On Simple Approximations of Semialgebraic Sets and their Applications

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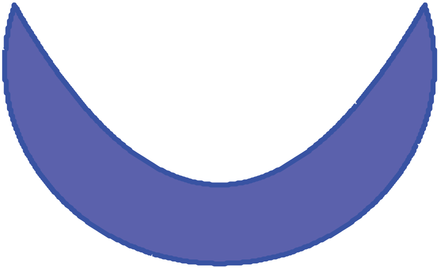
"It is the supreme art of the teacher to awaken joy in creative expression and knowledge."
Albert Einstein

the problem
approximating semialgebraic sets

Semialgebraic sets



- We are given a compact basic semialgebraic set

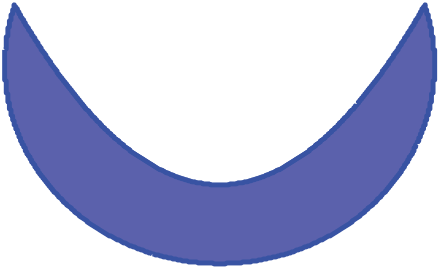
$$\mathcal{K} \doteq \{x \in \mathbb{R}^n : g_i(x) \geq 0, \; i = 1, 2, \ldots, m\}$$

where $g_i(x)$ are real multivariate polynomials

- In figure $\mathcal{K} \in \mathbb{R}^2$, with

$$g_1(x) = -(x_1 - 1)^2 - (x_2 - 1)^2 + 1$$
$$g_2(x) = (x_1 - 1)^2 - x_2 + .4$$

Semialgebraic sets



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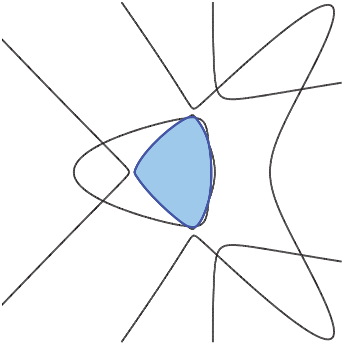
Semialgebraic sets are frequently encountered in (robust) control

- The feasible set of an LMI is a (convex) semialgebraic set

$$\mathcal{K}_{\text{LMI}} = \{x \in \mathbb{R}^n : F(x) \succeq 0\}$$

$$F(x) = F_0 + F_1 x_1 + \cdots + F_n x_n,\ F \in \mathbb{R}^{m,m}$$

  - $x \in \mathcal{K}_{\text{LMI}}$ if and only if **all the principal minors** of $F(x)$ are nonnegative
  - Set of m polynomial inequalities in x



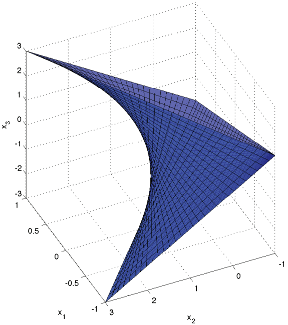
Semialgebraic sets

Stability sets

Semialgebraic sets are frequently encountered in (robust) control

- The Hurwitz and Schur stability regions of a polynomial are semiagebraic in the polynomial coefficients

Schur region of a 3rd order discrete-time polynomial



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[Bhattacharyya et. al(2009), Example 2.2]

Example 2.2
Consider the problem of determining stabilizing PID gains for the plant $P(s) = \frac{N(s)}{D(s)}$ where

$$N(s) = s^3 - 2s^2 - s - 1$$
$$D(s) = s^6 + 2s^5 + 32s^4 + 26s^3 + 65s^2 - 8s + 1.$$

In this example we use the PID controller with $T = 0$. The closed-loop characteristic polynomial is

$$\delta(s, k_p, k_i, k_d) = sD(s) + (k_i + k_d s^2)N(s) + k_p s N(s).$$

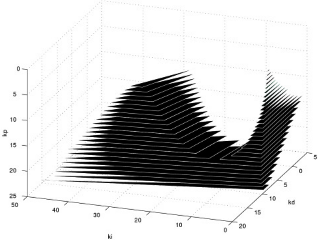

Figure 2.4

Set approximations

The problem of deriving reliable approximations of overly complicated sets by means of simpler geometrical shapes has a long history, and it arises in many research fields related to system identification, optimization and robust control

