Polyak Conference

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Moscow

Happy Birthday Boris

May you always remain young and productive

A Model Free Measurement Based Approach to Design

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Overview

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- Mathematical Preliminaries
- A Measurement-based Approach to Linear Systems
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  - Linear Mechanical Systems/Hydraulic Networks
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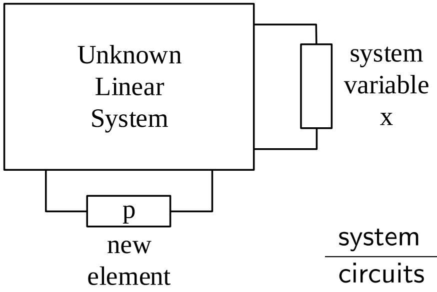
Introduction

- In most engineering design techniques, a mathematical model of the system is required.
- The behavior of a system operating at (or close to) the equilibrium point can be modeled by sets of linear equations.
- The current practice of control system design requires a model (transfer function, state-space equations) of the system.
- In practice, systems are very complex $\Rightarrow$ mathematical models will be complex/higher order.
- These observations motivate the search for a new approach whose objective is to determine the design variables directly from a small set of measurements and without producing a mathematical model of the system.

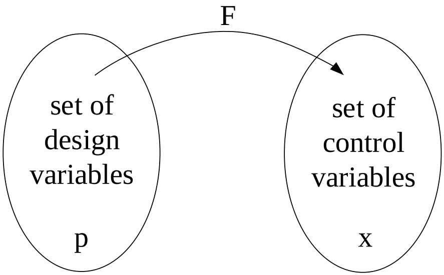


system	x	p
circuits	I, V, P	R, C, L
mechanical	δ, Q	k,A,r
control	$H(s), H(j\omega)$	$C(s), C(j\omega)$

Question:

How do you connect a new element p to an arbitrary location of an unknown linear system to control a system variable x at some other location?

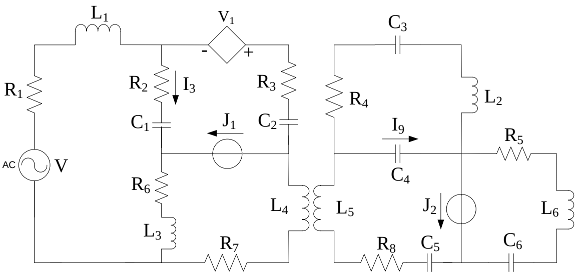
Functional Dependency and Measurements



$$x = \mathcal{F}(p)$$

Question:
Is it possible to determine $\mathcal{F}$ directly from measurements?

Functional Dependency and Measurements



Question:

Is it possible to find the functional dependencies $\mathcal{F}$, for example,

$$I_3 = \mathcal{F}_1(L_1, C_2), \quad I_9 = \mathcal{F}_2(L_1, C_2)$$

without knowing the system parameters and directly from a small set of measurements?

Objectives and Motivation

Objectives:

- To develop a methodology to find the parametrized functional dependency $\mathcal{F}$ of any system variable on any set of design variables without knowing the system parameters.
- To determine the design variables directly from a small set of measurements and without constructing a mathematical model of the system.

Motivation:

- To analyze and solve design problems in linear systems without requiring a mathematical model of the system.

Mathematical Preliminaries

Suppose that the scalar parameter p (real or complex) appears in the square matrix A *affinely*:

$$A(p) = A_0 + p\ A_1$$

Lemma 1

If $rank(A_1) = r$, *then*

$$|A(p)| = \alpha_r p^r + \alpha_{r-1} p^{r-1} + \cdots + \alpha_1 p + \alpha_0 \tag{1}$$

Consider $A(p_1, p_2, \ldots, p_l) = A_0 + p_1\ A_1 + p_2\ A_2 + \cdots + p_l\ A_l$:

Lemma 2

If $rank(A_i) = r_i,\ i = 1, 2, \ldots, l$, *then*

$$|A(p_1, p_2, \ldots, p_l)| = \sum_{i_l=0}^{r_l} \cdots \sum_{i_2=0}^{r_2} \sum_{i_1=0}^{r_1} \alpha_{i_1 i_2 \cdots i_l}\ p_1^{i_1} p_2^{i_2} \cdots p_l^{i_l} \tag{2}$$

Suppose that a linear system can be described as:

$$A(p)x = b(q) \tag{3}$$

- $A(p)$: system characteristic matrix
- p: vector of system parameters
- x: vector of unknown system variables
- $b(q)$: contains the system inputs, $b(q) = b_1 q_1 + b_2 q_2 + \cdots + b_m q_m$, where q_i's are the system inputs.

Assumption 1

$A(p)$ is a nonsingular matrix over the range of (physical) values of p.

- If there exists a physical value p_0 such that $A(p_0)$ becomes a singular matrix, then x will not be unique which is usually not the case for physical systems.

Applying Cramer's rule:

$$x_i(p,q) = \frac{|B_i(p,q)|}{|A(p)|}, \quad i = 1, 2, \ldots, n \tag{4}$$

- For *unknown* systems, $B_i(p,q)$ and $A(p)$ are *unknown*.
- But, if the rank dependencies of p are known, then the determinants in (4) can be expanded (Lemmas 1 and 2).

Theorem 1

For the system of linear equations

$$A(p)x = b(q)$$

suppose that p *appears in* $A(p)$ *affinely:*

$$A(p) = A_0 + p_1 A_1 + p_2 A_2 + \cdots + p_l A_l$$

and $\mathrm{rank}(A_i) = r_i,\ \ i = 1, 2, \ldots, l$, *and* $b(q) = b_1 q_1 + b_2 q_2 + \cdots + b_m q_m$, *then*

$$x_i(p, q) = \frac{\alpha_i(p, q)}{\beta(p)}, \qquad i = 1, 2, \ldots, n$$

where $\alpha_i(p, q)$ *and* $\beta(p)$ *are multivariate polynomials in* p, *and* $\alpha_i(p, q)$ *is linear in* q.

A Measurement-based Approach to Linear Systems

Theorem 1

Moreover, the coefficients of $\alpha_i(p,q)$ and $\beta(p)$ can be determined by setting (p,q) to μ (number of coefficients) linearly independent sets of values and measuring x_i.

Proof. The proof follows from Lemma 2 and the linear form of $b(q)$.

S. P. Bhattacharyya, L. H. Keel, and D. N. Mohsenizadeh. *Linear Systems: A Measurement Based Approach*. Springer-Verlag, 2014.

A General Linear Model

Consider the generMotivational Exampleal linear input-output model:

$$\begin{aligned} A(\mathbf{p})\mathbf{x} &= B\mathbf{u} \\ \mathbf{y} &= C(\mathbf{p})\mathbf{x} + D\mathbf{u} \end{aligned} \qquad (1)$$

with outputs $\mathbf{y}$ and inputs $\mathbf{u}$:

$$\mathbf{y} = \begin{pmatrix} y_1 & \cdots & y_m \end{pmatrix}', \mathbf{u} = \begin{pmatrix} u_1 & \cdots & u_r \end{pmatrix}'$$

and design parameters $\mathbf{p} = (p_1 \ \cdots \ p_\ell)'$. Let $\mathbf{z} := (\mathbf{x}\ \mathbf{y})'$ so that (1) may be rewritten as

$$\begin{pmatrix} A(\mathbf{p}) & 0 \\ C(\mathbf{p}) & -I \end{pmatrix} \mathbf{z} = \begin{pmatrix} B \\ -D \end{pmatrix} \mathbf{u}. \qquad (2)$$

A Parametrized Input-Output Solution

Let

$$\mathcal{P} = \{\mathbf{p} : p_k^- \leq p_k \leq p_k^+, k = 1, \cdots, \ell\}$$
$$\mathcal{U} = \{\mathbf{u} : u_j^- \leq u_j \leq u_j^+, j = 1, \cdots, r\}$$

Assumption

The parameter $\mathbf{p}$ *appears affinely in* $A(\mathbf{p})$ *and* $C(\mathbf{p})$*, that is*

$$A(\mathbf{p}) = A_0 + p_1 A_1 + \cdots + p_\ell A_\ell$$
$$C(\mathbf{p}) = C_0 + p_1 C_1 + \cdots + p_\ell C_\ell.$$

Assumption

$$|T(\mathbf{p})| \neq 0, p \in \mathcal{P}.$$

A Parametrized Input-Output Solution

Theorem

For the system described by (1), the input-output relationship is

$$y_i = \sum_{j=1}^{r} \frac{\beta_{ij}(\mathbf{p})}{\alpha(\mathbf{p})} u_j, \;\; i = 1, 2, \cdots, m, j = 1, 2, \cdots, r \tag{3}$$

with $\beta_{ij}(\mathbf{p})$ *and* $\alpha(\mathbf{p})$ *as already defined. In matrix form, the input outp ut relationship is:*

$$\mathbf{y} = \frac{1}{\alpha(\mathbf{p})} \begin{pmatrix} \beta_{11}(\mathbf{p}) & \cdots & \beta_{1r}(\mathbf{p}) \\ \vdots & & \vdots \\ \beta_{m1}(\mathbf{p}) & \cdots & \beta_{mr}(\mathbf{p}) \end{pmatrix} \mathbf{u}. \tag{4}$$

Measurement Based Approach

Knowledge $\alpha(\mathbf{p})$ and $\beta_{ij}(\mathbf{p})$ is sufficient to determine outputs y_i as a function of $\mathbf{p}$ and $\mathbf{u}$. Conducting experiments by setting $\mathbf{p}$ and input $\mathbf{u}$ to various values and measuring the corresponding y_i, these unknown coefficients can be determined. Let

$$y_1 = \frac{\beta_{11}(\mathbf{p})}{\alpha(\mathbf{p})} u_1 + \frac{\beta_{12}(\mathbf{p})}{\alpha(\mathbf{p})} u_2 \tag{6}$$

with

$$\begin{aligned} \beta_{1j}(\mathbf{p}) &= \beta_{1j0} + \beta_{1j1} p_1 + \beta_{1j2} p_2 + \beta_{1j3} p_1 p_2, j = 1, 2 \\ \alpha(\mathbf{p}) &= \alpha_0 + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_1 p_2 \end{aligned} \tag{7}$$

Measurement Based Approach

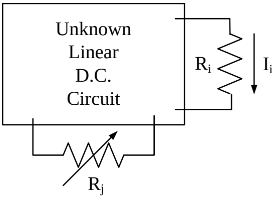
Set $u_2 = 0, u_1 = u_1^*$ and measure y_1 for seven different sets of values $(p_1,\ p_2)$ to determine the seven coefficients of $\alpha(\mathbf{p})$ and $\beta_{1j}(\mathbf{p}), j = 1, 2$, $y_1(i), i = 1, \cdots, 7$, $\mathbf{p}(i) = [p_1(i)\ p_2(i)]$

$$M \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \beta_{1j0} & \beta_{1j1} & \beta_{1j2} & \beta_{1j3} \end{pmatrix}' = -\begin{pmatrix} y_1(1)\ y_1(2)\ y_1(3)\ y_1(4)\ y_1(5)\ y_1(6)\ y_1(7) \end{pmatrix}' \tag{8}$$

M=

$$\begin{pmatrix}
y_1(1)p_1(1) & y_1(1)p_2(1) & y_1(1)p_1(1)p_2(1) & -u_1(1) & -u_1(1)p_1(1) & -u_1(1)p_2(1) & -u_1(1)p_1(1)p_2(1) \\
y_1(2)p_1(2) & y_1(2)p_2(2) & y_1(2)p_1(2)p_2(2) & -u_1(2) & -u_1(2)p_1(2) & -u_1(2)p_2(2) & -u_1(2)p_1(2)p_2(2) \\
y_1(3)p_1(3) & y_1(3)p_2(3) & y_1(3)p_1(3)p_2(3) & -u_1(3) & -u_1(3)p_1(3) & -u_1(3)p_2(3) & -u_1(3)p_1(3)p_2(3) \\
y_1(4)p_1(4) & y_1(4)p_2(4) & y_1(4)p_1(4)p_2(4) & -u_1(4) & -u_1(4)p_1(4) & -u_1(4)p_2(4) & -u_1(4)p_1(4)p_2(4) \\
y_1(5)p_1(5) & y_1(5)p_2(5) & y_1(5)p_1(5)p_2(5) & -u_1(5) & -u_1(5)p_1(5) & -u_1(5)p_2(5) & -u_1(5)p_1(5)p_2(5) \\
y_1(6)p_1(6) & y_1(6)p_2(6) & y_1(6)p_1(6)p_2(6) & -u_1(6) & -u_1(6)p_1(6) & -u_1(6)p_2(6) & -u_1(6)p_1(6)p_2(6) \\
y_1(7)p_1(7) & y_1(7)p_2(7) & y_1(7)p_1(7)p_2(7) & -u_1(7) & -u_1(7)p_1(7) & -u_1(7)p_2(7) & -u_1(7)p_1(7)p_2(7)
\end{pmatrix}$$

Linear DC Circuits: Current Control using a Single Resistor



- Design objective: control I_i (the current in the i-th branch)
- Design variable: resistor R_j at an arbitrary location

Theorem 2

In a linear DC circuit, the functional dependency of any current I_i on any resistance R_j can be determined by at most 3 measurements of the current I_i obtained for 3 different values of R_j.

