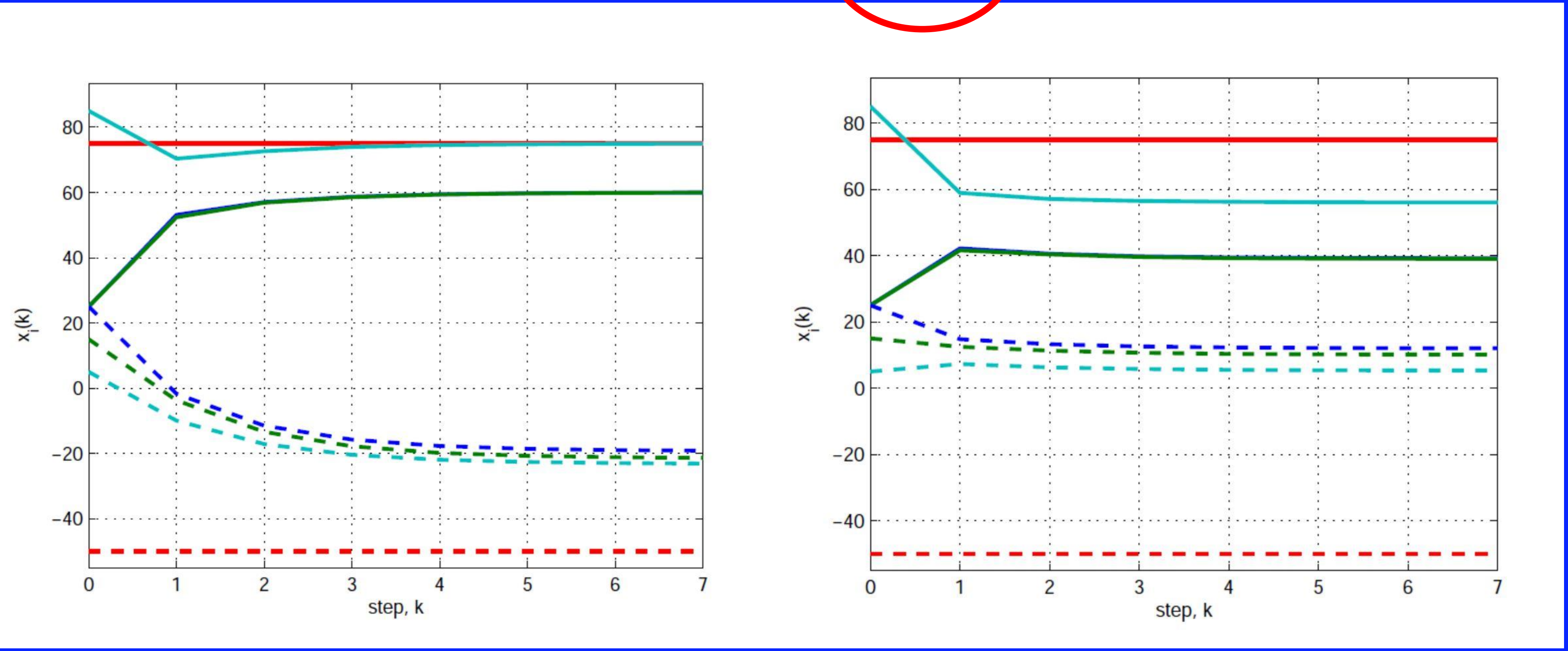
$u = [25\ 25\ 25\ 15\ 75\ -50\ 85\ 5]^T$