- Outer bounding sets
  - Ellipsoids of minimum volume (or trace)
  - Orthotopes
  - Polytopes / Zonotopes
- Inner bounding sets

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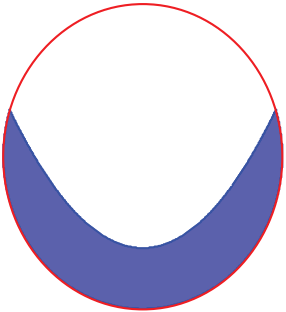

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Ellipsoidal parameter or state estimation under model uncertainty☆

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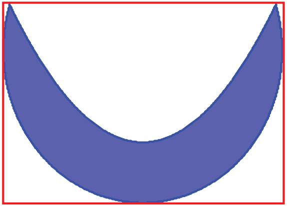

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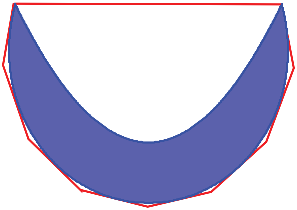

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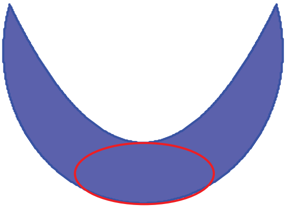


the tool

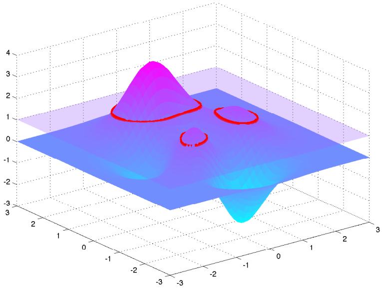
polynomial superlevel sets

What is a PSS?

Polynomial superlevel set

Given a polynomial $p \in \mathbb{P}_d$ of degree d, we define its polynomial superlevel set (PSS) as follows

$$\mathcal{U}(p) \doteq \{x \in \mathbb{R}^n : p(x) \geq 1\}$$



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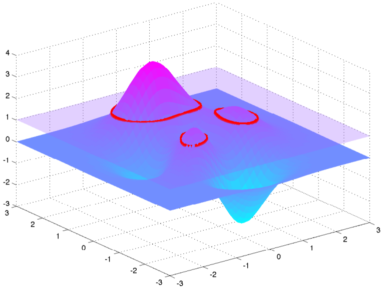

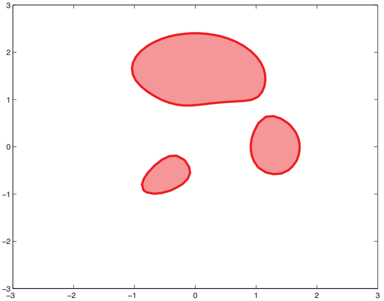

Polynomial superlevel set

- PSS has *enough dof* to capture the main characteristics of $\mathcal{K}$
  - can be non convex
  - can be non connected
- PSS has a much simpler description than the original set $\mathcal{K}$
  - single polynomial
  - can use low order polynomial $p(x)$
  - for $d = 2$, we recover classical ellipsoidal approximation

We look for the PSS of minimum volume that contains $\mathcal{K}$

Minimum volume PSS

Problem (Minimum volume PSS)

Given a compact basic semialgebraic set $\mathcal{K}$ *and a degree* d, *find a polynomial* $p \in \mathbb{P}_d$ *whose PSS* $\mathcal{U}(p)$ *is of minimum volume and contains* $\mathcal{K}$, *i.e.solve*

$$\begin{aligned} v_d^* := \inf_{p \in \mathbb{P}_d} \quad & \mathrm{vol}\,\mathcal{U}(p) \\ \text{s.t.} \quad & \mathcal{K} \subseteq \mathcal{U}(p) \end{aligned} \qquad \text{(MINVOL)}$$