N. Mohsenizadeh et al. "A Measurement Based Approach to Circuit Design". In: *IASTED International Conf. on Engineering and Applied Science*. Colombo, Sri Lanka, 2012, pp. 27–34.

Proof. Recall

$$A(p)x = b(q) \tag{5}$$

$$I_i = x_i = \frac{|B_i(p,q)|}{|A(p)|} \tag{6}$$

Each resistance R_j appears only in one column of $A(p)$; thus, we say it has rank 1 dependency.

Consider two cases:

- Case 1: $i \neq j$
- Case 2: $i = j$

Case 1 ($i \neq j$):

- $B_i(p,q)$ is of rank 1 w.r.t. R_j
- $A(p)$ is of rank 1 w.r.t. R_j

According to Lemma 1:

$$I_i(R_j) = \frac{\tilde{\alpha_0} + \tilde{\alpha_1} R_j}{\tilde{\beta_0} + \tilde{\beta_1} R_j} \tag{7}$$

where $\tilde{\alpha_0}, \tilde{\alpha_1}, \tilde{\beta_0}, \tilde{\beta_1}$ are constants.

- $\tilde{\beta}_0 = \tilde{\beta}_1 = 0 \Rightarrow I_i \to \infty, \ \forall R_j$, physically impossible $\Rightarrow$ Rule this out.
- $\tilde{\beta}_1 \neq 0$:

$$I_i(R_j) = \frac{\alpha_0 + \alpha_1 R_j}{\beta_0 + R_j} \tag{8}$$

  where $\alpha_0, \alpha_1, \beta_0$ are unknown constants.
- To determine $\alpha_0, \alpha_1, \beta_0$: Set R_j to 3 different values and measure I_i:

$$\underbrace{\begin{bmatrix} 1 & R_{j1} & -I_{i1} \\ 1 & R_{j2} & -I_{i2} \\ 1 & R_{j3} & -I_{i3} \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \beta_0 \end{bmatrix}}_{u} = \underbrace{\begin{bmatrix} I_{i1} R_{j1} \\ I_{i2} R_{j2} \\ I_{i3} R_{j3} \end{bmatrix}}_{m} \tag{9}$$

- $|M| \neq 0 \quad \Leftrightarrow$ unique solution for $\alpha_0, \alpha_1, \beta_0$
- $|M| = 0 \quad \Leftrightarrow$ last column of M is a linear combination of the first two columns:

$$I_i(R_j) = \alpha_0 + \alpha_1 R_j \tag{10}$$

  This corresponds to the case where $\tilde{\beta}_1 = 0$ in (7).

Case 2 $(i = j)$:

- $B_i(p, q)$ is of rank 0 w.r.t. R_i
- $A(p)$ is of rank 1 w.r.t. R_i

According to Lemma 1:

$$I_i(R_i) = \frac{\tilde{\alpha_0}}{\tilde{\beta}_0 + \tilde{\beta}_1 R_i} \tag{11}$$

- $\tilde{\beta}_1 \neq 0$:

$$I_i(R_i) = \frac{\alpha_0}{\beta_0 + R_i} \tag{12}$$

- To determine α_0, β_0: Set R_i to 2 different values and measure I_i:

$$\underbrace{\begin{bmatrix} 1 & -I_{i1} \\ 1 & -I_{i2} \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \alpha_0 \\ \beta_0 \end{bmatrix}}_{u} = \underbrace{\begin{bmatrix} I_{i1} R_{i1} \\ I_{i2} R_{i2} \end{bmatrix}}_{m} \tag{13}$$

- $|M| \neq 0 \quad \Leftrightarrow$ unique solution for α_0, β_0
- $|M| = 0 \quad \Leftrightarrow$

$$I_i(R_i) = \alpha_0 \tag{14}$$

Remarks

Consider

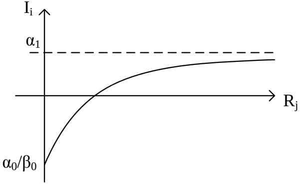
$$I_i(R_j) = \frac{\alpha_0 + \alpha_1 R_j}{\beta_0 + R_j} \quad (15)$$

Then

$$\frac{dI_i}{dR_j} = \frac{\alpha_1 \beta_0 - \alpha_0}{(\beta_0 + R_j)^2} \quad (16)$$

I_i is monotonic in R_j.

- $\frac{\alpha_0}{\beta_0} > \alpha_1 \Rightarrow I_i$ monotonically decrease
- $\frac{\alpha_0}{\beta_0} < \alpha_1 \Rightarrow I_i$ monotonically increase



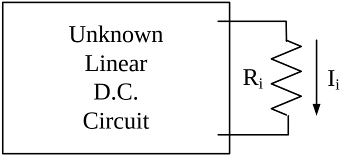
- The achievable range for I_i:

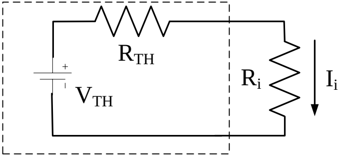
$$\min\{\frac{\alpha_0}{\beta_0}, \alpha_1\} < I_i < \max\{\frac{\alpha_0}{\beta_0}, \alpha_1\} \quad (17)$$

- Current control problem: I_i within a desired prescribed range, $I_i^- \leq I_i \leq I_i^+ \quad \Rightarrow \quad$ unique range for R_j, $R_j^- \leq R_j \leq R_j^+$

Thevenin's Theorem (special case of Theorem 2, $i = j$)

Thevenin's Theorem: The current in a resistor/impedance connected to an arbitrary network can be obtained by representing the network by a voltage source and a resistance/impedance and these can be determined from short circuit and open circuit measurements made at these terminals.





- Short circuit current: I_{sc}
- Open circuit voltage: $V_{oc} = V_{Th}$
- The Thevenin resistance: $R_{Th} = \frac{V_{oc}}{I_{sc}}$

$$I_i(R_i) = \frac{V_{Th}}{R_{Th} + R_i} \tag{18}$$

But this corresponds to the case $i = j$. Recall

$$I_i(R_i) = \frac{\alpha_0}{\beta_0 + R_i} \tag{19}$$

- Short circuit current: $I_{sc} = \frac{\alpha_0}{\beta_0}$
- Open circuit voltage: $V_{oc} = V_{Th} = \alpha_0$
- The Thevenin resistance: $R_{Th} = \frac{V_{oc}}{I_{sc}} = \beta_0$

Substituting into (19):

$$I_i(R_i) = \frac{V_{Th}}{R_{Th} + R_i} \tag{20}$$

which is exactly Thevenin's Theorem.

- It is not necessary to measure short circuit current or open circuit voltage, indeed two **arbitrary** measurements suffice.
- This has practical and useful implications in circuits where short circuiting and open circuiting may sometimes be impossible.

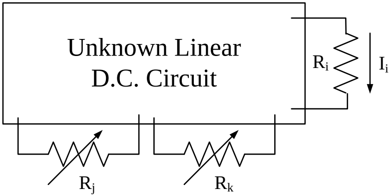
- Statement of Theorem 2 is the generalization of Thevenin's Theorem.
- For the general case $i \neq j$:

$$I_i(R_j) = \frac{\alpha_0 + \alpha_1 R_j}{\beta_0 + R_j} \tag{21}$$

- The resistor can be connected at a point different from the point where measurements are taken.
- The current can be predicted from arbitrary measurements, not necessarily from short and open circuit measurements.

N. Mohsenizadeh et al. "Linear Circuits: A Measurement Based Approach". In: *International Journal of Circuit Theory and Applications* 43.2 (2015). published online: July 2013, pp. 205–232.

Current Control using Two Resistors



- Design objective: control the current in the i-th branch, I_i
- Design variables: resistors R_j and R_k at arbitrary locations

Theorem 3

In a linear DC circuit, the functional dependency of any current I_i on any two resistances R_j and R_k can be determined by at most 7 measurements of the current I_i obtained for 7 different sets of values (R_j, R_k).

Proof. Recall

$$A(p)x = b(q) \tag{22}$$

$$I_i = x_i = \frac{|B_i(p,q)|}{|A(p)|} \tag{23}$$

Consider two cases:

- Case 1: $i \neq j, k$
- Case 2: $i = j$ or $i = k$

Case 1 ($i \neq j, k$):

- $B_i(p,q)$ is of rank 1 w.r.t. R_j and R_k
- $A(p)$ is of rank 1 w.r.t. R_j and R_k

According to Lemma 2:

$$I_i(R_j, R_k) = \frac{\tilde{\alpha_0} + \tilde{\alpha_1} R_j + \tilde{\alpha_2} R_k + \tilde{\alpha_3} R_j R_k}{\tilde{\beta_0} + \tilde{\beta_1} R_j + \tilde{\beta_2} R_k + \tilde{\beta_3} R_j R_k} \tag{24}$$

- $\tilde{\beta}_3 \neq 0$:

$$I_i(R_j, R_k) = \frac{\alpha_0 + \alpha_1 R_j + \alpha_2 R_k + \alpha_3 R_j R_k}{\beta_0 + \beta_1 R_j + \beta_2 R_k + R_j R_k} \tag{25}$$

- To determine $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2$: conduct 7 experiments

$$\underbrace{\begin{bmatrix} 1 & R_{j1} & R_{k1} & R_{j1}R_{k1} & -I_{i1} & -I_{i1}R_{j1} & -I_{i1}R_{k1} \\ 1 & R_{j2} & R_{k2} & R_{j2}R_{k2} & -I_{i2} & -I_{i2}R_{j2} & -I_{i2}R_{k2} \\ 1 & R_{j3} & R_{k3} & R_{j3}R_{k3} & -I_{i3} & -I_{i3}R_{j3} & -I_{i3}R_{k3} \\ 1 & R_{j4} & R_{k4} & R_{j4}R_{k4} & -I_{i4} & -I_{i4}R_{j4} & -I_{i4}R_{k4} \\ 1 & R_{j5} & R_{k5} & R_{j5}R_{k5} & -I_{i5} & -I_{i5}R_{j5} & -I_{i5}R_{k5} \\ 1 & R_{j6} & R_{k6} & R_{j6}R_{k6} & -I_{i6} & -I_{i6}R_{j6} & -I_{i6}R_{k6} \\ 1 & R_{j7} & R_{k7} & R_{j7}R_{k7} & -I_{i7} & -I_{i7}R_{j7} & -I_{i7}R_{k7} \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}}_{u} = \underbrace{\begin{bmatrix} I_{i1}R_{j1}R_{k1} \\ I_{i2}R_{j2}R_{k2} \\ I_{i3}R_{j3}R_{k3} \\ I_{i4}R_{j4}R_{k4} \\ I_{i5}R_{j5}R_{k5} \\ I_{i6}R_{j6}R_{k6} \\ I_{i7}R_{j7}R_{k7} \end{bmatrix}}_{m} \tag{26}$$

- $|M| \neq 0 \quad \Leftrightarrow$ unique solution for $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2$
- $|M| = 0 \quad \Rightarrow$ corresponding functional dependency can be obtained.

