Convex Relaxations of Chance Constrained Algebraic Problems

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PENNSTATE 1855

Electrical Engineering

Chance Optimization Problem

Let

- $q \in \mathbf{R}^m$ be a random variable with probability measure μ_q
- $x \in \mathbf{R}^n$ be a decision variable
- $p_j : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$, $j = 1, 2, \ldots, l$ be polynomials

Solve

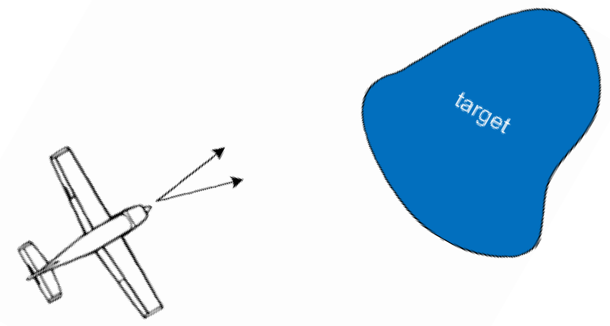
$$\mathbf{P_1}^* = \max_x Prob_{\mu_q} \{q : p_j(x, q) \geq 0,\ j = 1, 2, \ldots, l\ \}$$

This is, in general, a hard nonconvex problem

Example: Target Reaching Under Random Uncertainty

Take a polynomial system

$$x(t+1) = f[x(t), u(t), \eta(t)]$$



Find a polynomial state feedback law

$$u(t) = q[x(t)]$$

that maximizes the probability of reaching and remaining in the target.

Example: Explaining Data using Switched Systems

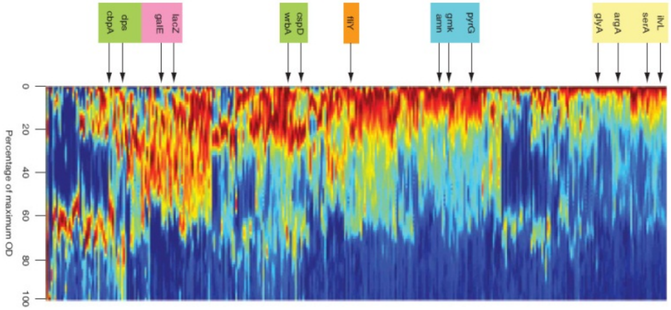
- Find "simple" explanations/models for data
- Find relations in between data collected
- Detect changes in data "behavior"

Examples:

gait/activity recognition

video shot change detection

Gene classification



Other Examples

- Chance Constrained Model Predictive Control
- Fixed Order Probabilistically Robust Controller
- Portfolio Optimization/Risk Minimization
- ...

Back to the Problem

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Previous Work

- Robust Optimization
- Upper Bounds/Approximations of Chance Constraints
- Scenario Approach

Our Objective

Find approximations of this problem that are

i) Convex Problems

ii) Asymptotically Exact

Assumptions

i) x belongs to the hyper-cube $\chi = [-1, 1]^n$

ii) probability measure μ_q satisfies $supp(\mu_q) \subseteq \mathcal{Q} = [-1, 1]^m$

Equivalent Problem in Space of Measures

Define the compact semialgebraic set

$$\mathcal{K} = \{(x,q) : p_j(x,q) \geq 0,\ j = 1,2,\ldots,l\ \} \bigcap (\chi \times \mathcal{Q})$$