Theorem

The sequence of infima of problem (MINVOL) monotically converges from above to $\mathrm{vol}\,\mathcal{K}$, *i.e. for all* $d \geq 1$

$$v_d^* \geq v_{d+1}^* \text{ and } \lim_{d\to\infty} v_d^* = \mathrm{vol}\,\mathcal{K}.$$

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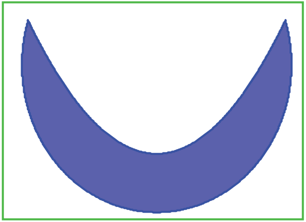
the heuristic
L^1-norm minimization

L^1-norm minimization

- It is assumed that a "simple set" $\mathcal{B}$ containing $\mathcal{K}$ is known
- This is a very mild assumption since it is easy to compute an hyper-rectangle containing $\mathcal{K}$
- We redefine the PSS as

$$\mathcal{U}(p) \doteq \{x \in \mathcal{B} : p(x) \geq 1\}$$



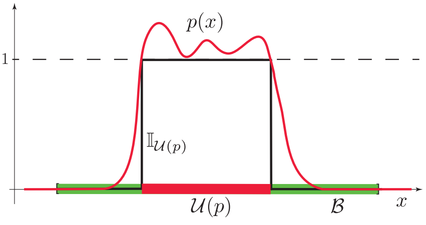
L^1-norm minimization

- Note that, by definition

$$p(x) \geq \mathbb{I}_{\mathcal{U}(p)}(x) \ \text{ on } \mathcal{B}$$

- Integrating both sides we get (Chebyshev's inequality)

$$\int_{\mathcal{B}} p(x)dx \geq \int_{\mathcal{B}} \mathbb{I}_{\mathcal{U}(p)}(x)dx = \mathrm{vol}\,\mathcal{U}(p)$$



If p is nonnegative on $\mathcal{B}$ then the above left-hand side is the L^1-norm $\|p\|_1$

L^1-norm minimization

Problem (Minimum L^1-norm PSS)

Given a semialgebraic set $\mathcal{K}$, a bounding set $\mathcal{B} \supseteq \mathcal{K}$, and a degree d, find a polynomial $p \in \mathbb{P}_d$ with minimum L^1-norm over $\mathcal{B}$, i.e. solve

$$\begin{array}{rcll} w_d^* & := & \min_{p \in \mathbb{P}_d} & \|p\|_1 \\ & & \text{s.t.} & p \geq 0 \text{ on } \mathcal{B} \\ & & & p \geq 1 \text{ on } \mathcal{K} \end{array} \qquad \text{(MINL1)}$$

A L^1-norm minimization approach was proposed in [Henrion et al.(2009)] for the numerical computation of the volume of a semialgebraic set

L^1-norm minimization

- First note that, for fixed d, when solving problem MINL1 we are minimizing an upper-bound on the volume of the PSS: the solution is expected to be a good approximation of the set $\mathcal{K}$
- Second, it can be shown that, as the degree d increases, the Chebyshev bound becomes increasingly tight

Theorem

The minimum of problem MINL1 *monotonically converges from above to the minimum of problem* MINVOL *for increasing values of d, that is*

$$w_{d-1}^* \geq w_d^* \geq v_d^* \quad \textit{for all } d$$

and

$$\lim_{d\to\infty} w_d^* = \lim_{d\to\infty} v_d^*$$

The trace interpretation

L^1-norm minimization

- If we express $p(x)$ with an orthonormal basis wrt the (scalar product induced by the) Lebesgue measure on $\mathcal{B}$

$$p(x) = \pi_d^T(x)p = \pi_{\lceil d/2 \rceil}^T(x) P \pi_{\lceil d/2 \rceil}(x)$$

  where P is the (symmetric) Gram matrix

- We have

$$\|p\|_1 = \text{trace } P$$

Extension of classical trace heuristic

In the case of quadratic polynomials ($d = 2$) we retrieve the **trace heuristic** used as a surrogate of volume minimization of ellipsoids

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Numerically Solving PSS: SoS relaxation