Case 2 ($i = j$ or $i = k$):

- $B_i(p, q)$ is of rank 0 w.r.t. R_i and is of rank 1 w.r.t. R_k
- $A(p)$ is of rank 1 w.r.t. R_i and R_k

According to Lemma 2:

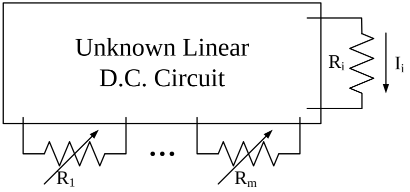
$$I_i(R_i, R_k) = \frac{\tilde{\alpha_0} + \tilde{\alpha_1} R_k}{\tilde{\beta_0} + \tilde{\beta_1} R_i + \tilde{\beta_2} R_k + \tilde{\beta_3} R_i R_k} \tag{27}$$

- $\tilde{\beta_3} \neq 0$:

$$I_i(R_i, R_k) = \frac{\alpha_0 + \alpha_1 R_k}{\beta_0 + \beta_1 R_i + \beta_2 R_k + R_i R_k} \tag{28}$$

- To determine $\alpha_0, \alpha_1, \beta_0, \beta_1, \beta_2$: conduct 5 experiments

Current Control using m Resistors



- Design objective: control the current in the i-th branch, I_i
- Design variables: resistors $R_1, R_2, \ldots, R_m$ at arbitrary locations

Theorem 4

In a linear DC circuit, the functional dependency of any current I_i on any m resistances R_j, $j = 1, 2, \ldots, m$, can be determined by at most $2^{m+1} - 1$ measurements of the current I_i obtained for $2^{m+1} - 1$ different sets on values $(R_1, R_2, \ldots, R_m)$.

Proof. Consider two cases:

- Case 1: $i \neq j$ for $j = 1, 2, \ldots, m$
- Case 2: $i = j$ for some $j = 1, 2, \ldots, m$

Case 1:

- $B_i(p, q)$ is of rank 1 w.r.t. R_j, $j = 1, 2, \ldots, m$
- $A(p)$ is of rank 1 w.r.t. R_j, $j = 1, 2, \ldots, m$

According to Lemma 2:

$$I_i(R_1, R_2, \ldots, R_m) = \frac{\sum_{i_m=0}^{1} \cdots \sum_{i_2=0}^{1} \sum_{i_1=0}^{1} \alpha_{i_1 i_2 \cdots i_m} \; R_1^{i_1} R_2^{i_2} \cdots R_m^{i_m}}{\sum_{i_m=0}^{1} \cdots \sum_{i_2=0}^{1} \sum_{i_1=0}^{1} \beta_{i_1 i_2 \cdots i_m} \; R_1^{i_1} R_2^{i_2} \cdots R_m^{i_m}} \quad (29)$$

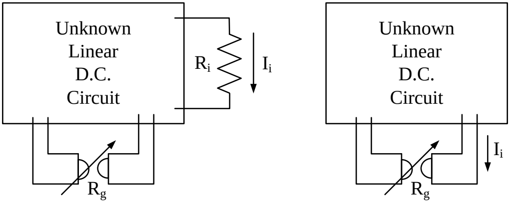
Case 2 (assume $i = m$):

- $B_i(p, q)$: rank 1 w.r.t. R_j, $j = 1, 2, \ldots, m-1$, and rank 0 w.r.t. R_m
- $A(p)$ is of rank 1 w.r.t. R_j, $j = 1, 2, \ldots, m$

According to Lemma 2:

$$I_i(R_1, R_2, \ldots, R_m) = \frac{\sum_{i_{m-1}=0}^{1} \cdots \sum_{i_2=0}^{1} \sum_{i_1=0}^{1} \alpha_{i_1 i_2 \cdots i_{m-1}} \; R_1^{i_1} R_2^{i_2} \cdots R_{m-1}^{i_{m-1}}}{\sum_{i_m=0}^{1} \cdots \sum_{i_2=0}^{1} \sum_{i_1=0}^{1} \beta_{i_1 i_2 \cdots i_m} \; R_1^{i_1} R_2^{i_2} \cdots R_m^{i_m}} \quad (30)$$

Current Control using Gyrator Resistance



- Design objective: control the current in the *i*-th branch, I_i
- Design variable: gyrator resistance R_g at an arbitrary location

Theorem 5

In a linear DC circuit, the functional dependency of any current I_i on any gyrator resistance R_g can be determined by at most 5 measurements of the current I_i obtained for 5 different values of R_g.

Proof.

Each gyrator resistance R_g appears in 2 columns of $A(p)$.

Consider two cases:

- Case 1: i-th branch is not connected to either port of the gyrator
- Case 2: i-th branch is connected to one port of the gyrator

Case 1:

- $B_i(p,q)$ is of rank 2 w.r.t. R_g
- $A(p)$ is of rank 2 w.r.t. R_g

According to Lemma 1:

$$I_i(R_g) = \frac{\alpha_0 + \alpha_1 R_g + \alpha_2 R_g^2}{\beta_0 + \beta_1 R_g + R_g^2} \tag{31}$$

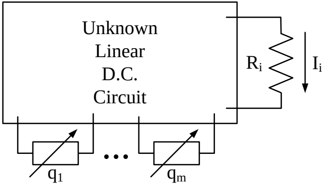
Case 2:

- $B_i(p,q)$ is of rank 1 w.r.t. R_g
- $A(p)$ is of rank 2 w.r.t. R_g

According to Lemma 1:

$$I_i(R_g) = \frac{\alpha_0 + \alpha_1 R_g}{\beta_0 + \beta_1 R_g + R_g^2} \tag{32}$$

Current Control using Independent Sources



- Design objective: control the current in the i-th branch, I_i
- Design variables: independent sources $q_1, q_2, \ldots, q_m$ at arbitrary locations

Theorem 6

In a linear DC circuit, the functional dependency of any current I_i on the independent sources can be determined by m measurements of the current I_i obtained for m linearly independent sets of values of the source vector q, where m is the number of independent sources.

Proof. Recall:

$$b(q) = b_1 q_1 + b_2 q_2 + \cdots + b_m q_m \tag{33}$$

$$\Rightarrow \quad B_i(p,q) = [A_1(p), \ldots, A_{i-1}(p), b(q), A_{i+1}(p), \ldots, A_n(p)] \tag{34}$$

- $B_i(p,q)$ is of rank 1 w.r.t. $q_1, q_2, \ldots, q_m$
- $|B_i(p,q)|$ can be written as a linear combination of $q_1, q_2, \ldots, q_m$:

$$|B_i(p,q)| = |B_{i1}(p)|q_1 + |B_{i2}(p)|q_2 + \cdots + |B_{im}(p)|q_m \tag{35}$$

where

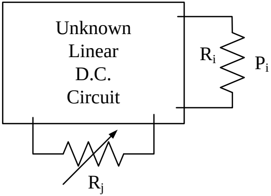
$$B_{ij}(p) = [A_1(p), \ldots, A_{i-1}(p), b_j, A_{i+1}(p), \ldots, A_n(p)], \; j = 1, 2, \ldots, m$$

- $A(p)$ is of rank 0 w.r.t. $q_1, q_2, \ldots, q_m \quad \Rightarrow \quad |A(p)| = const.$

The functional dependency will be:

$$I_i(q_1, q_2, \ldots, q_m) = \alpha_1 q_1 + \alpha_2 q_2 + \cdots + \alpha_m q_m \tag{36}$$

Power Level Control using a Single Resistor



- Design objective: control the power level P_i, in the resistor R_i, located in the i-th branch
- Design variable: resistor R_j at an arbitrary location

Theorem 7

In a linear DC circuit, the functional dependency of the power level P_i, in the resistor R_i, on any resistance R_j can be determined by at most 3 measurements of the current I_i (passing through R_i) obtained for 3 different values of R_j, and 1 measurement of the voltage across the resistor R_i, corresponding to one of the resistance settings.

Proof. Consider two cases:

- Case 1: $i \neq j$
- Case 2: $i = j$

Case 1 ($i \neq j$):

- $P_i(R_j) = \frac{V_i}{I_i}\ I_i^2(R_j)$
- The functional dependency $I_i(R_j)$ is as either forms (8) or (10)
- The ratio $\frac{V_i}{I_i}$ is the same for each experiment $\Rightarrow$ only 1 extra measurement of V_i, across the resistor R_i, is needed
- Assume one measures V_{i1} from the first experiment
- The functional dependency will be:
  - If $|M| \neq 0$ in (9):

$$P_i(R_j) = \frac{V_{i1}}{I_{i1}} \left(\frac{\alpha_0 + \alpha_1 R_j}{\beta_0 + R_j} \right)^2 \tag{37}$$

  - If $|M| = 0$ in (9):

$$P_i(R_j) = \frac{V_{i1}}{I_{i1}} (\alpha_0 + \alpha_1 R_j)^2 \tag{38}$$

Case 2 ($i = j$):

- $P_i(R_i) = R_i \; I_i^2(R_i)$
- The functional dependency $I_i(R_i)$ is as either forms (12) or (14)
- The functional dependency will be:
  - If $|M| \neq 0$ in (13):

$$P_i(R_i) = R_i \left(\frac{\alpha_0}{\beta_0 + R_i} \right)^2 \tag{39}$$

  - If $|M| = 0$ in (13):

$$P_i(R_i) = \alpha_0^2 R_i \tag{40}$$

Similarly, the functional dependency of P_i on any two or more resistances, or any gyrator resistance, can be obtained.

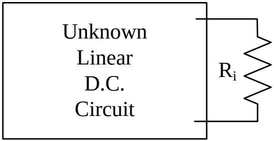
Maximum Power Transfer Theorem

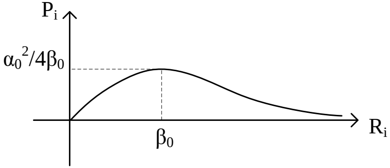
Maximum Power Transfer Theorem: To obtain maximum external power from a source with an internal resistance, the resistance of the load (R_i) must be equal to the resistance of the source as viewing from the output terminals (R_{Th}).

Recall

$$P_i(R_i) = R_i \left(\frac{\alpha_0}{\beta_0 + R_i} \right)^2 \tag{41}$$

and that $\beta_0 = R_{Th}$.

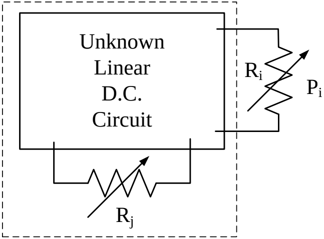


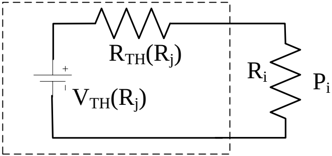


$$P_{i,max} = \frac{\alpha_0^2}{4\beta_0} \tag{42}$$

Question:

Is it possible to adjust R_j and set the load resistance $R_i = R_{Th}(R_j)$ to receive maximum power P_i?



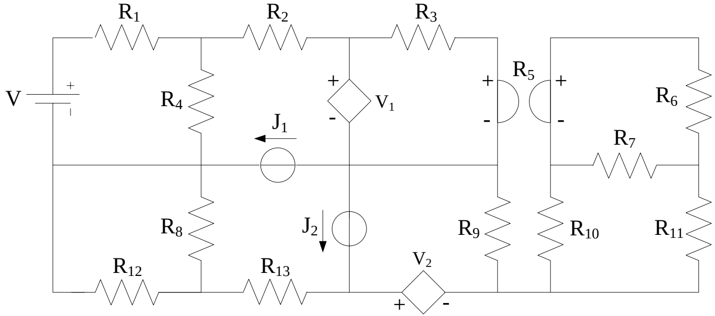


$$P_i(R_i, R_j) = R_i \left(\frac{\alpha_0 + \alpha_1 R_j}{\beta_0 + \beta_1 R_i + \beta_2 R_j + R_i R_j} \right)^2 \tag{43}$$

$$\begin{matrix} \frac{dP_i}{dR_i} = 0 \\ \frac{dP_i}{dR_j} = 0 \end{matrix} \Rightarrow \begin{cases} R_j = \frac{2\alpha_1\beta_0\beta_1 - \alpha_0\beta_1\beta_2 - \alpha_0\beta_0}{2\alpha_0\beta_2 - \alpha_1\beta_0 - \alpha_1\beta_1\beta_2} \\ R_i = \frac{\alpha_1\beta_0 - \alpha_0\beta_2}{\alpha_0 - \alpha_1\beta_1} \end{cases} \tag{44}$$

$$P_{i,\max} = \frac{\alpha_0\alpha_1\beta_0 + \alpha_0\alpha_1\beta_1\beta_2 - {\alpha_0}^2\beta_2 - {\alpha_1}^2\beta_0\beta_1}{(\beta_0 - \beta_1\beta_2)^2} \tag{45}$$

Example: A DC Circuit



- Design objective: control the power levels

$$40 \leq P_3 \leq 60 \; (W) \quad (46)$$
$$1 \leq P_6 \leq 8 \; (W) \quad (47)$$
$$0.5 \leq P_{11} \leq 5 \; (W) \quad (48)$$

- Design variables: resistors R_1 and R_6

- P_3 vs. R_1 and R_6:

$$P_3(R_1, R_6) = \frac{V_{3,1}}{I_{3,1}} \left(\frac{\alpha_0 + \alpha_1 R_1 + \alpha_2 R_6 + \alpha_3 R_1 R_6}{\beta_0 + \beta_1 R_1 + \beta_2 R_6 + R_1 R_6} \right)^2$$

7 measurements of current and 1 measurement of voltage is needed.