Then, one can define the equivalent problem

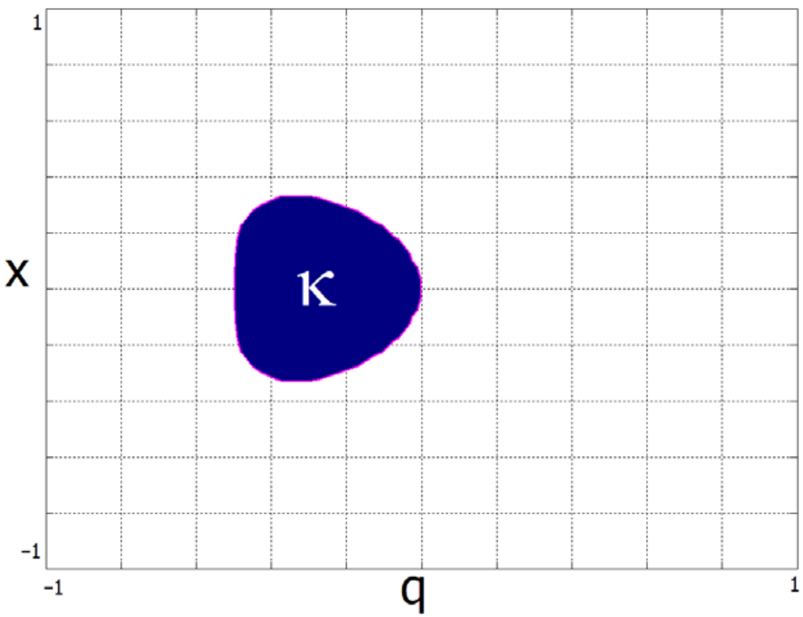
$$\mathbf{P}_2^* = \max_{\mu,\mu_x} \int d\mu$$

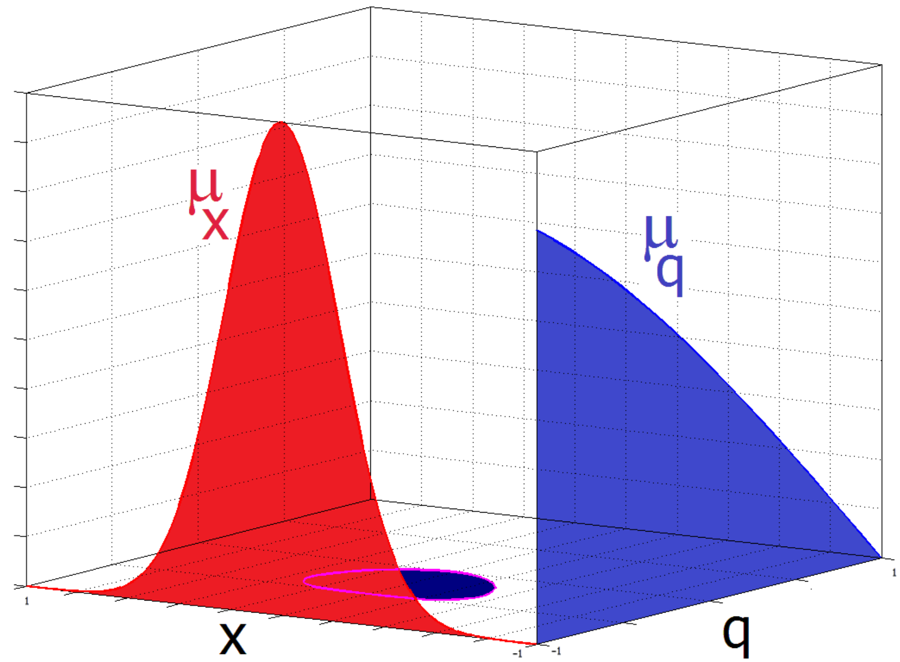
subject to

$$\mu \preccurlyeq \mu_x \times \mu_q$$

$$supp(\mu_x) \subseteq \chi,\ supp\,(\mu) \subseteq \mathcal{K}$$

Equivalent Problem (cont.)