Let the relaxation order be N, consider the sum-of-squares (SoS) problem

$$p^*_{N,d} = \arg\min_{p \in \mathbb{P}_d} \int_{\mathcal{B}} p(x)dx \quad s.t.$$

$$p(x) = s_{0,\mathcal{B}}(x) + \sum_{j=1}^{n} s_{j,\mathcal{B}}(x)(x_j - a_j)(b_j - x_j)$$

$$s_{0,\mathcal{B}} \in \mathbf{SoS}_N;\ s_{j,\mathcal{B}} \in \mathbf{SoS}_{N-2}; \quad j = 1, 2, \ldots, n$$

$$p(x) - 1 = s_{0,\mathcal{K}}(x) + \sum_{i=1}^{m} s_{i,\mathcal{K}}(x) g_i(x)$$

$$s_{0,\mathcal{K}} \in \mathbf{SoS}_N;\ s_{i,\mathcal{K}} \in \mathbf{SoS}_{N-d_i}; \quad i = 1, 2, \ldots, m$$

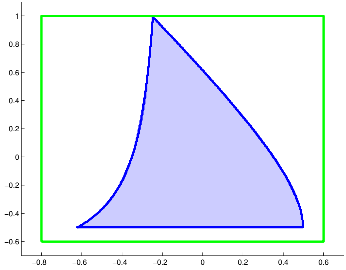
i) $p^*_{N,d}$ converges to the solution of problem MINL1 as $N \to \infty$.
ii) $p^*_{N,d}(x) \geq 0$ for all $x \in \mathcal{B}$
iii) $p^*_{N,d}(x) \geq 1$ for all $x \in \mathcal{K}$

Numerical example

- Consider [Henrion, Lasserre(2012), Example 4.4], which is a degree 4 discrete-time polynomial

$$z \in \mathbb{R} \mapsto x_2 + 2x_1 z - (2x_1 + x_2)z^3 + z^4$$

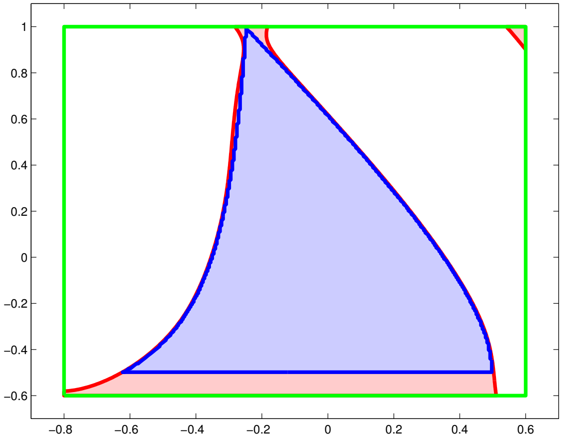
- We want to stabilize it with two real control parameters x_1, x_2



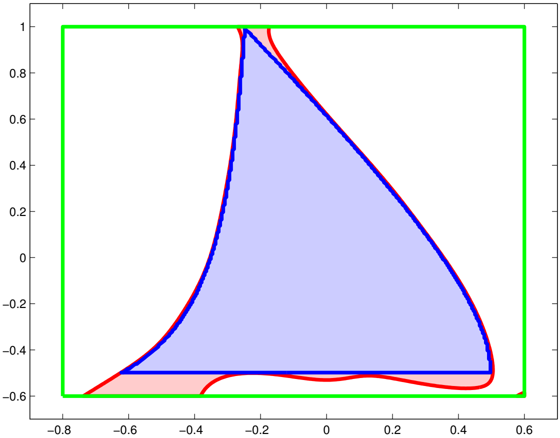
- We have the semialgebraic set

$$\begin{aligned}\mathcal{K} = \{x \in \mathbb{R}^3 \ : \ & g_1(x) = 1 + 2x_2 \geq 0,\\ & g_2(x) = 2 - 4x_1 - 3x_2 \geq 0,\\ & g_3(x) = 10 - 28x_1 - 5x_2 - 24x_1x_2 - 18x_2^2 \geq 0,\\ & g_4(x) = 1 - x_2 - 8x_1^2 - 2x_1x_2 - x_2^2 - 8x_1^2x_2 - 6x_1x_2^2 \geq 0\}\end{aligned}$$