Exp.No.	$R_1(\Omega)$	$R_6(\Omega)$	$I_3(A)$
1	7	1	3.33
2	13	8	2.71
3	21	19	2.47
4	35	26	2.57
5	40	32	2.52
6	52	45	2.47
7	59	56	2.44

Exp.No.	$R_1(\Omega)$	$R_6(\Omega)$	$V_{3,1}(V)$
1	7	1	33.3

- These numerical values yield: $\alpha_0 = 98.4,\ \alpha_1 = 36,\ \alpha_2 = 6.6,\ \alpha_3 = 2.4,\ \beta_0 = 58.5,\ \beta_1 = 5,\ \beta_2 = 11.7 \quad \Rightarrow$

$$P_3(R_1, R_6) = \frac{33.3}{3.33} \left(\frac{98.4 + 36 R_1 + 6.6 R_6 + 2.4 R_1 R_6}{58.5 + 5 R_1 + 11.7 R_6 + R_1 R_6} \right)^2$$

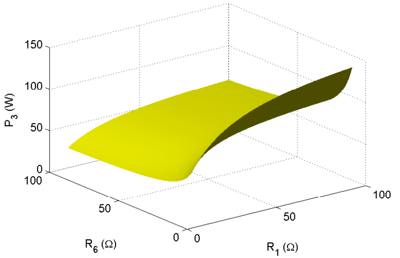


Figure: P_3 vs. R_1 and R_6

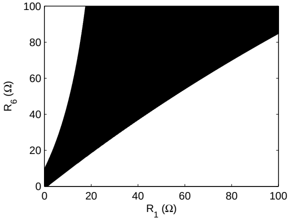


Figure: Region (in black color) where (46) is satisfied.

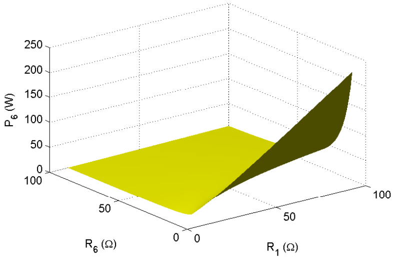


Figure: P_6 vs. R_1 and R_6

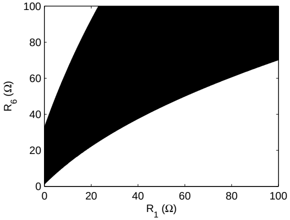


Figure: Region (in black color) where (47) is satisfied.

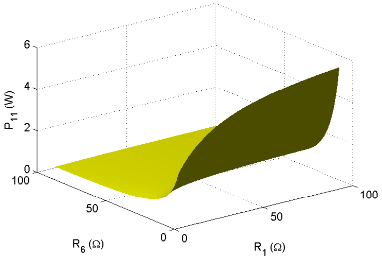


Figure: P_{11} vs. R_1 and R_6

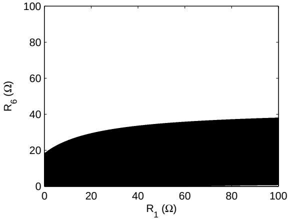


Figure: Region (in black color) where (48) is satisfied.

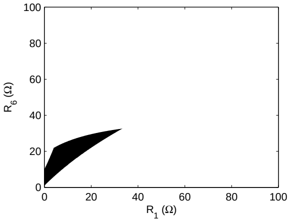


Figure: Region (in black color) where (46), (47) and (48) are satisfied.

Motivational Example

Consider the DC circuit

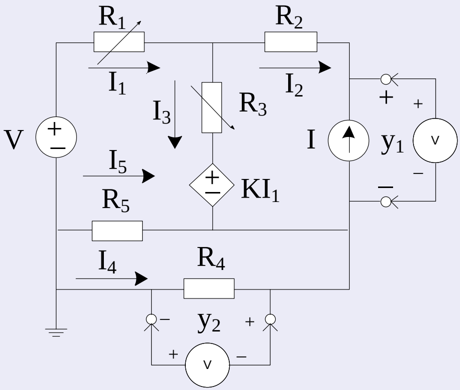


Figure: Circuit used in simulations.

Simulation Example Using PSpice

A simulation was performed with the circuit shown before by selecting R_1, R_3 as parameters as illustrated below.

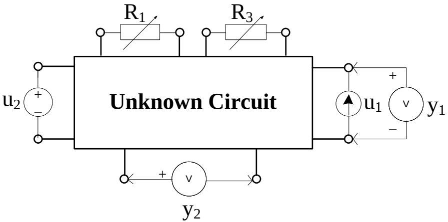


Figure: Circuit showing the parameters, inputs and outputs varied to obtain the results

Simulation Example Using PSpice

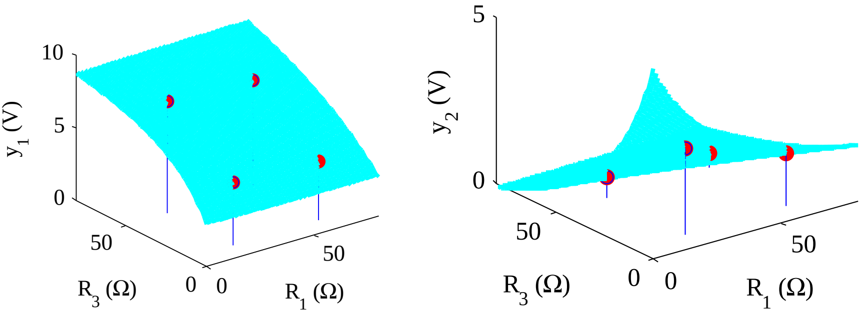


Figure: Solution surface for y_1 and y_2 showing the extremal points of obtained via PSpice.

Simulation Example Using PSpice

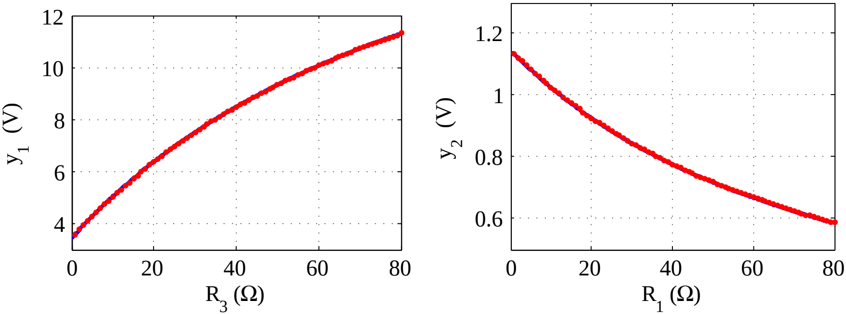
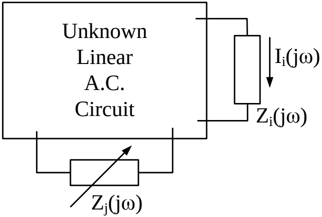


Figure: Solution for y_1 and y_2 with via Theorem 1 (solid line) and PSpice simulation (dashed line).

Linear AC Circuits:
Current Control using a Single Impedance

Consider a linear AC circuit operating at the steady-state and at a fixed frequency ω.



- Design objective: control the current in the i-th branch, I_i
- Design variable: impedance Z_j at an arbitrary location

Theorem 8

In a linear AC circuit, the functional dependency of any current phasor I_i on any impedance Z_j can be determined by at most 3 measurements of the current phasor I_i obtained for 3 different complex values of Z_j.

Proof. Similar to its DC circuit counterpart. The main difference is that the circuit signals/variables and constants appearing in the functional dependencies will be complex quantities rather than real numbers.

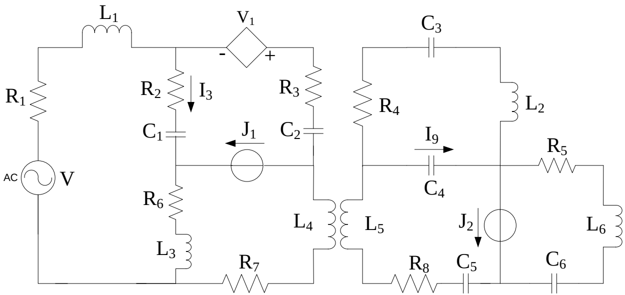
Case 1 ($i \neq j$):

$$I_i(Z_j) = \frac{\alpha_0 + \alpha_1 Z_j}{\beta_0 + Z_j} \tag{49}$$

Case 2 ($i = j$):

$$I_i(Z_i) = \frac{\alpha_0}{\beta_0 + Z_i} \tag{50}$$

Example: An AC Circuit



- Design objective: control the currents

$$0 \leq |I_3| \leq 4\ (A), \tag{51}$$

$$10 \leq \angle I_3 \leq 30\ (deg), \tag{52}$$

$$0 \leq |I_9| \leq 2.5\ (A), \tag{53}$$

$$-30 \leq \angle I_9 \leq -10\ (deg). \tag{54}$$

- Design variables: inductor L_1 and capacitor C_2

- I_3 vs. L_1 and C_2:

$$I_3(L_1, C_2) = \frac{\alpha_0 + \alpha_1 L_1 j\omega_0 + \alpha_2/(C_2 j\omega_0) + \alpha_3 L_1/C_2}{\beta_0 + \beta_1 L_1 j\omega_0 + \beta_2/(C_2 j\omega_0) + L_1/C_2}$$

7 measurements of current phasor I_3 for 7 different sets of values (L_1, C_2) is needed.

Exp.No.	$L_1(mH)$	$C_2(\mu F)$	$I_3(A)$
1	13	10	3.3-2.9i
2	25	20	2.7-3.2i
3	32	23	2.3-3.4i
4	45	29	1.4-3.6i
5	54	33	.7-3.5i
6	68	40	-.5-2.9i
7	90	47	-1.4-1.3i

$$I_3(L_1, C_2) = \frac{(-1502 - 2772j) + (173 + 74j)L_1 j\omega_0 + (106 + 151j)/(C_2 j\omega_0)}{(-481 - 316j) + (13 + 13j)L_1 j\omega_0 + (30 + 15j)/(C_2 j\omega_0) + L_1/C_2}$$

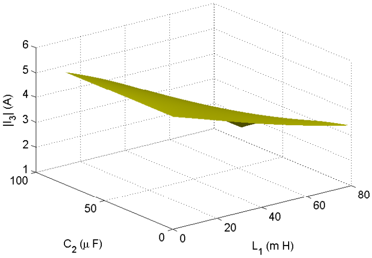


Figure: $|I_3(j\omega_0)|$ vs. L_1 and C_2

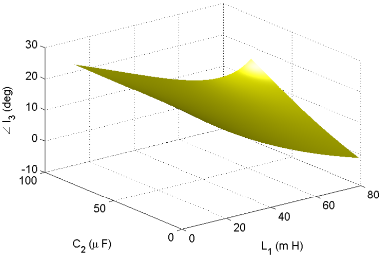


Figure: $\angle I_3(j\omega_0)$ vs. L_1 and C_2

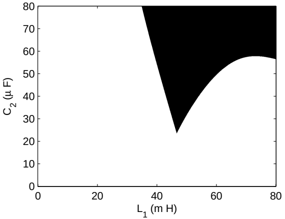


Figure: Region (in black color) where (51) and (52) are satisfied.

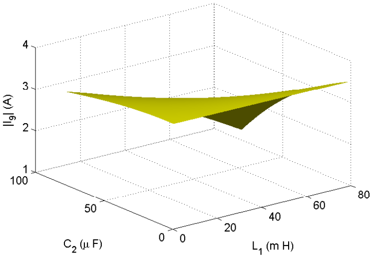


Figure: $|I_9(j\omega_0)|$ vs. L_1 and C_2

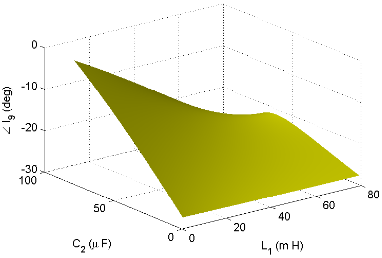


Figure: $\angle I_9(j\omega_0)$ vs. L_1 and C_2

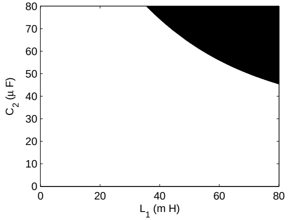


Figure: Region (in black color) where (53) and (54) are satisfied.