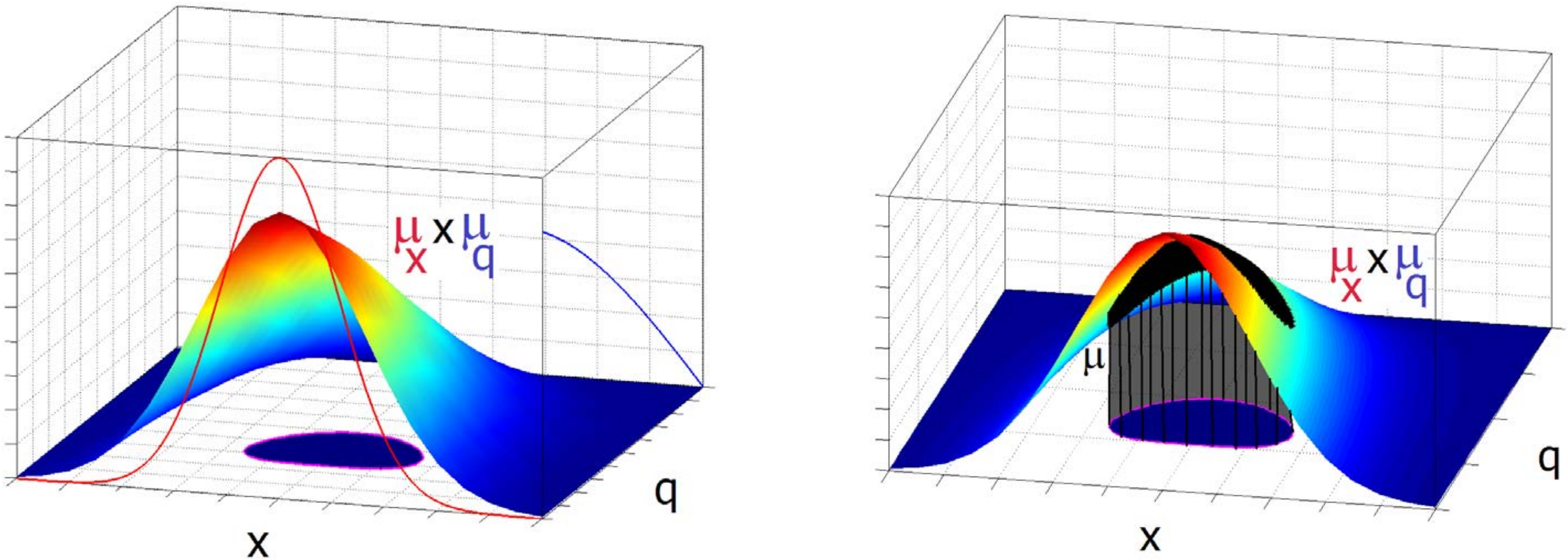




$$\mathbf{P_1}^* = \max_x Prob_{\mu_q} \{q : p_j(x,q) \geq 0,\ j = 1, 2, \ldots, l\}$$

$$\mathcal{K} = \{(x,q) : p_j(x,q) \geqslant 0,\ j = 1, 2, \ldots, l\} \bigcap (\chi \times \mathcal{Q})$$

Equivalent Problem



$$Prob_{\mu_q}\{q : p_j(x,q) \geq 0,\ j = 1, 2, \ldots, l\ \} = \int_{\mathcal{K}} d(\mu_x \times \mu_q)$$

$$= \max_{\mu} \int d\mu \text{ subject to } \mu \preccurlyeq \mu_x \times \mu_q \text{ and } supp(\mu) \subseteq \mathcal{K}$$

Equivalence Result

Theorem: Problem 1 and Problem 2 are equivalent in the following sense

- The optimal values are the same.
- If μ_x^* be a solution of Problem 2, then, any $x^* \in supp(\mu_x^*)$ is a solution of Problem 1
- If x^* be a solution of Problem 1, then $\mu_x^* = \delta_{x^*}$ is a solution of Problem 2

Comments

We have transformed our problem from a hard nonconvex one into a linear program in measure space. Hence

- It is convex (linear in the measures)
- But, it is infinite dimensional

Lets work with the moments of the measures instead

Moments of Measures

Consider a sequence **y**. We say that this sequence is a moment sequence if there exists a measure μ such that

$$y_\alpha = \int x^\alpha d\mu$$

Under some technical conditions, a sequence **y** is a moment sequence of some measure μ supported in the set

$$\mathbf{K} = \{x \in \mathbf{R}^n : \ p_j(x) \geq 0, \ j = 1, 2, \ldots, m \}$$

if the following holds

$$M_d(\mathbf{y}) \succcurlyeq 0, M_d(\ p_j \mathbf{y}) \succcurlyeq 0, \quad j = 1, \ldots, m$$

for all integer *d.*

Two Dimensional Example

$$M_N(m) = \begin{bmatrix} M_{0,0}(m) & M_{0,1}(m) & \cdots & M_{0,N}(m) \\ M_{1,0}(m) & M_{1,1}(m) & \cdots & M_{1,N}(m) \\ \vdots & \vdots & \ddots & \vdots \\ M_{N,0}(m) & M_{N,1}(m) & \cdots & M_{N,N}(m) \end{bmatrix}$$

$$M_{j,k}(m) = \begin{bmatrix} m_{j+k,0} & m_{j+k-1,1} & \cdots & m_{j,k} \\ m_{j+k-1,1} & m_{j+k-2,2} & \cdots & m_{j-1,k+1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{k,j} & m_{k-1,j+1} & \cdots & m_{0,j+k} \end{bmatrix}$$