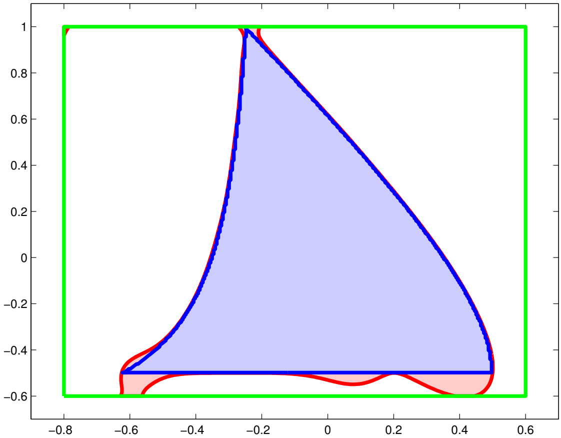
- This set is nonconvex and it is included in the box $\mathcal{B} = [-0.8, 0.6] \times [-0.6, 1]$



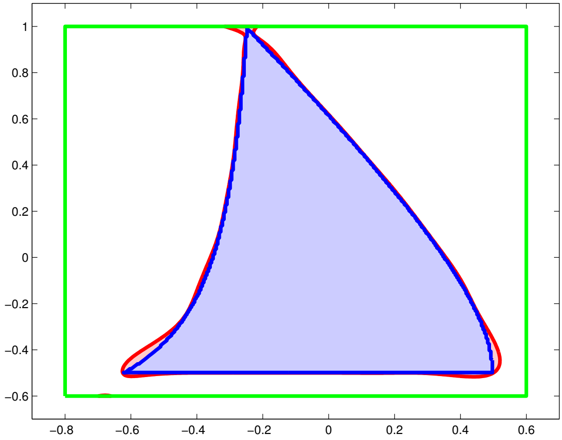
PSS for $d = 6$



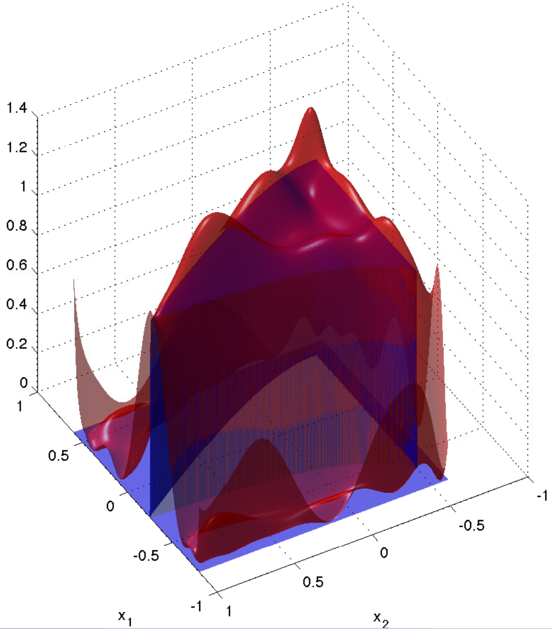
PSS for $d = 8$



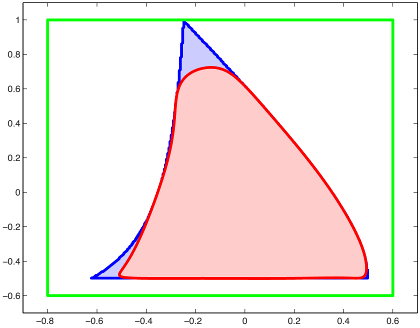
PSS for $d = 10$

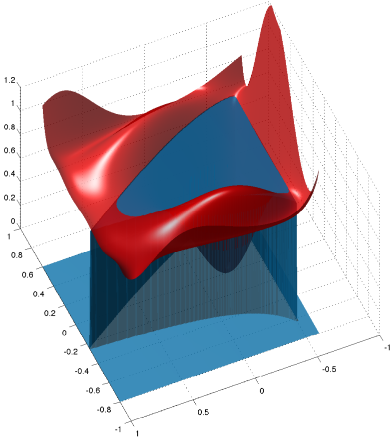


PSS for $d = 12$

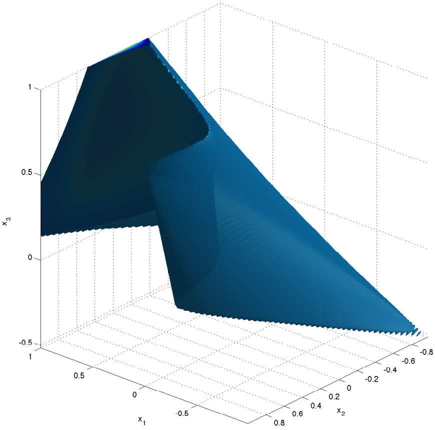

1.4
1.2
1
0.8
0.6
0.4
0.2
0
1
0.5
0
-0.5
-1
1
0.5
0
-0.5
-1
x_1
x_2

Inner approximations

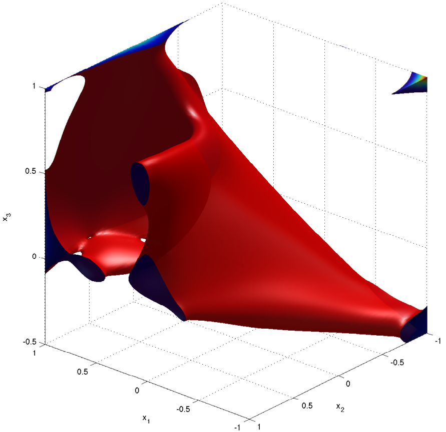




Shankar's set: PSS for $d = 14$

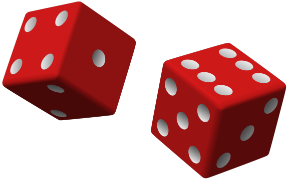


Set of stabilizing PID gains.



Degree 14 optimal outer PSS approximation.

Sampling over Semialgebraic Sets



Uniform sample generation in semialgebraic sets

- We aim at developing systematic procedures for **generating samples in a given set $\mathcal{K}$**

Problem (Generating Uniform Samples in $\mathcal{K}$)

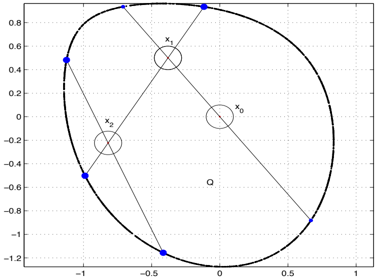
Given a semialgebraic set $\mathcal{K}$, generate N independent identically distributed (i.i.d.) random samples

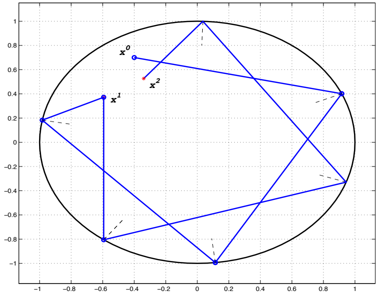
$$x^{(1)}, \ldots, x^{(N)}$$