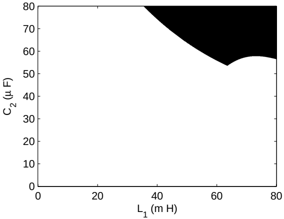
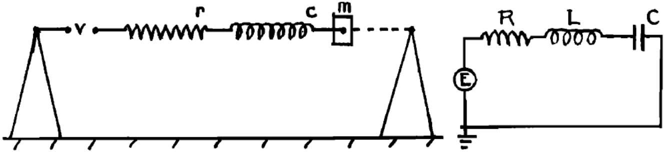


Figure: Region (in black color) where (51)-(54) are satisfied.

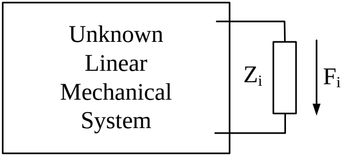
Linear Mechanical Systems: Analogy btw. Electrical and Mechanical Systems

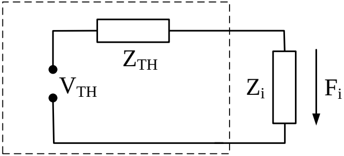
Electrical Sys.	Mechanical Sys.
Voltage, V	Velocity, V
Current, I	Force, F
Resistance, R	Lubricity, 1/B
Capacitance, C	Mass, M
Inductance, L	Compliance, 1/K

- Electrical systems: $V = IZ_{\text{elec}} \quad , \quad Z_{\text{elec}} = R + i(\omega L - 1/\omega C)$
- Mechanical systems: $V = F\bar{Z}_{\text{mech}} \quad , \quad \bar{Z}_{\text{mech}} = r + i(\omega c - 1/\omega m)$
- $1/B = r$ and $1/K = c$



Equivalent Thevenin's Theorem

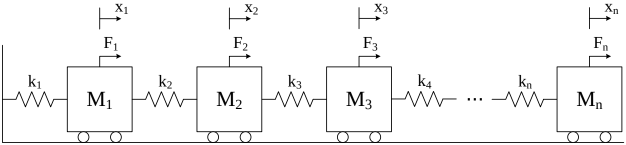




$$F_i(\bar{Z}_i) = \frac{V_{Th}}{\bar{Z}_{Th} + \bar{Z}_i} \tag{55}$$

- $P_i(\bar{Z}_i) = F_i V_i = \bar{Z}_i F_i^2 = \bar{Z}_i(\frac{V_{Th}}{\bar{Z}_{Th}+\bar{Z}_i})^2$
- $P_i(\bar{Z}_i)$ is maximum if $\bar{Z}_i = Z_{Th}$.

Network of Springs



- Design objective: control the displacements of the masses
- Design variable: some set of spring stiffness at arbitrary locations

$$\underbrace{\begin{bmatrix} k_1+k_2 & -k_2 & 0 & \cdots & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & \cdots & 0 & 0 \\ 0 & -k_3 & k_3+k_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -k_{n-1} & k_n \end{bmatrix}}_{A(p)} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_n \end{bmatrix}}_{b(q)} \tag{56}$$

N. Mohsenizadeh et al. "A Measurement Based Approach to Mechanical Systems". In: *9th Asian Control Conference*. Istanbul, Turkey, 2013.

- $p = [k_1, k_2, \ldots, k_n]^T$
- x: vector of unknown displacements
- q: vector of external forces

Theorem 9

In a network of linear springs, the functional dependency of any displacement x_i on any spring stiffness k_j can be determined by 3 measurements of the displacement x_i obtained for 3 different values of k_j.

Proof.

- $B_i(p, q)$ is of rank 1 w.r.t. k_j
- $A(p)$ is of rank 1 w.r.t. k_j

According to Lemma 1:

$$x_i(k_j) = \frac{\tilde{\alpha}_0 + \tilde{\alpha}_1 k_j}{\tilde{\beta}_0 + \tilde{\beta}_1 k_j} \tag{57}$$

- $\tilde{\beta}_0 = \tilde{\beta}_1 = 0 \Rightarrow x_i \to \infty, \ \forall k_j$, physically impossible $\Rightarrow$ Rule this out.

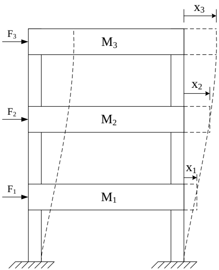
- $\tilde{\beta}_1 \neq 0$:

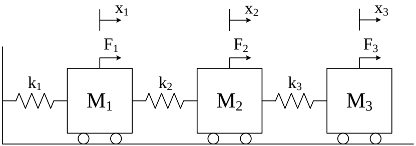
$$x_i(k_j) = \frac{\alpha_0 + \alpha_1 k_j}{\beta_0 + k_j} \tag{58}$$

- To determine $\alpha_0, \alpha_1, \beta_0$:

$$\underbrace{\begin{bmatrix} 1 & k_{j1} & -x_{i1} \\ 1 & k_{j2} & -x_{i2} \\ 1 & k_{j3} & -x_{i3} \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \beta_0 \end{bmatrix}}_{u} = \underbrace{\begin{bmatrix} x_{i1} k_{j1} \\ x_{i2} k_{j2} \\ x_{i3} k_{j3} \end{bmatrix}}_{m} \tag{59}$$

Example: A Network of Springs





- Design objective: control x_2

$$-5 \leq x_2 \leq -3 \quad (cm) \tag{60}$$

- Design variables: spring stiffness k_2

- Based on Theorem 9: 3 measurements of x_2 are needed for 3 different values of k_2:

$$x_2 = \frac{\alpha_0 + \alpha_1 k_2}{\beta_0 + k_2} \tag{61}$$

Exp. #	k_2 (kN/m)	x_2 (cm)
1	200	-3.5
2	300	-3.0
3	500	-2.6

- Using these numerical values:

$$x_2 = \frac{-3000 - 0.02k_2}{k_2} \tag{62}$$

- Applying the constraint on x_2:

$$100 \leq k_2 \leq 300 \quad (kN/m) \tag{63}$$

Truss Structures

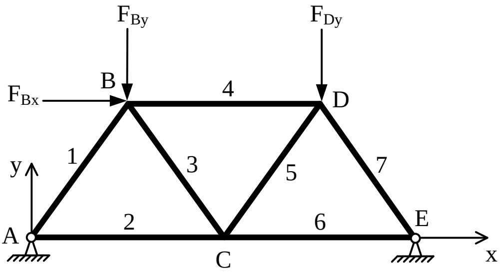
- Governing equations for static deflections:

$$A(p)x = b(q) \tag{64}$$

- $A(p)$: global stiffness matrix
- x: vector of joint deflections
- $b(q)$: vector of external forces (including reaction forces)
- $A(p)$ will be generated by assembling element-wise stiffness matrices K_e:

$$K_e = \frac{E_e A_e}{L_e} \begin{bmatrix} cos^2\theta_e & \frac{1}{2}sin2\theta_e & -cos^2\theta_e & -\frac{1}{2}sin2\theta_e \\ \frac{1}{2}sin2\theta_e & sin^2\theta_e & -\frac{1}{2}sin2\theta_e & -sin^2\theta_e \\ -cos^2\theta_e & -\frac{1}{2}sin2\theta_e & cos^2\theta_e & \frac{1}{2}sin2\theta_e \\ -\frac{1}{2}sin2\theta_e & -sin^2\theta_e & \frac{1}{2}sin2\theta_e & sin^2\theta_e \end{bmatrix} \tag{65}$$

- For complex truss structures the size of $A(p)$ becomes very large



$$\underbrace{\begin{bmatrix} A_{11}(p) & A_{12}(p) \\ A_{12}^T(p) & A_{22}(p) \end{bmatrix}}_{A(p)} \underbrace{\begin{bmatrix} \delta_{Ax} \\ \delta_{Ay} \\ \vdots \\ \delta_{Ex} \\ \delta_{Ey} \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} F_{Ax} \\ F_{Ay} \\ \vdots \\ F_{Ex} \\ F_{Ey} \end{bmatrix}}_{b(q)} \tag{66}$$

$$A_{11}(p) = \begin{bmatrix} R_1 c^2\theta_1 + R_2 c^2\theta_2 & R_1 s\theta_1 + R_2 s\theta_2 & -R_1 c^2\theta_1 & -R_1 s\theta_1 & -R_2 c^2\theta_2 \\ & R_1 s^2\theta_1 + R_2 s^2\theta_2 & -R_1 s\theta_1 & -R_1 s^2\theta_1 & -R_2 s\theta_2 \\ & & R_1 c^2\theta_1 + R_3 c^2\theta_3 + R_4 c^2\theta_4 & R_1 s\theta_1 + R_3 s\theta_3 + R_4 s\theta_4 & -2R_3 c^2\theta_3 \\ & & & R_1 s^2\theta_1 + R_3 s^2\theta_3 + R_4 s^2\theta_4 & -2R_3 s\theta_3 \\ & & & & R_2 c^2\theta_2 + R_3 c^2\theta_3 + R_5 c^2\theta_5 + R_6 c^2\theta_6 \end{bmatrix}$$

$$A_{12}(p) = \begin{bmatrix} -R_2 s\theta_2 & 0 & 0 & 0 & 0 \\ -R_2 s^2\theta_2 & 0 & 0 & 0 & 0 \\ -2R_3 s\theta_3 & -R_4 c^2\theta_4 & -R_4 s\theta_4 & 0 & 0 \\ -2R_3 s^2\theta_3 & -R_4 s\theta_4 & -R_4 s^2\theta_4 & 0 & 0 \\ R_2 s\theta_2 + R_3 s\theta_3 + R_5 s\theta_5 + R_6 s\theta_6 & -R_5 c^2\theta_5 & -R_5 s\theta_5 & -R_6 c^2\theta_6 & -R_6 s\theta_6 \end{bmatrix}$$

$$A_{22}(p) = \begin{bmatrix} R_2 s^2\theta_2 + R_3 s^2\theta_3 + R_5 s^2\theta_5 + R_6 s^2\theta_6 & -R_5 s\theta_5 & -R_5 s^2\theta_5 & -R_6 s\theta_6 & -R_6 s^2\theta_6 \\ & R_4 c^2\theta_4 + R_5 c^2\theta_5 + R_7 c^2\theta_7 & R_4 s\theta_4 + R_5 s\theta_5 + R_7 s\theta_7 & -R_7 c^2\theta_7 & -R_7 s\theta_7 \\ & & R_4 s^2\theta_4 + R_5 s^2\theta_5 + R_7 s^2\theta_7 & -R_7 s\theta_7 & -R_7 s^2\theta_7 \\ & & & R_6 c^2\theta_6 + R_7 c^2\theta_7 & R_6 s\theta_6 + R_7 s\theta_7 \\ & & & & R_6 s^2\theta_6 + R_7 s^2\theta_7 \end{bmatrix}$$

- Design objective: control the deflection of the joints
- Design variable: mechanical properties of elements (for example: cross section area)

Theorem 10

In a truss structure, the functional dependency of any joint displacement δ_i, in any direction, on any element cross section area A_j can be determined by 3 measurements of the joint displacement δ_i, in the respective direction, obtained for 3 different values of A_j.

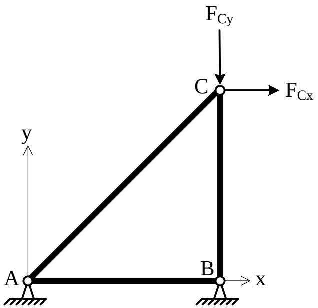
Proof.