Moment localization matrix: $M_{N_i}(p_i m)(i,j) = \sum_{\alpha} p_{i,\alpha} m(\beta(i,j) + \alpha)$

Measure "Coverage"

Recall that given two measures μ_1 and μ_2

$\mu_1 \preccurlyeq \mu_2$ denotes $\mu_1(\mathcal{A}) \leq \mu_2(\mathcal{A})$ for any measurable set $\mathcal{A}$

Given two measures μ_1 and μ_2 on a compact set $\mathbf{K}$, with moment sequences $\mathbf{y_1} = (y_{1\alpha})$ and $\mathbf{y_2} = (y_{2\alpha})$, we have $\mu_1 \preccurlyeq \mu_2$ if :

$$M_d(\mathbf{y_2} - \mathbf{y_1}) \succcurlyeq 0, M_d(\ p_j(\mathbf{y_2} - \mathbf{y_1})) \succcurlyeq 0\ ,\ j = 1, \ldots, m$$

for every $d \in \mathbb{N}$

Equivalent Problem in "Moment Space"

Let $\mathbf{y}$, $\mathbf{y_x}$, and $\widehat{\mathbf{y}}$ be the infinite sequence of all moments of measures μ, μ_x, and $\widehat{\mu} = \mu_x \times \mu_q$, respectively

Solve

$$\mathbf{P}_3^* = \sup_{\mathbf{y},\mathbf{y}_x} \; y_0$$

subject to

$$M_\infty(\mathbf{y}) \succcurlyeq 0$$

$$M_\infty\left(p_j \mathbf{y}\right) \succcurlyeq 0 \;,\; j = 1, 2, \ldots, l$$

$$M_\infty(\mathbf{y_x}) \succcurlyeq 0$$

$$M_\infty(\widehat{\mathbf{y}} - \mathbf{y}) \succcurlyeq 0$$

Finite Dimensional Approximation

Solve

$$\mathbf{P}_4^{*i} = \sup_{\mathbf{y},\mathbf{y}_x} \; y_0$$

subject to

$$M_i(\mathbf{y}) \succcurlyeq 0$$

$$M_{i-r_j}(p_j \mathbf{y}) \succcurlyeq 0 \;, \;\; j = 1, 2, \ldots, l$$

$$M_i(\mathbf{y_x}) \succcurlyeq 0$$

$$M_i(\widehat{\mathbf{y}} - \mathbf{y}) \succcurlyeq 0$$

Theorem: Optimal value of problem $\mathbf{P_4^i}$ converges to optimal value of problem $\mathbf{P_3}$ as i $\rightarrow \infty$.