independent issues

$C = I$

interdependent issues

$$C = \begin{bmatrix} .8 & .2 \\ .3 & .7 \end{bmatrix}$$

Opinion Dynamics: Multidimensional DeGroot Model

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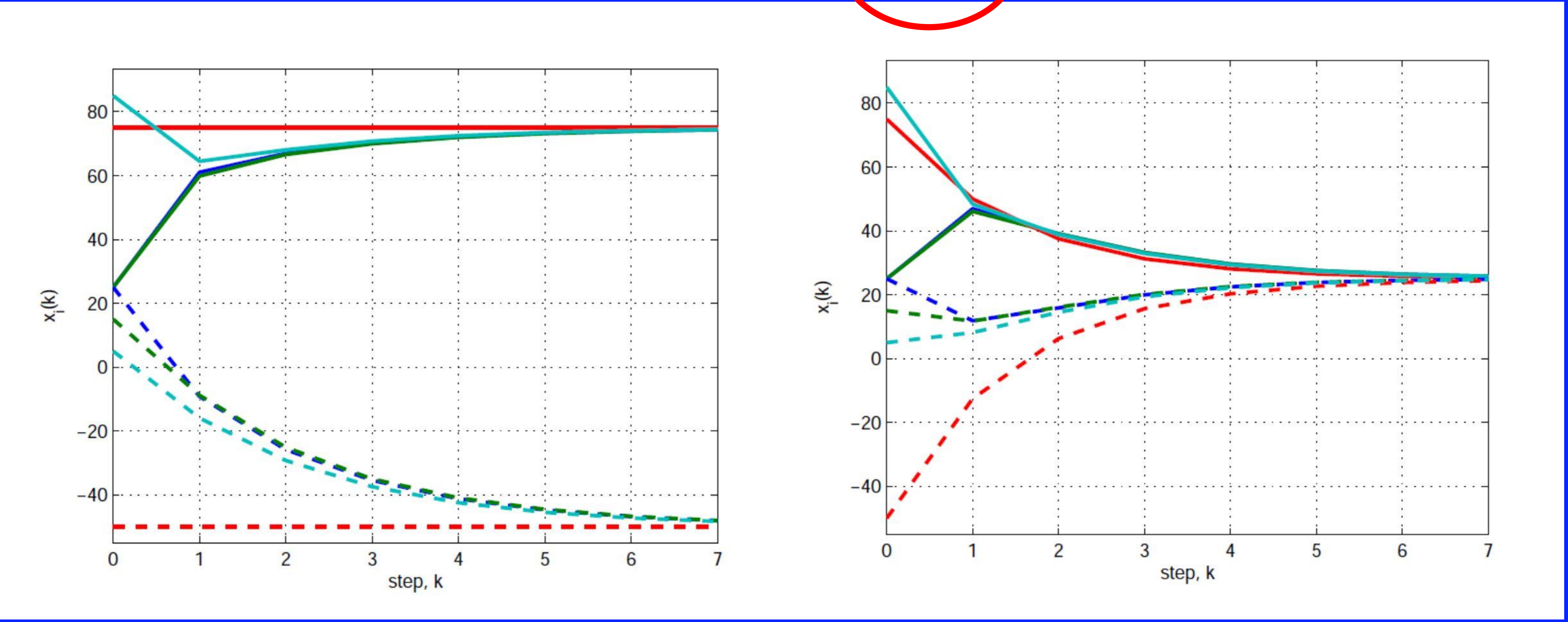
$u = [25\ 25\ 25\ 15\ 75\ \text{-}50\ 85\ 5]^T$



independent issues	interdependent issues
$C = I$	$C = \begin{bmatrix} .8 & .2 \\ .3 & .7 \end{bmatrix}$

Randomization and Gossiping

Model of Interpersonal Influences

"… this model of social influence will be imperfect at some level… it is obvious that interpersonal influences do not occur in the simultaneous way and that there are complex sequences of interpersonal influences in a group…"

N.E. Friedkin and E.C. Johnsen (1999)

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Communications between Humans are becoming Increasingly Asynchronous

- Examples of asynchronous communications: text-based messages, email, bulletin boards, blogs, forum, …

- Delivered via modern technological tools and they are independent of time and place

- Examples of synchronous communications: phone and conference calls which require humans to decide a common time

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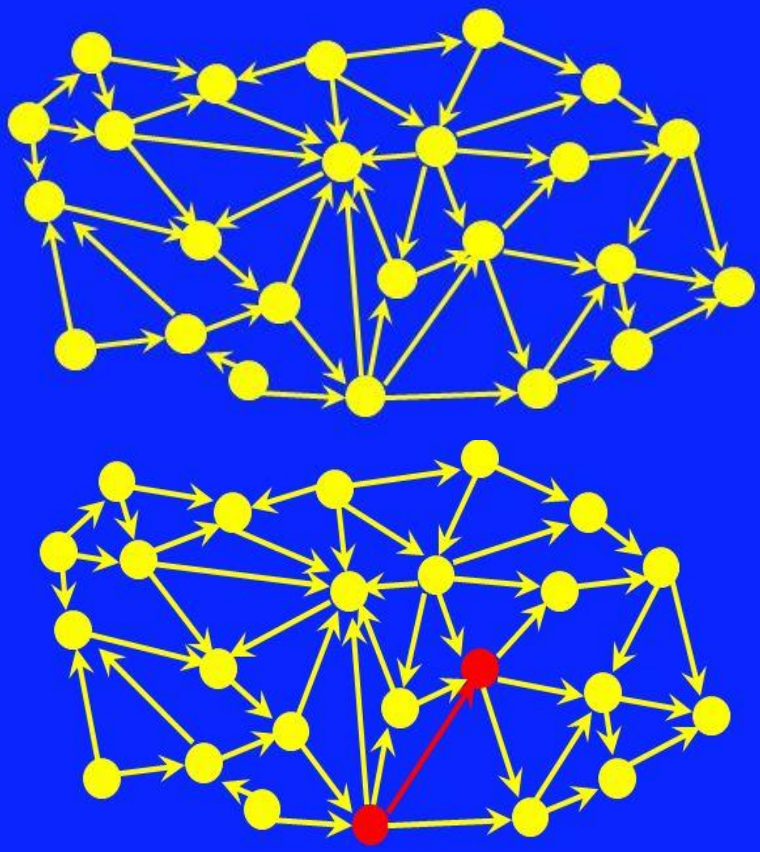
Randomization and Gossiping

- Synchronous model where all the agents (nodes) *simultaneously* exchange information through links
- Asynchronous model based on a *random* and *gossip* communication protocol
- Ergodicity properties



C. Ravazzi, P. Frasca, R. Tempo and H. Ishii (2015)

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Ergodicity

- Theorem (convergence properties):
- Assume that there are no oblivious agents
- Time average random gossip-based opinions are *almost sure ergodic*

Conclusions

From consensus to disagreement