- $B_i(p, q)$ is of rank 1 w.r.t. A_j
- $A(p)$ is of rank 1 w.r.t. A_j

According to Lemma 1:

$$\delta_i(A_j) = \frac{\alpha_0 + \alpha_1 A_j}{\beta_0 + A_j} \tag{67}$$

Example: A Truss Structure



- Design objective: control δ_{Cx}

$$0 \leq \delta_{Cx} \leq 2 \quad (cm) \tag{68}$$

- Design variables: cross section area of the link AC, A_{AC}

- General form of the functional dependency:

$$\delta_{Cx} = \frac{\alpha_0 + \alpha_1 A_{AC}}{\beta_0 + A_{AC}} \tag{69}$$

Exp. #	A_{AC} (mm^2)	δ_{Cx} (cm)
1	100	1.8
2	150	1.4
3	200	1.2

- Using these numerical values:

$$\delta_{Cx} = \frac{1.1 \times 10^{-6} + 6.67 \times 10^{-3} A_{AC}}{A_{AC}} \tag{70}$$

- Applying the constraint on δ_{Cx}:

$$A_{AC} \geq 83 \ (mm^2) \tag{71}$$

Linear Hydraulic Networks

- If all the flows are in the laminar state $\Rightarrow$ linear equations

$$A(p)x = b(q)$$

- Pressure drop:

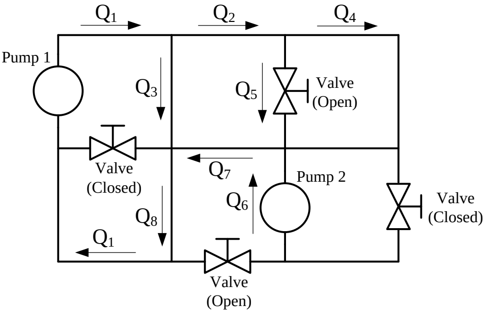
$$\Delta P = \frac{8\mu LQ}{\pi r^4} \tag{72}$$

μ: dynamic viscosity, L: length of pipe, Q: flow rate, r: inner radius

- Rewrite (72) as: $\Delta P = RQ$, where pipe resistance constant is:

$$R = \frac{8\mu L}{\pi r^4} \tag{73}$$

- $A(p)$: contains the mechanical properties of pipes
- x: vector of unknown flow rates
- $b(q)$: inputs to the system (such as pump pressures)



$$\underbrace{\begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ R_1 & 0 & R_3 & 0 & 0 & 0 & 0 & R_8 \\ 0 & -R_2 & R_3 & 0 & -R_5 & R_6 & 0 & R_8 \\ 0 & 0 & 0 & -R_4 & R_5 & 0 & 0 & 0 \\ 0 & -R_2 & R_3 & -R_4 & 0 & 0 & -R_7 & 0 \end{bmatrix}}_{A(p)} \underbrace{\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ P_1 \\ P_2 \\ 0 \\ 0 \end{bmatrix}}_{b(q)} \tag{74}$$

Flow Rate Control using a Single Pipe Resistance

- Design objective: control the flow rates
- Design variable: any pipe resistance R_j at an arbitrary location

Theorem 11

In a linear hydraulic network, the functional dependency of any flow rate Q_i on any pipe resistance R_j can be determined by at most 3 measurements of the flow rate Q_i obtained for 3 different values of R_j.

Proof. Similar to DC circuits:

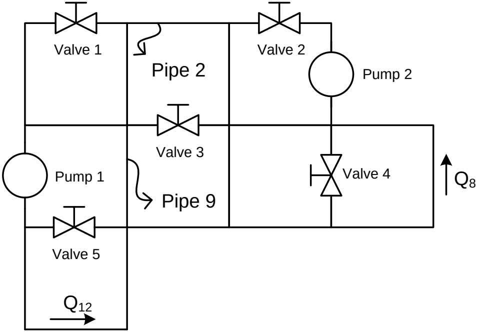
- Case 1 ($i \neq j$): $B_i(p,q)$ and $A(p)$ are both of rank 1 w.r.t. R_j:

$$Q_i(R_j) = \frac{\alpha_0 + \alpha_1 R_j}{\beta_0 + R_j} \tag{75}$$

- Case 2 ($i = j$): $B_i(p,q)$: rank 0 w.r.t. R_i, $A(p)$: rank 1 w.r.t. R_i:

$$Q_i(R_i) = \frac{\alpha_0}{\beta_0 + R_i} \tag{76}$$

Example: A Linear Hydraulic Network



- Design objective: control flow rates Q_8 and Q_{12}

$$0.045 \leq Q_8 \leq 0.055 \ (m^3/s) \tag{77}$$

$$0.01 \leq Q_{12} \leq 0.03 \ (m^3/s) \tag{78}$$

- Design variables: radii of the pipes numbered 2 and 9, r_2 and r_9

- General form of the functional dependency:

$$Q_i(R_j, R_k) = \frac{\alpha_0 + \alpha_1 R_j + \alpha_2 R_k + \alpha_3 R_j R_k}{\beta_0 + \beta_1 R_j + \beta_2 R_k + R_j R_k} \tag{79}$$

Exp.#	r_2 (m)	R_2 $(Pa.s/m^3)$	r_9 (m)	R_9 $(Pa.s/m^3)$	Q_8 (m^3/s)
1	0.05	408	0.05	408	0.038
2	0.07	107	0.08	62	0.043
3	0.09	39	0.11	17	0.049
4	0.1	26	0.13	9	0.051
5	0.12	12	0.15	5	0.054
6	0.14	6	0.17	3	0.055
7	0.17	3	0.2	1.6	0.056

- Using these numerical values:

$$Q_8(r_2, r_9) = \frac{8.7 \times 10^7 + \frac{1600}{r_2^4} + \frac{3500}{r_9^4} + \frac{0.034}{r_2^4 r_9^4}}{1.5 \times 10^9 + \frac{48000}{r_2^4} + \frac{75000}{r_9^4} + \frac{1}{r_2^4 r_9^4}} \tag{80}$$

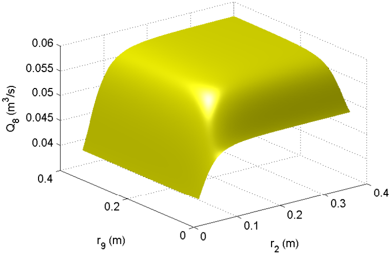


Figure: Q_8 vs. r_2 and r_9

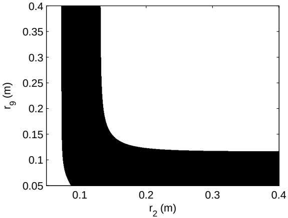


Figure: Region where (77) is satisfied

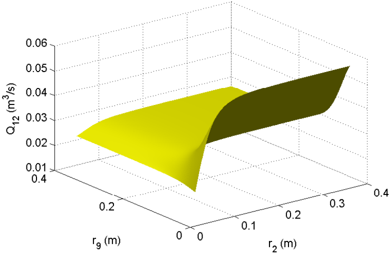


Figure: Q_{12} vs. r_2 and r_9

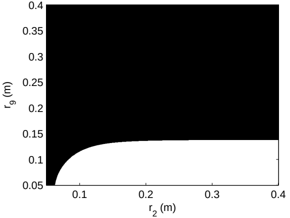


Figure: Region where (78) is satisfied

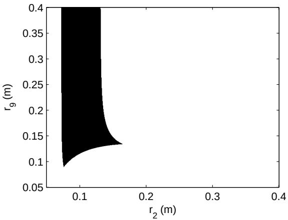


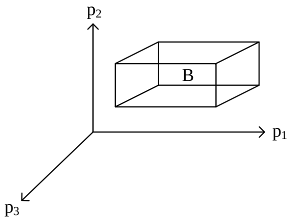
Figure: Region where (77) and (78) are satisfied

Application to Robust Stability Analysis

In an interval linear system:

$$A(p)x = b(q)$$

p and q are varying in intervals (box $\mathcal{B}$):



In physical systems, p *usually* appears in $A(p)$ with rank one dependency:

- circuits: resistors, impedances and dependent sources
- truss structures: mechanical properties of the links
- hydraulic networks: pipe resistances
- signal flow block diagrams: each block

Recalling the *monotonic* behavior of the solution set x, we want to characterize the extremal values of x over $\mathcal{B}$.

Theorem 12

Suppose $A(p) = A_0 + p_1 A_1$, $\mathrm{rank}(A_1) = 1$, *and* p_1 *is varying in an interval,* $\mathcal{I} = [p_1^-, p_1^+]$, *then the extremal values of* x_i *can be obtained from:*

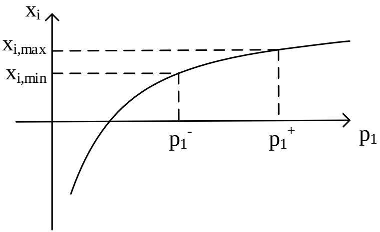
$$\min_{p_1 \in \mathcal{I}} x_i(p_1) = \min\{x_i(p_1^-), x_i(p_1^+)\}$$

$$\max_{p_1 \in \mathcal{I}} x_i(p_1) = \max\{x_i(p_1^-), x_i(p_1^+)\}$$

Proof. For $p = p_1$ and $\mathrm{rank}(A_1) = 1$:

$$x_i(p_1) = \frac{\alpha_0 + \alpha_1 p_1}{\beta_0 + p_1}$$

which is monotonic in p_1.

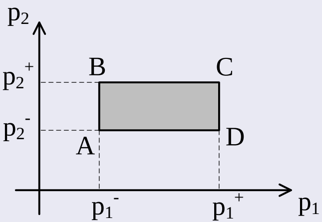


D. N. Mohsenizadeh, L. H. Keel, and S. P. Bhattacharyya. "An Extremal Result for Unknown Interval Linear Systems". In: *19th IFAC World Congress*. Cape Town, South Africa, 2014, pp. 6502–6507.

Theorem 13

Suppose that $A(p) = A_0 + p_1 A_1 + p_2 A_2$, $\mathrm{rank}(A_1) = \mathrm{rank}(A_2) = 1$, *and* p_1 *and* p_2 *are varying in a rectangle,* $\mathcal{R}$*:*

$$\mathcal{R} = \{(p_1, p_2) \mid p_1^- \leq p_1 \leq p_1^+,\ p_2^- \leq p_2 \leq p_2^+\}$$



then the extremal values of x_i *happen at the vertices of* $\mathcal{R}$*:*

$$\min_{p_1, p_2 \in \mathcal{R}} x_i(p_1, p_2) = \min\{x_i(A), x_i(B), x_i(C), x_i(D)\}$$

$$\max_{p_1, p_2 \in \mathcal{R}} x_i(p_1, p_2) = \max\{x_i(A), x_i(B), x_i(C), x_i(D)\}$$

Theorem 14

If $A(p) = A_0 + p_1 A_1 + p_2 A_2 + \cdots + p_l A_l$, $\mathrm{rank}(A_i) = 1$, $i = 1, 2, \ldots, l$, $b(q) = b_1 q_1 + b_2 q_2 + \cdots + b_m q_m$, *and* (p, q) *are varying in a box,* $\mathcal{B}$*:*

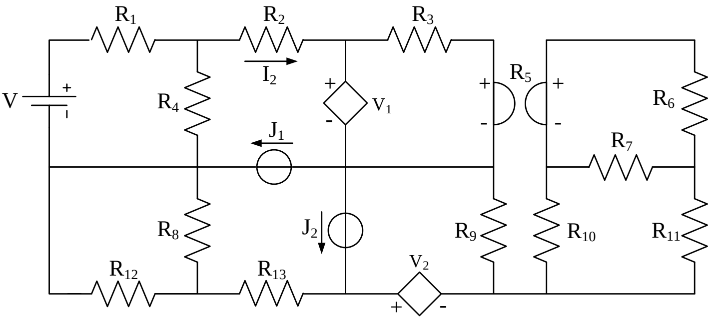
$$\mathcal{B} = \{(p, q) \mid p_i^- \leq p_i \leq p_i^+,\ i = 1, \ldots, l,\ q_j^- \leq q_j \leq q_j^+,\ j = 1, \ldots, m\}$$

with $v := 2^{l+m}$ *vertices, labeled* $V_1, V_2, \ldots, V_v$*, then:*

$$\min_{p,q \in \mathcal{B}} x_i(p, q) = \min\{x_i(V_1), x_i(V_2), \ldots, x_i(V_v)\}$$

$$\max_{p,q \in \mathcal{B}} x_i(p, q) = \max\{x_i(V_1), x_i(V_2), \ldots, x_i(V_v)\}$$

Example: Electrical Circuits



Problem: Find the extremal values of I_2, if R_1 is varying in:

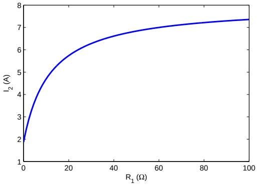
$$\mathcal{I} = [R_1^-, R_1^+] = [10, 30] \ (\Omega)$$

Solution: Using Theorem 12, the extremal values of I_2 occur at $R_1^- = 10\ (\Omega)$ and $R_1^+ = 30\ (\Omega)$:

$$I_{2,\min} = 4.7\ (A), \quad I_{2,\max} = 6.3\ (A)$$

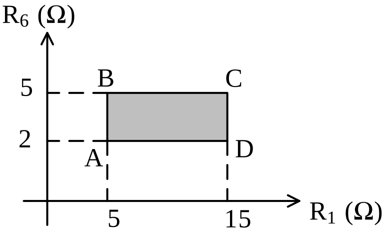
Alternative Approach: First find $I_2(R_1)$ by 3 measurements, and then evaluate I_2 at the extremes of $\mathcal{I}$:

Exp. No.	R_1 (Ω)	I_2 (A)
1	7	4.2
2	18	5.6
3	32	6.4



$$I_2(R_1) = \frac{21.9 + 8R_1}{11.7 + R_1}$$

Problem: Considering the same circuit, find the extremal values of P_3 (in $R_3 = 10\ (\Omega)$) over $\mathcal{R}$:



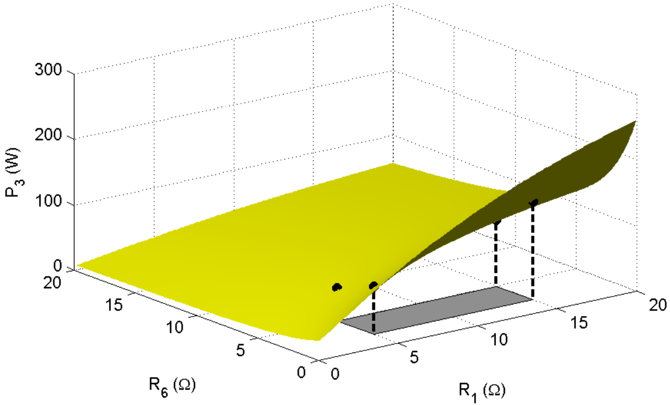
Solution:

$$P_3(R_1, R_6) = R_3\ I_3^2(R_1, R_6)$$

$I_3(R_1, R_6)$ is monotonic in R_1 and R_6, thus our extremal result gives:

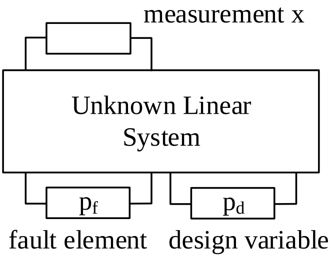
$$P_{3,\min} = 49.4\ (W) \text{ at vertex B} = (5, 5)$$
$$P_{3,\max} = 150\ (W) \text{ at vertex D} = (15, 2)$$

Alternative Approach: First find $P_3(R_1, R_6)$ using 7 measurements and then evaluate P_3 by setting (R_1, R_6) to the values of vertices of $\mathcal{R}$:



Fault-tolerant System Design: Single Failure

Consider the following system



Problem: Design p_d such that the system performance x stays within an acceptable range as p_f undergoes normal and failure states.

- For example: $p_f = p_f^*$ (normal state), $p_f = 0$ or $p_f = \infty$ (failure state)

P. Kallakuri, L. H. Keel, and S. P. Bhattacharyya. "Reliable Measurement-Based System Design: A New Paradigm". In: *19th IFAC World Congress*. Cape Town, South Africa, 2014, pp. 9394–9399.

- Acceptable range of x: $x \in [x_{\min}, x_{\max}]$
- Supposing that p_f and p_d appear in the system characteristic matrix with rank 1 dependency:

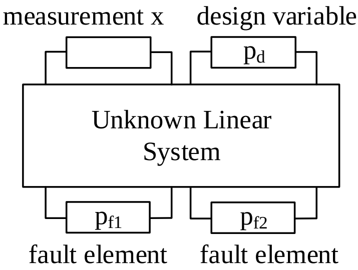
$$x(p_f, p_d) = \frac{\alpha_0 + \alpha_1 p_f + \alpha_2 p_d + \alpha_3 p_f p_d}{\beta_0 + \beta_1 p_f + \beta_2 p_d + p_f p_d}$$

where α's and β's can be determined from 7 measurements.

- The design task is to determine p_d such that:

$$x_{\min} \leq \min_{p_f} x(p_f, p_d), \qquad x_{\max} \geq \max_{p_f} x(p_f, p_d)$$

Fault-tolerant System Design: Two Failures



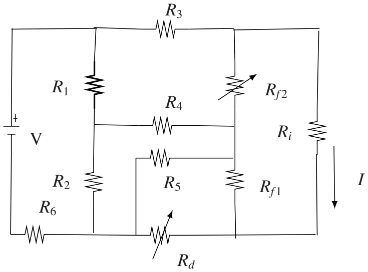
- If p_{f1}, p_{f2}, p_d appear with rank 1 dependency:

$$x(p_{f1}, p_{f2}, p_d) = \frac{\alpha_0 + \alpha_1 p_{f1} + \alpha_2 p_{f2} + \alpha_3 p_d + \alpha_4 p_{f1} p_{f2} + \alpha_5 p_{f1} p_d + \alpha_6 p_{f2} p_d + \alpha_7 p_{f1} p_{f2} p_d}{\beta_0 + \beta_1 p_{f1} + \beta_2 p_{f2} + \beta_3 p_d + \beta_4 p_{f1} p_{f2} + \beta_5 p_{f1} p_d + \beta_6 p_{f2} p_d + p_{f1} p_{f2} p_d}$$

- Design task: determine p_d such that:

$$x_{\min} \leq \min_{p_{f1}, p_{f2}} x(p_{f1}, p_{f2}, p_d), \qquad x_{\max} \geq \max_{p_{f1}, p_{f2}} x(p_{f1}, p_{f2}, p_d)$$

Example: A Fault-tolerant System Design



- Resistors R_{f1} and R_{f2} are most vulnerable to faults.
- Design R_d so that I (current through R_i) stays within:

$$[I_{\min}, I_{\max}] = [0.5, 4] \quad (A)$$

- Functional dependency $I(R_d, R_{f1}, R_{f2})$:

$$I(R_d, R_{f1}, R_{f2}) = \frac{\alpha_0 + \alpha_1 R_d + \alpha_2 R_{f1} + \alpha_3 R_{f2} + \alpha_4 R_d R_{f1} + \alpha_5 R_{f1} R_{f2} + \alpha_6 R_{f2} R_d + \alpha_7 R_d R_{f1} R_{f2}}{\beta_0 + \beta_1 R_d + \beta_2 R_{f1} + \beta_3 R_{f2} + \beta_4 R_d R_{f1} + \beta_5 R_{f1} R_{f2} + \beta_6 R_{f2} R_d + R_d R_{f1} R_{f2}}$$

Measurements to determine $I(R_d, R_{f1}, R_{f2})$:

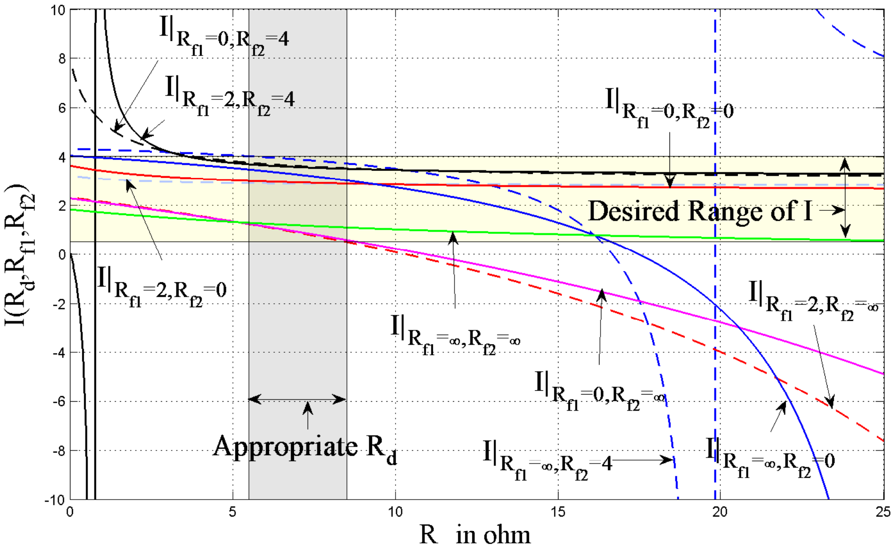
Exp.No	$R_d(\Omega)$	$R_{f1}(\Omega)$	$R_{f2}(\Omega)$	$I(R_d, R_{f1}, R_{f2})(A)$
1	1	1	1	3.53
2	2	3	4	3.62
3	7	7	7	4.35
4	10	9	12	3.99
5	13	12	17	3.68
6	18	16	20	3.54
7	21	20	25	3.24
8	24	28	31	2.85
9	29	35	35	2.65
10	34	42	43	2.38
11	39	51	56	2.06
12	43	67	67	1.77
13	51	75	70	1.71
14	58	83	79	1.59
15	75	90	85	1.53

$$I(R_d, R_{f1}, R_{f2}) = \frac{7.5 - 2.1R_d - 0.9R_{f1} + 1.28R_{f2} + 0.03R_dR_{f1} + 0.08R_{f1}R_{f2} + 0.11R_{f2}R_d - 0.0002R_dR_{f1}R_{f2}}{2 - 0.5R_d - 0.4R_{f1} + 0.5R_{f2} + 0.01R_dR_{f1} + 0.006R_{f1}R_{f2} + 0.02R_{f2}R_d + 0.001R_dR_{f1}R_{f2}}$$

Failure conditions:

Fault Condition	R_{f1}	R_{f2}
1	Short	Normal
2	Open	Normal
3	Normal	Short
4	Normal	Open
5	Short	Short
6	Short	Open
7	Open	Short
8	Open	Open
9	Normal	Normal

I vs. R_d for different values of R_{f1}, and R_{f2}:



- The appropriate range of R_d for fault tolerance has to be selected such that

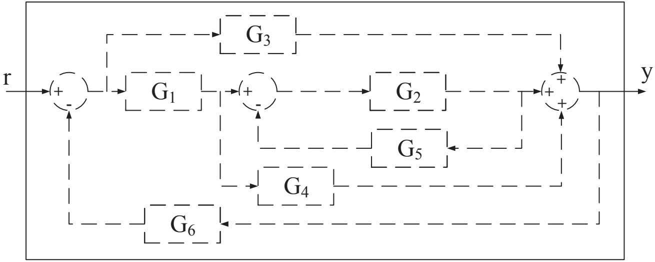
$$I(R_{d,\min}, R_{f1}, R_{f2}) \in [0.5, 4] \ \ (A), \quad I(R_{d,\max}, R_{f1}, R_{f2}) \in [0.5, 4] \ \ (A)$$

for all the considered fault conditions.

- Design range for R_d will be:

$$R_d \in [5.5, 8.5] \ \ (\Omega)$$

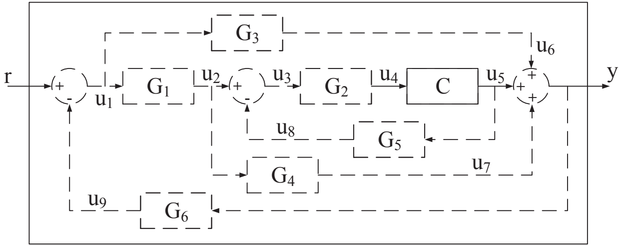
Consider a general unknown complex linear control system:



Problem: How can one synthesize a controller, at a specific location of an unknown complex system, such that desired stability margins and performance specifications can be attained?

A. Datta et al. "Towards Data Based Adaptive Control". In: *International Journal of Adaptive Control and Signal Processing* 27.1-2 (2013), pp. 122–135.

Introduce a new controller C at an arbitrary location:



$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ G_1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & G_2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ G_3 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & G_4 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & G_5 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & G_6 \end{bmatrix}}_{A(p)} \underbrace{\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_7 \\ U_8 \\ U_9 \\ Y \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} R \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{b(q)} \tag{81}$$

Results

Theorem 15

Let H denote the frequency response between any two arbitrary loop-breaking points of an unknown linear control system, and C represent a design controller at an arbitrary location, then

$$H = \frac{\alpha_0 + \alpha_1 C}{1 + \beta C} \tag{82}$$

where α_0, α_1 and β are complex quantities.

To determine α_0, α_1 and β: Connect 3 stabilizing controllers, C^1, C^2 and C^3, in the system, measure $H(j\omega)$ at a finite set of frequencies $\omega_k,\ k = 1, 2, \ldots, N$, and solve:

$$\begin{bmatrix} 1 & C^1(j\omega_k) & -H^1(j\omega_k)C^1(j\omega_k) \\ 1 & C^2(j\omega_k) & -H^2(j\omega_k)C^2(j\omega_k) \\ 1 & C^3(j\omega_k) & -H^3(j\omega_k)C^3(j\omega_k) \end{bmatrix} \begin{bmatrix} \alpha_0(j\omega_k) \\ \alpha_1(j\omega_k) \\ \beta(j\omega_k) \end{bmatrix} = \begin{bmatrix} H^1(j\omega_k) \\ H^2(j\omega_k) \\ H^3(j\omega_k) \end{bmatrix} \tag{83}$$

at each frequency $\omega_k,\ k = 1, 2, \ldots, N$.