Comments on Implementation

Solve

$$\inf_{\mathbf{y},\mathbf{y}_x} \|M_i(\mathbf{y_x})\|_*$$

Nuclear norm

subject to

$$y_0 \geq \gamma$$

Do line search

$$M_i(\mathbf{y}) \succcurlyeq 0$$

$$M_{i-r_j}(p_j\mathbf{y}) \succcurlyeq 0 \ , \ j = 1, 2, \ldots, l$$

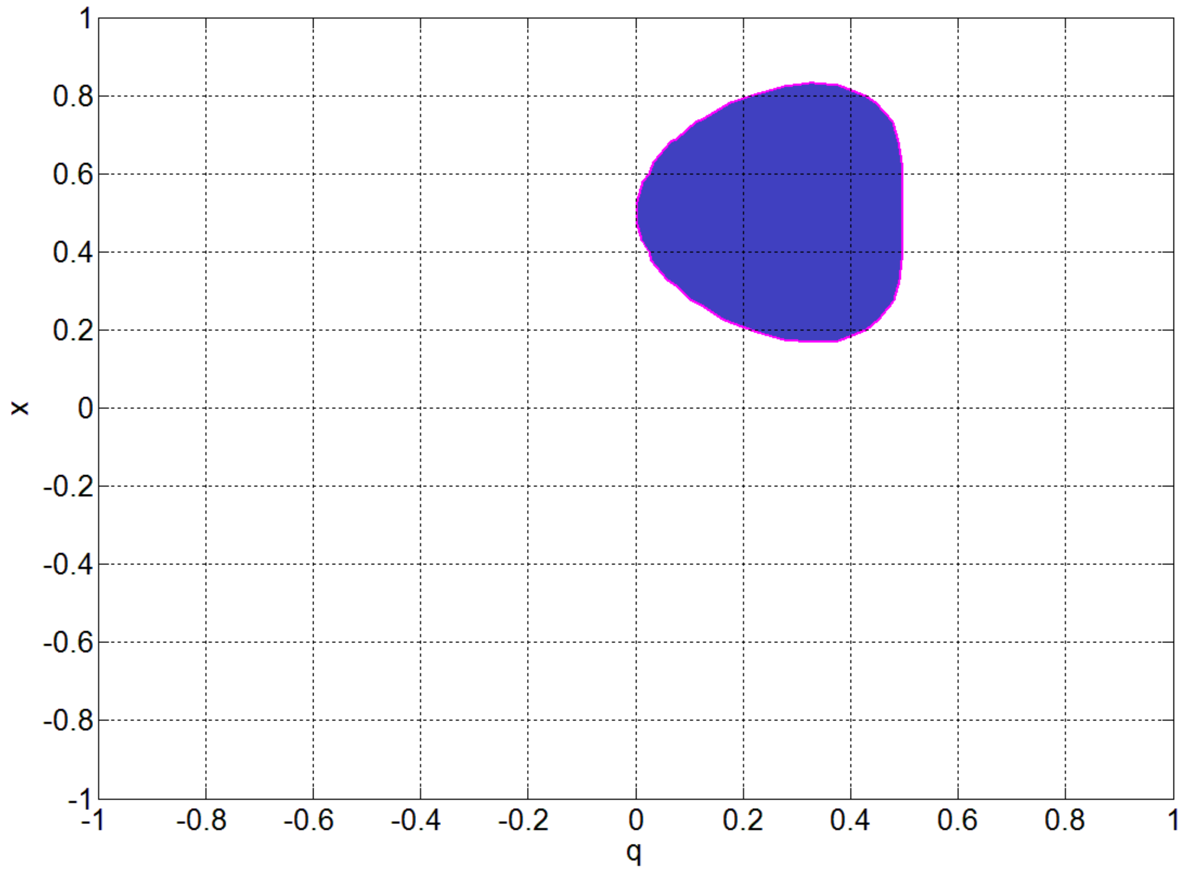
$$M_i(\mathbf{y_x}) \succcurlyeq 0$$

$$M_i(\widehat{\mathbf{y}} - \mathbf{y}) \succcurlyeq 0$$

Example 1

$$\max_x Prob_{\mu_q}\left\{q : \quad p(x,q) = -\frac{1}{2}q(q^2+(x-\frac{1}{2})^2) + (q^4+q^2(x-\frac{1}{2})^2+(x-\frac{1}{2})^4) \geq 0\right\}$$

The uncertain parameter q : $\mu_q = U[-1,1]$



Example 1: Moment Vectors

Moment vector of measure μ

$$\mathbf{y} = [y_{00}|y_{10}, y_{01}|y_{20}, y_{11}, y_{02}|y_{30}, y_{21}, y_{12}, y_{03}|y_{40}, y_{31}, y_{22}, y_{13}, y_{04}]$$

Moment vector of measure μ_x

$$\mathbf{y_x} = [1, y_{x1}, y_{x2}, y_{x3}, y_{x4}]$$

Moment vector of measure μ_q

$$\mathbf{y_q} = [1, y_{q1}, y_{q2}, y_{q3}, y_{q4}] = [1, 0, \frac{1}{3}, 0, \frac{1}{5}, 0]$$