uniformly distributed in $\mathcal{K}$

It is assumed, as before, that there exists a bounding set $\mathcal{B} \supset \mathcal{K}$, where integration of polynomials is easily done

Uniform sample generation in semialgebraic sets





- **Hit and Run**[Turchin(1971), Smith(1984)]
- Random walk in $\mathcal{K}$

- **Billiard Walk** [Gryazina, Polyak(2014)]

Uniform sample generation in semialgebraic sets

- The uniform density over the set $\mathcal{K}$ is defined by $\mathbb{U}_{\mathcal{K}}(x) := \dfrac{\mathbb{I}_{\mathcal{K}}(x)}{\mathrm{vol}(\mathcal{K})}$.
- Idea: the PSS polynomial p_d^* is a good approximation of the indicator function $\mathbb{I}_{\mathcal{K}}(x)$:
  - i) $p_d^*(x) \geq \mathbb{I}_{\mathcal{K}}(x)$ for all $x \in \mathcal{B}$
  - ii) As $d \to \infty$, $p_d^* \to \mathbb{I}_{\mathcal{K}}$ both in the $L^1(\mathcal{B})$ and almost uniformly in $\mathcal{B}$.
- This polynomial is a so-called dominating density of the uniform density $\mathbb{U}_{\mathcal{K}}(x)$ for all $x \in \mathcal{B}$; i.e., there exists a value $\beta > 0$ such that $\beta p_d^*(x) \geq \mathbb{U}_{\mathcal{K}}(x)$ for all $x \in \mathcal{B}$