Theorem 15 can be generalized to the case where two controllers involve:

Theorem 16

Let H denote the frequency response between any two arbitrary loop-breaking points of an unknown linear control system. Suppose that C_1 and C_2 are two design controllers at arbitrary locations. Then

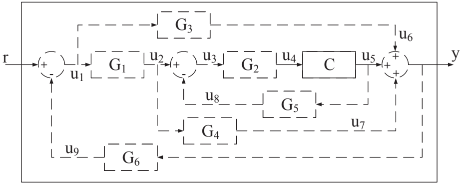
$$H = \frac{\alpha_0 + \alpha_1 C_1 + \alpha_2 C_2 + \alpha_3 C_1 C_2}{1 + \beta_1 C_1 + \beta_2 C_2 + \beta_3 C_1 C_2}, \tag{84}$$

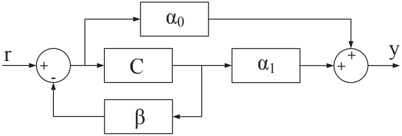
where α's and β's are complex quantities.

Realization of

$$H = \frac{\alpha_0 + \alpha_1 C}{1 + \beta C} \tag{85}$$

is an equivalent single-loop representation of the original unknown system:





- $\alpha_0, \alpha_1, \beta$: can be determined from measurements
- β: *equivalent plant* of the original unknown system
- Stability margins of the original unknown complex system can be assessed from $\beta(j\omega)$

D. N. Mohsenizadeh, L. H. Keel, and S. P. Bhattacharyya. "An Equivalent Plant Representation for Unknown Control Systems". In: *7th ASME Dynamic Systems and Control Conf.* San Antonio, TX, 2014.

- Knowledge of one stabilizing controller suffices to find p^+ and z^+, the number of RHP poles and zeros of the (equivalent) plant.
- Any given controller can then be checked for stability: if $C(j\omega)$ and $\beta(j\omega)$ satisfy certain conditions at a set of specific frequencies, stability is guaranteed.

Control Design Problem: Design a stabilizing controller C such that a desired set of stability margins can be attained.

Approach:

- Based on the desired stability margins, define: $H^*(j\omega)$
- Perform measurements, find α's and β's, and solve (85) for $C(j\omega)$:

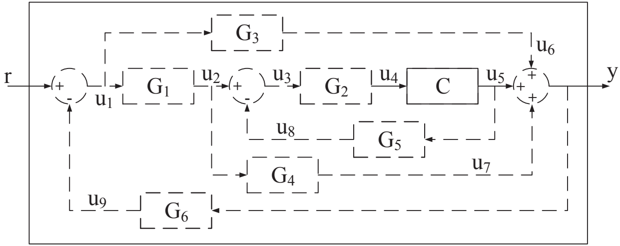
$$C(j\omega) = \frac{H^*(j\omega) - \alpha_0(j\omega)}{\alpha_1(j\omega) - H^*(j\omega)\beta(j\omega)} \tag{86}$$

S. P. Bhattacharyya, A. Datta, and L. H. Keel. *Linear Control Theory: Structure, Robustness, and Optimization*. CRC Press, Boca Raton, FL, 2009.

L. H. Keel and S. P. Bhattacharyya. "A Bode Plot Characterization of All Stabilizing Controllers". In: *IEEE Transactions on Automatic Control* 55.11 (2010), pp. 2650–2654.

Example: A Controller Design

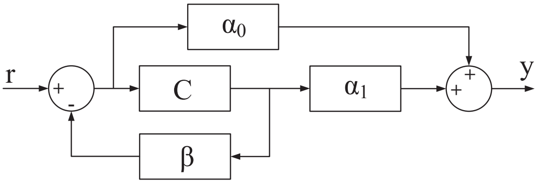
Consider the following unknown complex control system:



- What are the stability margins of the system?
- Design a controller to satisfy the followings for closed-loop system:

$$\begin{aligned} \text{GM} &= \text{Inf} \\ \text{PM} &> 60 \text{ deg} \\ \text{Bandwidth} &> 10 \text{ rad/sec} \end{aligned} \tag{87}$$

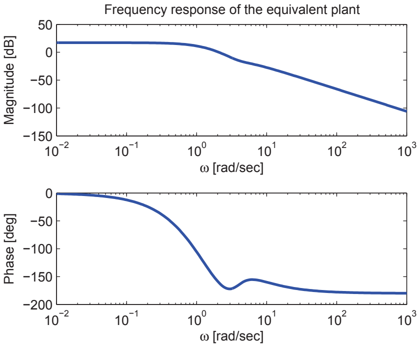
The equivalent single-loop system of the original system:



To find $\alpha_0(j\omega), \alpha_1(j\omega)$ and $\beta(j\omega)$ we connected the following 3 stabilizing PID controllers to the system and measured $H(j\omega)$, between r and y. Then we solved (83) for $\alpha_0(j\omega), \alpha_1(j\omega)$ and $\beta(j\omega)$.

$$C^1(s) = \frac{s^2+9s+8}{s}, \quad C^2(s) = \frac{2s^2+8s+1}{s}, \quad C^3(s) = \frac{s^2+2s+1}{s}$$

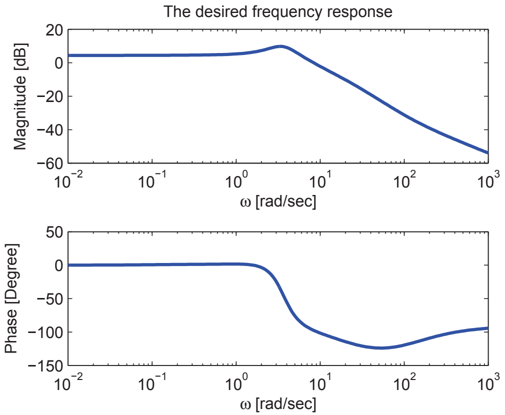
$\beta(j\omega)$: the frequency response of the equivalent plant



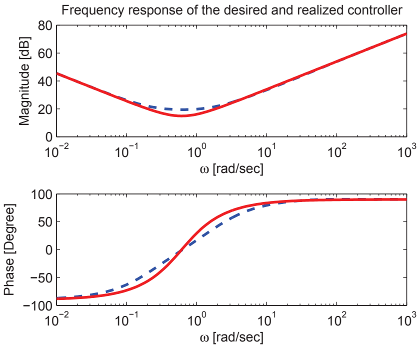
Stability margins and bandwidth of the original system:

$$\begin{aligned} \text{GM} &= \text{Inf} \\ \text{PM} &= 20 \text{ deg} \\ \text{Bandwidth} &= 2.4 \text{ rad/sec} \end{aligned} \quad (88)$$

- The problem is to design C so that the desired specifications are met.
- Define $H^*(j\omega)$ which meets the specifications (plotted below)



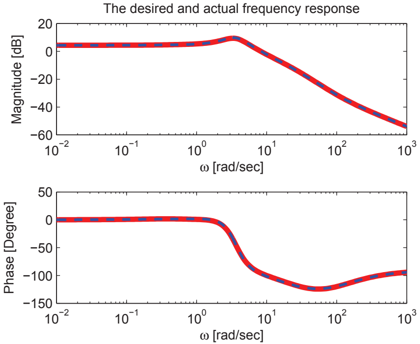
Replacing $\alpha_0(j\omega), \alpha_1(j\omega), \beta(j\omega)$ and $H^*(j\omega)$ into (86), one finds $C^*(j\omega)$ (dashed lines):



$C^*(j\omega)$ can be realized by a transfer function representation, $C^r(s)$, using system identification methods (freq response of C^r is shown by solid lines)

$$C^r(s) = \frac{5s^2 + 5.6s + 1.9}{s} \tag{89}$$

Embedding C^r in the original system we obtained the frequency response between r and y (solid lines). $H^*(j\omega)$ is also shown by dashed lines.



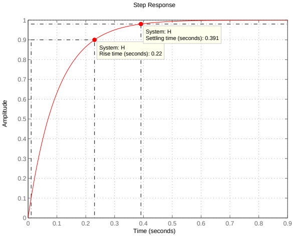
The new stability margins and bandwidth using C^r:

$$\begin{aligned} \text{GM} &= \text{Inf} \\ \text{PM} &= 83 \text{ deg} \\ \text{Bandwidth} &= 10.8 \text{ rad/sec} \end{aligned} \tag{90}$$

Experimental PI Controller Design

Problem: Design a PI controller for the following unknown servo motor so that the motor speed response to a step input has:

$$t_r = 0.22 \quad \text{sec}, \qquad t_s = 0.4 \quad \text{sec}$$



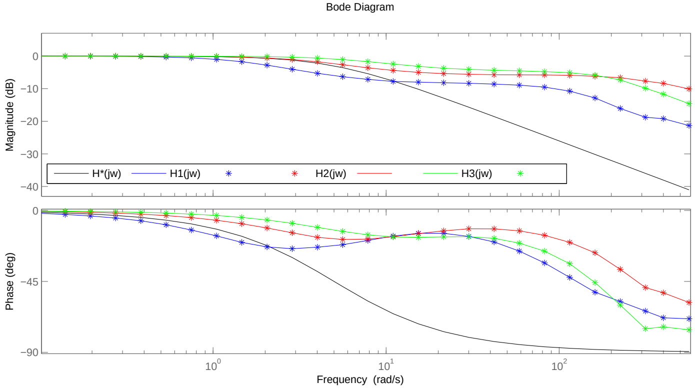
- Define a desired closed loop frequency response:

$$H^*(j\omega) = \frac{1}{\tau j\omega + 1}, \quad \tau = 0.1 \tag{91}$$

I. Diaz-Rodriguez, D. N. Mohsenizadeh, and S. P. Bhattacharyya. "Experimental PID Controller Design: A New Frequency Domain Approach based on Desired Performance". In: *7th ASME Dynamic Systems and Control Conf.* San Antonio, TX, 2014.

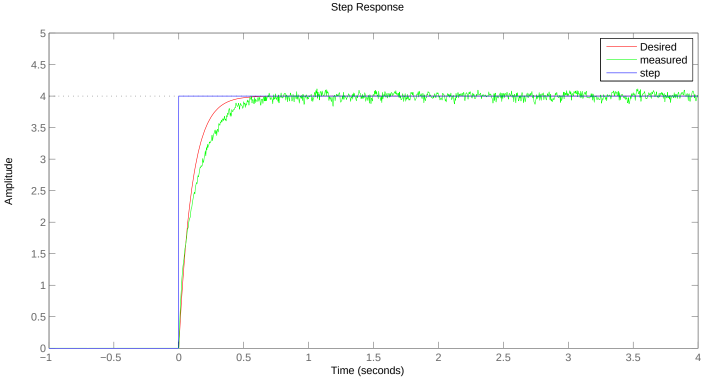
- Connect 3 controllers and measure the closed loop frequency response.

$$C_1(s) = \frac{0.5s + 2}{s}, \quad C_2(s) = \frac{0.8s + 6}{s}, \quad C_3(s) = \frac{s + 12}{s}$$



- Calculate $\alpha_0(j\omega), \alpha_1(j\omega)$ and $\beta(j\omega)$
- Find $C^*(j\omega)$ and realize it by a PI transfer function to get:

$$C^r(s) = \frac{0.2s + 4.477}{s}$$



Current Research

The proposed method can be applied to a broad range of problems in engineering and science, such as

- Power distribution networks
- Bioinformatics and biological systems
- Distributed network of multi-agent systems

These topics are currently under research.

D. N. Mohsenizadeh et al. "A New Measurement Based Approach to the Study of Biological Systems". In: *6th International Symposium on Communications, Control and Signal Processing*. Athens, Greece, 2014, pp. 48–52.

Conclusions

- We generalized the Thevenin's Theorem.
- In unknown linear systems, the functional dependency of any system variable on any set of the design elements, at arbitrary locations of the system, can be determined by a small number of measurements.
- The obtained functional dependency can be used to solve design problems.
- For control systems, an equivalent single-loop representation of a general unknown complex system can be found. The stability margins can be evaluated and a controller design can be accomplished.
- For interval linear systems (with parameters with rank one dependency), the extremal values of system variables occur at the vertices of the box in the parameter space.
- Fault-tolerant system design can be achieved using our proposed approach and based on a small set of measurements.

Thank you