Moment vector of measure $\hat{\mu} = \mu_x \times \mu_q$

$$\mathbf{y_x}\mathbf{y_q} = [1|y_{x1}, y_{q1}|y_{x2}, y_{x1}y_{q1}, y_{q2}|y_{x3}, y_{x2}y_{q1}, y_{x1}y_{q2}, y_{q3}|y_{x4}, y_{x3}y_{q1}, y_{x2}y_{q2}, y_{x1}y_{q3}, y_{q4}]$$

$$= [1|y_{x1}, 0|y_{x2}, 0, \frac{1}{3}1|y_{x3}, 0, \frac{1}{3}y_{x1}, 0|y_{x4}, 0, \frac{1}{3}y_{x2}, 0, \frac{1}{5}1]$$

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Example 1: Optimization Problem

$$min_{\gamma, y_{ij}, y_{xk}} \| M_4(y_x) \|_* = \left\| \begin{pmatrix} 1 & y_{x1} & y_{x2} \\ y_{x1} & y_{x2} & y_{x3} \\ y_{x2} & y_{x3} & y_{x4} \end{pmatrix} \right\|_*$$

$$y_{00} \geq \gamma$$

$$M_4(y) \succeq 0 \Rightarrow \begin{pmatrix} y_{00} & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{pmatrix} \succeq 0$$

$$M_4(y_x y_q) - M_4(y) \succeq 0 \Rightarrow \begin{pmatrix} 1 & y_{x1} & 0 & y_{x2} & 0 & 1/3 \\ y_{x1} & y_{x2} & 0 & y_{x3} & 0 & 1/3 y_{x1} \\ 0 & 0 & 1/3 & 0 & 1/3 y_{x1} & 0 \\ y_{x2} & y_{x3} & 0 & y_{x4} & 0 & 1/3 y_{x2} \\ 0 & 0 & 1/3 y_{x1} & 0 & 1/3 y_{x2} & 0 \\ 1/3 & 1/3 y_{x1} & 0 & 1/3 y_{x2} & 0 & 2/5 \end{pmatrix} - M_4(y) \succeq 0$$

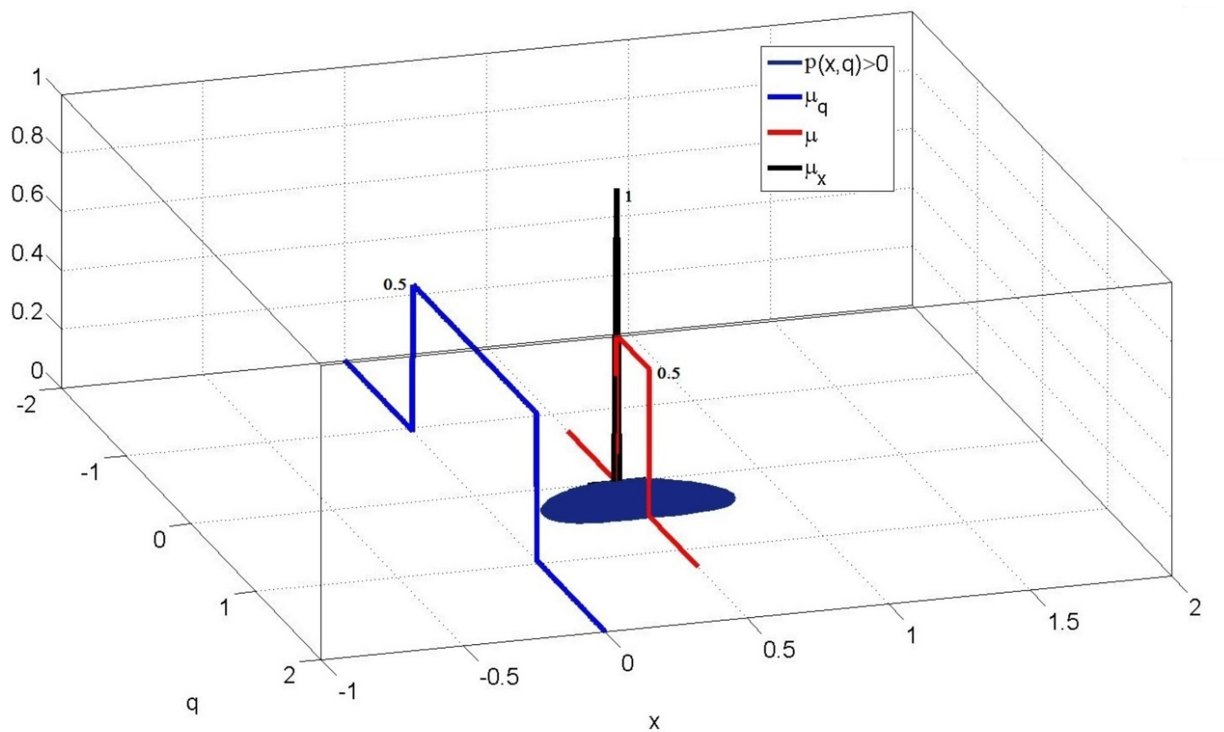
$$M_4(py) \succeq 0 \Rightarrow -y_{04} + \frac{1}{2} y_{03} - y_{22} + y_{12} - \frac{1}{4} y_{02} + \frac{1}{2} y_{21} - \frac{1}{2} y_{11} + \frac{1}{8} y_{01} - y_{40} + 2 y_{30} - \frac{3}{2} y_{20} + \frac{1}{2} y_{10} - \frac{1}{16} \succeq 0$$