Uniform sample generation in semialgebraic sets

Rejection method from a dominating density [Tempo et al.(2013), Section 14.3.1]

Algorithm (Uniform Sample Generation in $\mathcal{K}$)

1. *For a given integer $d > 0$, compute the solution of*

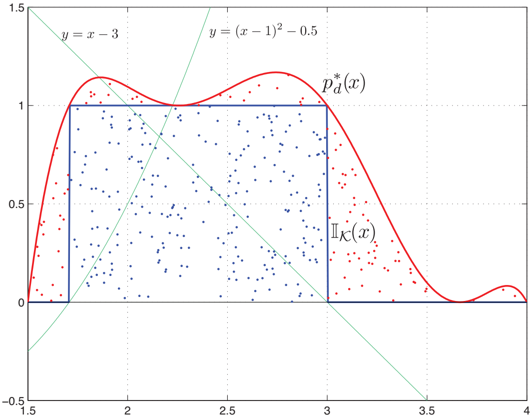
$$p_d^*(x) \quad := \arg \quad \min_{p \in \mathbb{P}_d} \quad \int_{\mathcal{B}} p(x)dx \qquad (1)$$
$$\text{s.t.} \quad p \geq 1 \text{ on } \mathcal{K}$$
$$p \geq 0 \text{ on } \mathcal{B}$$

2. *Generate a random sample $\xi^{(i)}$ with density proportional to $p_d^*(x)$ over $\mathcal{B}$*
3. *If $\xi^{(i)} \notin \mathcal{K}$ go to step 1*
4. *Generate a sample u uniform on $[0, 1]$*
5. *If $u\, p_d^*(\xi^i) \leq 1$ return $x^{(i)} = y$, else go to step 1*

Uniform sample generation in semialgebraic sets

Consider the simple one-dimensional set

$$\mathcal{K} = \left\{ x \in \mathbb{R} \,:\, (x-1)^2 - 0.5 \geq 0, x - 3 \leq 0 \right\}.$$



Problem (MINL1) is solved (for $d = 8$ and $\mathcal{B} = [1.5, 4]$), yielding

$$\begin{aligned} p_8^*(x) = {} & 0.06947x^8 - 2.0515x^7 \\ & + 23.43x^6 - 139.5x^5 \\ & + 477.9x^4 - 961.9x^3 \\ & + 1091x^2 - 606.1x + 107.3 \end{aligned}$$

Uniform sample generation in semialgebraic sets

Theorem

Algorithm 1 returns a sample uniformly distributed in $\mathcal{K}$. Moreover, the acceptance rate of the algorithm is given by

$$\gamma_d = \frac{\mathrm{vol}(\mathcal{K})}{w_d^*}$$

where w_d^* is the optimal solution of problem (MINL1), that is

$$w_d^* := \int_{\mathcal{B}} p_d^*(x)dx.$$

Corollary

Let d be the degree of the polynomial approximation of the indicator function of the set $\mathcal{K}$. Then, the acceptance rate tends to one as degree d increases; i.e.,

$$\lim_{d\to\infty} \gamma_d = 1$$

Uniform sample generation in semialgebraic sets

Theorem

Algorithm 1 returns a sample uniformly distributed in $\mathcal{K}$. Moreover, the acceptance rate of the algorithm is given by

$$\gamma_d = \frac{\mathrm{vol}(\mathcal{K})}{w_d^*}$$

where w_d^* is the optimal solution of problem (MINL1)*, that is*

$$w_d^* := \int_{\mathcal{B}} p_d^*(x)dx.$$

Corollary

Let d be the degree of the polynomial approximation of the indicator function of the set $\mathcal{K}$. Then, the acceptance rate tends to one as degree d increases; i.e.,

$$\lim_{d\to\infty} \gamma_d = 1$$

Numerical example

Sampling in a nonconvex semialgebraic set

Numerical example introduced in [Cerone et al.(2012)]
The considered semialgebraic set $\mathcal{K}$ is the two-dimensional nonconvex region described as:

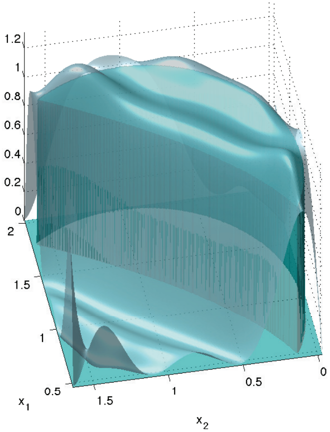
$$\mathcal{K} := \left\{ (x_1, x_2) \in \mathbb{R}^2 \; : \right. \\ \left. (x_1 - 1)^2 + (x_2 - 1)^2 \leq 1, \;\; x_2 \leq 0.5 x_1^2 \right\}.$$

In [Cerone et al.(2012)], the outer-bounding box was considered

$$\mathcal{B} = \left\{ (x_1, x_2) \in \mathbb{R}^2 \; : \; 0.46 \leq x_1 \leq 2.02, \;\; 0.03 \leq x_2 \leq 1.64 \right\}$$

Numerical example

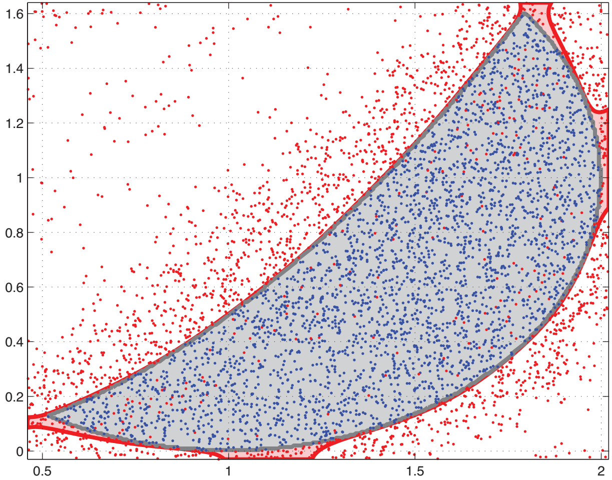
Sampling in a nonconvex semialgebraic set



Indicator function $\mathbb{I}_{\mathcal{K}}(x)$ and corresponding optimal solution $p_d^*(x)$ for $d = 8$

Numerical example

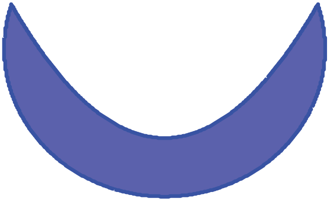
Sampling in a nonconvex semialgebraic set



- $N = 8,000$ random samples. The red points are discarded
- Some point inside $\mathcal{K}$ has been rejected: fundamental for uniformity

Conclusions

- We proposed a new family of set approximations of semialgebraic sets
- PSS can capture nonconvexity
- They have several applications in robust control / identification
- They can be used to generate uniform samples in semialgebraic sets
- **They are fun!**



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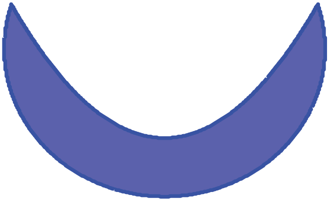

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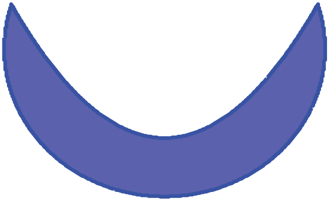

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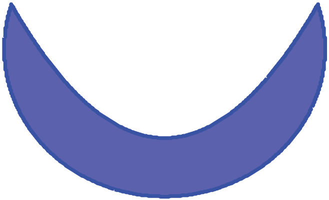

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