Example 1: Results

Obtained Moments

$$\mathbf{y} = [0.58,\ 0.49, -0.15, 0, 0, 0.18, 0, 0, 0, -0.07, 0, 0, 0, 0]$$

Eigenvalues of $M_4(x)$: $[0, 0, 0, 0, 0.1, 1]$: Rank $(M_4(x)) \simeq 1$: $\mu_x \simeq$ Dirac measure
Optimal x^* : $y_{x_1} = 0.499$
Optimal Probability : $y_{00} = 0.58$



Example 2

$$\sup_{x \in R^5} \mu_q \left(\{ q \in R^5 : \; \mathcal{P}(x,q) \geq 0 \; \} \right)$$

where

$$\begin{aligned} \mathcal{P}(x,q) = \quad & 0.185 + 0.5x_1 - 0.5x_2 + x_3 - x_4 + 0.5q_1 - 0.5q_2 + q_3 - q_4 - x_1^2 \\ & 2x_1 q_1 - x_2^2 - 2x_2 q_2 - x_3^2 - 2x_3 q_3 - x_4^2 - 2x_4 q_4 - x_5^2 + 2x_5 q_5 \\ & q_1^2 - q_2^2 - q_3^2 - q_4^2 - q_5^2, \end{aligned}$$

$$q_1 \sim U[-1,0], \; q_2 \sim U[0,1], \; q_3 \sim U[-0.5,1], \; q_4 \sim U[-1,0.5], \; q_5 \sim U[0,1]$$

Optimum (Monte Carlo):

$$x_1^* = 0.75, \; x_2^* = -0.75, \; x_3^* = 0.25, \; x_4^* = -0.25, \; x_5^* = 0.5$$

$$P^* = 0.75$$

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Example 2: Numerical Results

ALCC			
d	1	2	3
n_{var}	87	1127	8463
x_1	0.742	0.745	0.757
x_2	-0.777	-0.701	-0.721
x_3	0.213	0.226	0.216
x_4	-0.239	-0.250	-0.236
x_5	0.500	0.551	0.557
P_d	0.991	0.971	0.961
iter	169	624	1207
cpu	0.9	28.1	785.9

GloptiPoly			
d	1	2	3
n_{var}	87	1127	8463
x_1	0.467	0.710	0.742
x_2	-0.467	-0.710	-0.742
x_3	0.163	0.245	0.249
x_4	-0.163	-0.245	-0.249
x_5	0.319	0.475	0.495
P_d	1	1	1
iter	18	25	41
cpu	0.5	12.3	15324.3

Concluding Remarks

In this work

- Proposed convex relaxations for a very general class of chance optimization problems

- It is asymptotically exact

Future work

- But it is computationally expensive – Need better "optimization tools”

- Further work is needed in exploiting structure

Thank you

Questions?